axioms

Article

New Interval-Valued Soft Separation Axioms

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Abstract: Our research's main aim is to study two viewpoints: First, we define partial interval-valued soft T_i(j)-spaces (i = 0, 1, 2, 3, 4; j = i, ii), study some of their properties and some of relationships among them, and give some examples. Second, we introduce the notions of partial total interval-valued soft T_j(i)-spaces (i = 0, 1, 2, 3, 4; j = i, ii) and discuss some of their properties. We present some relationships among them and give some examples.

Keywords: interval-valued soft topological space; interval-valued soft continuous mapping; Partial interval-valued soft T-spaces; Partial total interval-valued soft T-spaces

MSC: 54A40; 54C05; 54D10; 54D15

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1. Introduction

Topology has been studied as a generalization of real systems. There are six types of separation axioms frequently used in classical topology. These axioms are very helpful in distinguishing topological spaces. From this viewpoint, we need to study separation axioms in interval-valued soft topological spaces.

In 1999, Molodtsov [1] proposed the concept of soft sets, which has practically been applied to several fields as a tool for solving uncertainties. Afterward, Maji et al. [2] defined various basic operations on soft sets and investigated some of their properties (see [3–6] for further studies). Furthermore, many researchers have applied the notion of soft sets to decision-making problems (see [7–9]), topological groups (see [10–15]), topology (see [16–27]), etc.

In 2021, Lee et al. [28] studied interval-valued soft topological structures as a generalization of soft topologies. Recently, Alcantud [29] discussed some relationships between fuzzy soft topologies and soft topologies. Ghour and Ameen [30] dealt with maximum of compactness and connectedness in soft topological spaces. Garg et al. [31] introduced the concept of spherical fuzzy soft topologies, studied separation axioms in a spherical fuzzy soft topological space, and applied them to group decision-making problems. Alajlan and Alghamdi [32] proposed a new soft topology from an ordinary topology and investigated separation axioms in the new soft topological spaces. Furthermore, Baek et al. [33] introduced the concepts of separation axioms in interval-valued soft topological spaces and investigated some of their properties and some relationships among them.

We would like to define and study new separation axioms in interval-valued soft topological spaces by modifying the separation axioms in the soft topological spaces introduced by El-Shafei et al. [34] and Al-Shami and El-Shafe [35]. This article is composed of six sections. In Section 2, we recall some basic concepts required in the subsequent sections. In Section 3, we define the relationships between interval-valued points and interval-valued soft sets and deal with some of their properties. Also, we define an interval-valued soft mapping and study some of its properties. In Section 4, we introduce the concept of an interval-valued soft continuous mapping and study its various properties. In Section 5, we introduce the notions of partial interval-valued soft T_i(j)-spaces (j = 0, 1, 2, 3, 4; j = i, ii) and discuss some of their properties, as well as relationships among them, and provide some examples. In Section 6, we propose the concepts of partial total interval-valued soft T_j(i)-spaces (i = 0, 1, 2, 3, 4; j = i, ii), study their properties and the relationships among them, and provide some examples.

2. Preliminaries

This section provides basic concepts and a result needed in the next sections. Throughout this paper, let $X, Y, Z, \cdots$ denote nonempty universe sets; $E, E', \cdots$, sets of parameters; and 2^X, the power set of X.

Definition 1 ([36,37])**.** *The collection of subsets of* X,

$$\{B \in 2^X : A^- \subset B \subset A^+\},$$

denoted by $[A^-, A^+]$, *is called an interval-valued set (briefly, IVS) or interval set in* X, *if* $A^-, A^+ \in 2^X$ *and* $A^- \subset A^+$. *The interval-valued empty [resp. whole] set in* X, *denoted by* $\widetilde{\varnothing}$ *[resp.* $\widetilde{X}$*], is an interval-valued set in* X *defined as*

$$\widetilde{\varnothing} = [\varnothing, \varnothing] \text{ [resp. } \widetilde{X} = [X, X]].$$

We will denote the set of all IVSs in X *as* $IVS(X)$ *(see [36,37] for definitions of the inclusion, the intersection, and the union of two IVSs).*

An interval-valued set in X is a special case of an interval-valued fuzzy set introduced by Zadeh [38] and can be considered a generalization of classical subsets of X.

Definition 2 ([36])**.** *Let* $a \in X$ *and* $A \in IVS(X)$. *Then, the IVS* $[\{a\}, \{a\}]$ *[resp.* $[\varnothing, \{a\}]$*] in* X *is called an interval-valued [resp. vanishing] point in* X *and is denoted by* a_1 *[resp.* a_0*]. We denote the set of all interval-valued points in* X *as* $IV_P(X)$ *and have the following:*

(i) We say that a_1 *belongs to* A, *denoted by* $a_1 \in A$, *if* $a \in A^-$.

(ii) We say that a_0 *belongs to* A, *denoted by* $a_0 \in A$, *if* $a \in A^+$.

Definition 3 ([36])**.** *Let* $\tau \subset IVS(X)$. *Then,* τ *is called an interval-valued topology (briefly, IVT) on* X, *if it satisfies the following axioms:*

(IVO$_1$) $\widetilde{\varnothing}, \widetilde{X} \in \tau$;

(IVO$_2$) $A \cap B \in \tau$ *for any* $A, B \in \tau$;

(IVO$_3$) $\bigcup_{j \in J} A_j \in \tau$ *for each* $(A_j)_{j \in J} \subset \tau$.

The pair (X, τ) *is called an interval-valued topological space (briefly, IVTS), and each member of* τ *is called an interval-valued open set (briefly, IVOS) in* X. *An IVS* A *is called an interval-valued closed set (briefly, IVCS) in* X, *if* $A^c \in \tau$.

$IVT(X)$ *denotes the set of all IVTs on* X. *For an IVTS* X, $IVO(X)$ *[resp.* $IVC(X)$*] denotes the set of all IVOSs [resp. IVCSs] in* X.

Definition 4 ([1,17])**.** *For each* $A \in 2^E$, *an* F_A *is called a soft set over* X *if* $F_A : A \to 2^X$ *is a mapping such that* $F_A(e) = \varnothing$ *for each* $e \notin A$.

We will denote the set of all soft sets over X *as* $SS(X)$, *while* $SS(X)_E$ *will denote the set of all soft sets over* X *with respect to a fixed set* E *of parameters.*

Definition 5 ([2,3])**.** $F_A \in SS(X)$ *is called the following:*

(i) A null soft set or a relative null soft set (with respect to A), denoted by $\emptyset_A$, if $F_A(e) = \emptyset$, for each $e \in A$;

(ii) An absolute soft set or a relative whole soft set (with respect to A), denoted by X_A, if $F_A(e) = X$ for each $e \in A$.

We will denote the null [resp. absolute] soft set in $SS_E(X)$ by X_E [resp. $\emptyset_E$].

Definition 6 ([16])**.** *Let τ be a collection of members of $SS_E(X)$. Then, τ is called a soft topology on X if it satisfies the following conditions:*

(i) $\emptyset_E,\ X_E \in \tau$;

(ii) $A \cap B \in \tau$ for any $A,\ B \in \tau$;

(iii) $\bigcup_{j \in J} A_j \in \tau$ for each $(A_j)_{j \in J} \subset \tau$, where J denotes an index set.

The triple (X, τ, E) is called a soft topological space over X. Each member of τ is called a soft open set in X, and a soft set A over X is called a closed soft set in X if $A^c \in \tau$, where A^c is a soft set over X defined by $A^c(e) = X - A(e)$ for each $e \in E$ (see [16]).

Definition 7 ([28])**.** *For each $A \in 2^E$, an $\mathbf{F}_A = [F^-_A, F^+_A]$ is called an interval-valued soft set (briefly, IVSS) over X if $\mathbf{F}_A : A \to IVS(X)$ is a mapping such that $\mathbf{F}_A(e) = \widetilde{\emptyset}$ for each $e \notin A$, i.e., $F^-_A,\ F^+_A \in SS(X)$ such that $F^-_A(e) \subset F^+_A(e)$ for each $e \in A$.*

We can see that an IVSS over X is a generalization of soft sets over X and the special case of an interval-valued fuzzy soft set proposed by Yang et al. [39].

Definition 8 ([28])**.** *Let $A \in 2^E$ and $\mathbf{F}_A \in IVSS(X)$. $\mathbf{F}_A$ is called the following:*

(i) A relative null interval-valued soft set (with respect to A), denoted by $\widetilde{\emptyset}_A$, if $\mathbf{F}_A(e) = \widetilde{\emptyset}$ for each $e \in A$;

(ii) A relative whole interval-valued soft set (with respect to A), denoted by $\widetilde{X}_A$, if $\mathbf{F}_A(e) = \widetilde{X}$ for each $e \in A$.

We denote the set of all IVSSs over X with respect to the fixed parameter set A as $IVSS_A(X)$.

The members of $IVSS_E(X)$ will be denoted by $\mathbf{A},\ \mathbf{B},\ \mathbf{C},\ \cdots$. The *interval-valued soft empty* [resp. *whole*] *set* over X with respect to E, denoted by $\widetilde{\emptyset}_E$ [resp. $\widetilde{X}_E$], is an IVS in X defined as follows: for each $e \in E$,

$$\widetilde{\emptyset}_E(e) = \widetilde{\emptyset} \text{ [resp. } \widetilde{X}_E(e) = \widetilde{X}].$$

See [28] for definitions of the inclusion, the intersection, and the union of two IVSSs.

Definition 9 ([28])**.** *$\tau \subset SS_E(X)$ is called an interval-valued soft topology (briefly, IVST) on X with respect to E if it satisfies the following axioms:*

[IVSO$_1$] $\widetilde{\emptyset}_E,\ \widetilde{X}_E \in \tau$;

[IVSO$_2$] If $\mathbf{A},\ \mathbf{B} \in \tau$, then $\mathbf{A} \cap \mathbf{B} \in \tau$;

[IVSO$_3$] If $(\mathbf{A}_j)_{j \in J} \subset \tau$, then $\bigcup_{j \in J} \mathbf{A}_j \in \tau$.

The triple (X, τ, E) is called an interval-valued soft topological space (briefly, IVSTS). Every member of τ is called an interval-valued soft open set (briefly, IVSOS), and the complement of an IVSOS is called an interval-valued soft closed set (briefly, IVSCS) in X. $IVSO(X)$ [resp. $IVSC(X)$] denotes the set of all IVSOSs [resp. IVSCSs] in X. The IVST $\{\widetilde{\emptyset}_E, \widetilde{X}_E\}$ [resp. $IVSS_E(X)$] is called an interval-valued soft indiscrete [resp. discrete] topology on X and is denoted by $\widetilde{\tau}_0$ [resp. $\widetilde{\tau}_1$]. We will denote the set of all IVSTSs over X with respect to E as $IVSTS_E(X)$ and denote the set of all IVSCSs in an IVSTS (X, τ, E) by τ^c.

We can easily see that an IVST is a special case of an interval-valued fuzzy soft topology in the sense of Ali et al. [40]. Moreover, (X, τ^-, τ^+) can be considered soft bi-topological space in the viewpoint of Kelly [41] for each $\tau \in IVST_E(X)$, where

$$\tau^- = \{U^- \in SS_E(X) : \mathbf{U} \in \tau\},\ \tau^+ = \{U^+ \in SS_E(X) : \mathbf{U} \in \tau\}.$$

Result 1 (Proposition 4.5, [28]). *Let (X, τ, E) be an IVSTS, and for each $e \in E$,*

$$\tau_e = \{\mathbf{U}(e) \in IVS(X) : \mathbf{U} \in \tau\}.$$

Then, τ_e is an interval-valued topology (briefly, IVT) *on X, as proposed by Kim et al. [36].* In this case, τ_e will be called a *parametric interval-valued topology*, and (X, τ_e) will be called a *parametric interval-valued topological space*.

Furthermore, we obtain two classical topologies on X for each IVSTS (X, τ, E), and each $e \in E$ is given as follows (see Remark 4.6 (1), [28]):

$$\tau_e^- = \{A(e)^- \in 2^X : \mathbf{A}(e) \in \tau_e\} \text{ and } \tau_e^+ = \{A(e)^+ \in 2^X : \mathbf{A}(e) \in \tau_e\}.$$

In this case, τ_e^- and τ_e^+ will be called *parametric topologies* on X.

3. Basic Properties of Interval-Valued Soft Sets

In this section, we define relationships between an interval-valued point and an interval-valued soft set and deal with some of their basic set theoretical properties. Also, we introduce the concept of interval-valued soft mappings and obtain some of their properties.

Definition 10 ([33]). *Let $\mathbf{A} \in IVSS_E(X)$ and $x \in X$. We then have the following:*
(i) x_1 is said to belong or totally belong to $\mathbf{A}$, denoted by $x_1 \in \mathbf{A}$, if $x \in A^-(e)$ for each $e \in E$.
(ii) x_0 is said to belong or totally belong to $\mathbf{A}$, denoted by $x_0 \in \mathbf{A}$, if $x \in A^+(e)$ for each $e \in E$.

Note that for any $x \in X$, $x_1 \notin \mathbf{A}$ [resp. $x_0 \notin \mathbf{A}$] if $x \notin A^-(e)$ [resp. $x \notin A^+(e)$] for some $e \in E$. It is obvious that if $x_1 \in \mathbf{A}$, then $x_0 \in \mathbf{A}$. But the converse is not true in general (see Example 3.2, [33]).

Definition 11. *Let $\mathbf{A} \in IVSS_E(X)$ and $x \in X$. Then we say the following:*
(i) x_1 partially belongs to $\mathbf{A}$, denoted by $x_1 \in_P \mathbf{A}$, if $x_1 \in \mathbf{A}(e)$, i.e., $x \in A^-(e)$ for some $e \in E$;
(ii) x_1 does not totally belong to $\mathbf{A}$, denoted by $x_1 \notin_T \mathbf{A}$, if $x_1 \notin \mathbf{A}(e)$, i.e., $x \notin A^-(e)$ for each $e \in E$;
(iii) x_0 partially belongs to $\mathbf{A}$, denoted by $x_0 \in_P \mathbf{A}$, if $x_0 \in \mathbf{A}(e)$, i.e., $x \in A^+(e)$ for some $e \in E$;
(vi) x_0 does not totally belong to $\mathbf{A}$, denoted by $x_0 \notin_T \mathbf{A}$, if $x_0 \notin \mathbf{A}(e)$, i.e., $x \notin A^+(e)$ for each $e \in E$.

It is obvious that if $x_1 \in_P \mathbf{A}$ [resp. $x_0 \notin_T \mathbf{A}$], then $x_0 \in_P \mathbf{A}$ [resp. $x_1 \notin_T \mathbf{A}$]. But the converse is not true in general (see Example 1).

Example 1. *Let $X = \{a, b, c, x, y, z\}$ be a universe set and $E = \{e, f, g\}$ a set of parameters. Consider the IVSS $\mathbf{A}$ defined by*

$$\mathbf{A}(e) = [\{a, b\}, \{a, b, c\}],\ \mathbf{A}(f) = [\{a\}, \{a, c, z\}],\ \mathbf{A}(g) = [\{a, c, x\}, \{a, c, x\}].$$

Then, clearly, a_1, $b_1 \in_P \mathbf{A}$ but c_1, x_1, y_1, $z_1 \notin_T \mathbf{A}$. Also, a_0, b_0, c_0, x_0, $z_0 \in_P \mathbf{A}$, but $y_0 \notin_T \mathbf{A}$.

Proposition 1. *Let $\mathbf{A}$, $\mathbf{B} \in IVSS_E(X)$ and $x \in X$. Then, we have the following:*
(1) *If $x_1 \in \mathbf{A}$* [resp. $x_0 \in \mathbf{A}$], *then $x_1 \in_P \mathbf{A}$* [resp. $x_0 \in_P \mathbf{A}$].
(2) *$x_1 \notin_T \mathbf{A}$* [resp. $x_0 \notin_T \mathbf{A}$] *if and only if $x_1 \in \mathbf{A}^c$* [resp. $x_0 \in \mathbf{A}^c$].
(3) *$x_1 \in_P \mathbf{A} \cup \mathbf{B}$* [resp. $x_0 \in_P \mathbf{A} \cup \mathbf{B}$] *if and only if $x_1 \in_P \mathbf{A}$ or $x_1 \in_P \mathbf{B}$* [resp. $x_0 \in_P \mathbf{A}$ *or* $x_0 \in_P \mathbf{B}$].
(4) *If $x_1 \in_P \mathbf{A} \cap \mathbf{B}$* [resp. $x_0 \in_P \mathbf{A} \cap \mathbf{B}$], *then $x_1 \in_P \mathbf{A}$ and $x_1 \in_P \mathbf{B}$* [resp. $x_0 \in_P \mathbf{A}$ *and* $x_0 \in_P \mathbf{B}$].

(5) *If* $x_1 \in \mathbf{A}$ *or* $x_1 \in \mathbf{B}$ [resp. $x_0 \in \mathbf{A}$ *or* $x_0 \in \mathbf{B}$], *then* $x_1 \in \mathbf{A} \cup \mathbf{B}$ [resp. $x_0 \in \mathbf{A} \cup \mathbf{B}$].

(6) $x_1 \in \mathbf{A} \cap \mathbf{B}$ [resp. $x_0 \in \mathbf{A} \cap \mathbf{B}$] *if and only if* $x_1 \in \mathbf{A}$ *and* $x_1 \in \mathbf{B}$ [resp. $x_0 \in \mathbf{A}$ *and* $x_0 \in \mathbf{B}$].

Proof. The proofs follow from Definitions 10 and 11. □

Remark 1. *The converses of Proposition 1 (1), (3), and (5) need not be true in general (see Example 2).*

Example 2. *Let* $X = \{a, b\}$ *be a universe set and* $E = \{e, f\}$ *a set of parameters, and consider two IVSSs* $\mathbf{A}$ *and* $\mathbf{B}$ *over* X *defined by*

$$\mathbf{A}(e) = [\{a\}, X],\ \mathbf{A}(f) = [\{b\}, X],\ \mathbf{B}(e) = [\varnothing, \{a\}],\ \mathbf{B}(f) = [X, X].$$

Then, we can easily check that $a_1 \in_P \mathbf{A}$ *but* $a_1 \notin \mathbf{A}$. *Also,* $b_1 \in_P \mathbf{A}$ *and* $b_1 \in_P \mathbf{B}$, *but* $b_1 \notin_T \mathbf{A} \cap \mathbf{B}$. *Furthermore,* $a_1 \in \mathbf{A} \cup \mathbf{B}$, *but neither* $a_1 \in \mathbf{A}$ *nor* $a_1 \in \mathbf{B}$.

Definition 12. *Let* $\mathbf{A},\ \mathbf{B} \in IVSS_E(X)$. *Then, the Cartesian product of* $\mathbf{A}$ *and* $\mathbf{B}$, *denoted by* $\mathbf{A} \times \mathbf{B}$, *is an IVSS over* $X \times X$ *defined as follows: for each* $(e, f) \in E \times E$,

$$(\mathbf{A} \times \mathbf{B})(e, f) = \mathbf{A}(e) \times \mathbf{B}(f) = [A^-(e) \times B^-(e), A^+(e) \times B^+(e)].$$

From Definitions 2 and 12, it is obvious that for each $(x, y) \in X \times X$,

$$(\mathbf{x}, \mathbf{y})_1 = \mathbf{x}_1 \times \mathbf{y}_1 \text{ and } (\mathbf{x}, \mathbf{y})_0 = \mathbf{x}_0 \times \mathbf{y}_0 = \mathbf{x}_0 \times \mathbf{y}_1 = \mathbf{x}_1 \times \mathbf{y}_0.$$

Example 3. *Consider* $\mathbf{A},\ \mathbf{B} \in IVSS_E(X)$ *given by Example 2. Then,* $\mathbf{A} \times \mathbf{B}$ *is given as follows:*

$$(\mathbf{A} \times \mathbf{B})(e, e) = [\varnothing, \{(a, a), (b, a)\},\ (\mathbf{A} \times \mathbf{B})(e, f) = [\{(a, a), (a, b)\}, X \times X],$$

$$(\mathbf{A} \times \mathbf{B})(f, e) = [\varnothing, \{(a, a), (b, a)\},\ (\mathbf{A} \times \mathbf{B})(f, f) = [\{(b, a), (b, b)\}, X \times X].$$

Definition 13. *Let* $A,\ B \in IVS(X)$. *Then, the Cartesian product of* A *and* B, *denoted by* $A \times B$, *is an IVS in* $X \times X$ *defined as follows:*

$$A \times B = [A^- \times B^-, A^+ \times B^+].$$

It is clear that $(x, y)_1 = [(x, y), (x, y)] = x_1 \times y_1$ and $(x, y)_0 = [\varnothing, (x, y)] = x_0 \times y_0$.

Lemma 1. *Let* $A,\ B \in IVS(X)$ *and* $x,\ y \in X$. *Then,* $(x, y)_1 \in A \times B$ [resp. $(x, y)_0 \in A \times B$] *if and only if* $x_1 \in A$ *and* $y_1 \in B$ [resp. $x_0 \in A$ *and* $y_0 \in B$].

Proof. The proof follows from Definitions 2 and 13. □

We obtain a similar consequence for Lemma 1.

Proposition 2. *Let* $\mathbf{A},\ \mathbf{B} \in IVSS_E(X)$ *and* $x,\ y \in X$. *Then, we have the following:*

(1) $(x, y)_1 \in_P \mathbf{A} \times \mathbf{B}$ [resp. $(x, y)_0 \in_P \mathbf{A} \times \mathbf{B}$] *if and only if* $x_1 \in_P \mathbf{A}$ *and* $y_1 \in_P \mathbf{B}$ [resp. $x_0 \in_P \mathbf{A}$ *and* $y_0 \in_P \mathbf{B}$];

(2) $(x, y)_1 \in \mathbf{A} \times \mathbf{B}$ [resp. $(x, y)_0 \in \mathbf{A} \times \mathbf{B}$] *if and only if* $x_1 \in \mathbf{A}$ *and* $y_1 \in \mathbf{B}$ [resp. $x_0 \in \mathbf{A}$ *and* $y_0 \in \mathbf{B}$].

Proof. (1) Suppose $(x, y)_1 \in_P \mathbf{A} \times \mathbf{B}$. Then, there is $(e, f) \in E \times E$ such that $(x, y)_1 \in \mathbf{A}(e) \times \mathbf{B}(f)$. Thus, by Lemma 1, $x_1 \in \mathbf{A}(e)$ and $y_1 \in \mathbf{B}(f)$. So, $x_1 \in_P \mathbf{A}$ and $y_1 \in_P \mathbf{B}$. The proof of the converse is obvious. Also, the proof of the second part is similar.

(2) The proof is similar to (1). □

Lemma 2. *Let* $A,\ B,\ C,\ D \in IVS(X)$. *Then, we have the following:*

(1) $A \times (B \cap C) = (A \times B) \cap (A \times C)$;
(2) $A \times (B \cup C) = (A \times B) \cup (A \times C)$;
(3) $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$;
(4) $(A \times B) \cup (C \times D) \subset (A \cup C) \times (B \cup D)$.

Proof. (1)
$$\begin{aligned} A \times (B \cap C) &= [A^-, A^+] \times ([B^-, B^+] \cap [C^-, C^+] \\ &= [A^-, A^+] \times [B^- \cap C^-, B^+ \cap C^+] \\ &= [A^- \times B^- \cap C^-], A^+ \times B^+ \cap C^+] \text{ [By Definition 13].} \end{aligned}$$

Let $(x,y)_1 \in A \times (B \cap C) = [A^-, A^+] \times ([B^-, B^+] \cap [C^-, C^+]$. Then clearly,

$$(x,y)_1 \in [A^- \times B^- \cap C^-], A^+ \times B^+ \cap C^+].$$

Thus, $(x,y) \in A^- \times B^- \cap C^-$, i.e., $(x,y) \in A^-,\ B^-$ and $(x,y) \in A^-,\ C^-$. So, $(x,y) \in (A^- \times B^-) \cap (A^- \times C^-)$, i.e., $(x,y) \in [(A \times B) \cap (A \times C)]^-$. Hence, $(x,y)_1 \in (A \times B) \cap (A \times C)$. Therefore, $A \times (B \cap C) \subset (A \times B) \cap (A \times C)$. The converse inclusion is proved similarly.

(2) The proof is similar to (1).

(3)
$$\begin{aligned} (A \times B) \cap (C \times D) &= ([A^- \times B^-, A^+ \times B^+]) \cap ([C^- \times D^-, C^+ \times D^+] \\ &= [(A^- \times B^-) \cap (C^- \times D^-), (A^+ \times B^+) \cap (C^+ \times D^+)] \\ &= [(A^- \cap C^-) \times (B^- \cap D^-), (A^+ \cap C^+) \times (B^+ \cap D^+)] \\ &= [((A \cap C) \times (B \cap D))^-, ((A \cap C) \times (B \cap D))^+] \\ &= (A \cap C) \times (B \cap D). \end{aligned}$$

(4)
$$\begin{aligned} & (A \cup C) \times (B \cup D) \\ &= [(A \cup C)^- \times (B \cup D)^-, (A \cup C)^+ \times (B \cup D)^+ \\ &= [(A^- \cup C^-) \times (B^- \cup D^-), (A^+ \cup C^+) \times (B^+ \cup D^+)] \\ &= [(A^- \times B^-) \cup (A^- \times D^-) \cup (C^- \times B^-) \cup (C^- \times D^-), \\ &\quad (A^+ \times B^+) \cup (A^+ \times D^+) \cup (C^+ \times B^+) \cup (C^+ \times D^+)] \\ &\supset [(A^- \times B^-) \cup (C^- \times D^-), (A^+ \times B^+) \cup (C^+ \times D^+)] \\ &= [((A \times B) \cup (C \times D))^-, ((A \times B) \cup (C \times D))^+] \\ &= (A \times B) \cup (C \times D). \end{aligned}$$

Note that (3) and (4) can be proved using Definition 2. □

We have a similar consequence for Lemma 2.

Proposition 3. *Let* $\mathbf{A},\ \mathbf{B},\ \mathbf{C},\ \mathbf{D} \in IVSS_E(X)$. *Then, we have the following:*
(1) $\mathbf{A} \times (\mathbf{B} \cap \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) \cap (\mathbf{A} \times \mathbf{C})$;
(2) $\mathbf{A} \times (\mathbf{B} \cup \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) \cup (\mathbf{A} \times \mathbf{C})$;
(3) $(\mathbf{A} \times \mathbf{B}) \cap (\mathbf{C} \times \mathbf{D}) = (\mathbf{A} \cap \mathbf{C}) \times (\mathbf{B} \cap \mathbf{D})$;
(4) $(\mathbf{A} \times \mathbf{B}) \cup (\mathbf{C} \times \mathbf{D}) \subset (\mathbf{A} \cup \mathbf{C}) \times (\mathbf{B} \cup \mathbf{D})$.

Proof. The proofs follow from Lemma 2 and Definitions 10–12. □

Definition 14. *Let X and Y be nonempty sets and E and $E^{'}$ sets of parameters. Let $f : X \to Y$ and $\varphi : E \to E^{'}$ be mappings, $\mathbf{A} \in IVSS_E(X)$, and $\mathbf{B} \in IVSS_F(Y)$. Then, we have the following:*

(i) The image of $\mathbf{A}$ under f with respect to φ, denoted by $f_\varphi(\mathbf{A})$, is an IVSS over X defined as follows: for each $e^{'} \in E^{'}$,

$$f_\varphi(\mathbf{A})(e^{'}) = \begin{cases} \bigvee_{e \in \varphi^{-1}(e^{'})} f(\mathbf{A}(e)) & \text{if } \varphi^{-1}(e^{'}) \neq \emptyset \\ \emptyset & \text{otherwise.} \end{cases} \quad (1)$$

(ii) The pre-image of $\mathbf{B}$ *under* f *with respect to* φ*, denoted by* $f_\varphi^{-1}(\mathbf{B})$*, is an IVSS over* X *defined as follows: for each* $e \in E$*,*

$$f_\varphi^{-1}(\mathbf{B})(e) = f^{-1}(\mathbf{B}(\varphi(e))). \tag{2}$$

In this case, the mapping $f_\varphi : IVSS_E(X) \to IVSS_{E'}(Y)$ *will be called an interval-valued soft mapping.*

It is clear that $f_\varphi(e_{a_1}) = e_{f(a_1)} = e_{f(a)_1}$ *and* $f_\varphi(e_{a_0}) = e_{f(a_0)} = e_{f(a)_0}$.

Definition 15. *An interval-valued soft mapping* $f_\varphi : IVSS_E(X) \to IVSS_{E'}(Y)$ *is said to be injective [resp. surjective, bijective] if* f *and* φ *are injective [resp. surjective, bijective].*

Proposition 4. *Let* $f_\varphi : IVSS_E(X) \to IVSS_{E'}(Y)$ *be an interval-valued soft mapping and* $\mathbf{B} \in IVSS_{E'}(Y)$*. Then, we have the following:*

(1) *If* φ *is surjective and* $y_1 \in_P f_\varphi^{-1}(\mathbf{B})$ [resp. $y_0 \in_P f_\varphi^{-1}(\mathbf{B})$], *then* $x_1 \in_P$ [resp. $x_0 \in_P$] *for each* $x \in f^{-1}(y)$.

(2) *If* $y_1 \notin_T \mathbf{B}$ [resp. $y_0 \notin_T \mathbf{B}$], *then* $x_1 \notin_T f_\varphi^{-1}(\mathbf{B})$ [resp. $x_0 \notin_T f_\varphi^{-1}(\mathbf{B})$] *for each* $x \in f^{-1}(y)$.

(3) *If* $y_1 \in \mathbf{B}$ [resp. $y_0 \in \mathbf{B}$], *then* $x_1 \in f_\varphi^{-1}(\mathbf{B})$ [resp. $x_0 \in f_\varphi^{-1}(\mathbf{B})$] *for each* $x \in f^{-1}(y)$.

(4) *If* φ *is surjective and* $y_1 \notin \mathbf{B}$ [resp. $y_0 \notin \mathbf{B}$], *then* $x_1 \notin f_\varphi^{-1}(\mathbf{B})$ [resp. $x_0 \notin f_\varphi^{-1}(\mathbf{B})$] *for each* $x \in f^{-1}(y)$.

Proof. (1) Suppose φ is surjective and $y_1 \in_P \mathbf{B}$ and let $x \in f^{-1}(y)$. Since $y_1 \in_P \mathbf{B}$, there is $e' \in E'$ such that $y_1 \in \mathbf{B}(e')$, i.e., $y \in B^-(e')$. Since φ is surjective, there is $e \in E$ such that $e' = \varphi(e)$. Then, $y \in B^-(e') = B^-(\varphi(e))$. Thus, we obtain

$$f^{-1}(y) \subset f^{-1}(B^-(\varphi(e))) = f_\varphi^{-1}(B^-)(e).$$

So, $x \in f_\varphi^{-1}(B^-)(e)$. Hence, $x_1 \in_P f_\varphi^{-1}(\mathbf{B})$.

(2) Suppose $y_1 \notin_T \mathbf{B}$ and let $x \in f^{-1}(y)$. Since $y_1 \notin_T \mathbf{B}$, $y_1 \notin \mathbf{B}(e')$, i.e., $y \notin B^-(e')$ for each $e' \in E'$. Thus, $y \notin B^-(\varphi(e))$ for each $e \in E$. So, we have

$$f^{-1}(y) \cap f^{-1}(B^-(\varphi(e))) = f^{-1}(\{y\} \cap B^-(\varphi(e))) = \varnothing.$$

Hence, $x \notin f^{-1}(B^-(\varphi(e)))$. Therefore, $x_1 \notin_T f_\varphi^{-1}(\mathbf{B})$.

(3) Suppose $y_1 \in \mathbf{B}$ and let $x \in f^{-1}(y)$. Since $y_1 \in \mathbf{B}$, $y_1 \in \mathbf{B}(e')$, i.e., $y \in B^-(e')$ for each $e' \in E'$. Thus, $y \in B^-(\varphi(e))$ for each $e \in E$. So, $x \in f^{-1}(B^-(\varphi(e)))$ for each $e \in E$. Hence, $x_1 \in f_\varphi^{-1}(\mathbf{B})$.

(4) Suppose φ is surjective and $y_1 \notin \mathbf{B}$ and let $x \in f^{-1}(y)$. Since $y_1 \notin \mathbf{B}$, there is $e' \in E'$ such that $y_1 \notin \mathbf{B}(e')$, i.e., $y \notin B^-(e')$. Since φ is surjective, there is $e \in E$ such that $e' = \varphi(e)$. Then, $y \notin B^-(e') = B^-(\varphi(e))$. Thus, $f^{-1}(y) \cap f^{-1}(B^-(\varphi(e)) = \varnothing$. So, $x \notin f^{-1}(B^-(\varphi(e))$. Hence, $x_1 \notin f_\varphi^{-1}(\mathbf{B})$.

Note that the proof of the second part in (1), (2), (3), and (4) is similar to each proof. □

The following is an immediate consequence of Definition 14.

Proposition 5. *Let* $f_\varphi : IVSS_E(X) \to IVSS_{E'}(Y)$ *be an interval-valued soft mapping,* $\mathbf{A}$, $\mathbf{A}_1, \mathbf{A}_2 \in IVSS_E(X)$*, and let* $(\mathbf{A}_j)_{j \in J}$ *be a family of IVSSs over* X*, where* J *is an index set. Then, we have the following:*

(1) $f_\varphi(\widetilde{\varnothing}_E) = \widetilde{\varnothing}_{E'}$;

(2) $f_\varphi(\widetilde{X}_E) \subset \widetilde{Y}_{E'}$;

(3) $f_\varphi(\bigcup_{j \in J} \mathbf{A}_j) = \bigcup_{j \in J} f_\varphi(\mathbf{A}_j)$;

(4) $f_\varphi(\bigcap_{j \in J} \mathbf{A}_j) \subset \bigcap_{j \in J} f_\varphi(\mathbf{A}_j)$;

(5) *If* $\mathbf{A}_1 \subset \mathbf{A}_2$, *then* $f_\varphi(\mathbf{A}_1) \subset f_\varphi(\mathbf{A}_2)$.

Proposition 6. *Let* $f_\varphi : IVSS_E(X) \to IVSS_{E'}(Y)$ *be a bijective interval-valued soft mapping and* $\mathbf{A} \in IVSS_E(X)$. *Then,* $(f_\varphi(\mathbf{A}))^c = f_\varphi(\mathbf{A}^c)$.

Proof. The proof follows from Definition 14 (i). □

Remark 2. *In Proposition 5 (4), if* f_φ *is injective, then the equality holds.*

Also, from Definition 14, we obtain the following.

Proposition 7. *Let* $f_\varphi : IVSS_E(X) \to IVSS_{E'}(Y)$ *be an interval-valued soft mapping,* $\mathbf{A} \in IVSS_E(X)$, $\mathbf{B}$, $\mathbf{B}_1$, $\mathbf{B}_2 \in IVSS_E(Y)$, *and* $(\mathbf{B}_j)_{j\in J}$ *be a family of IVSSs over* Y. *Then, we have the following:*

(1) $\mathbf{A} \subset f_\varphi^{-1}(f(\mathbf{A}))$;
(2) $f_\varphi(f_\varphi^{-1}(\mathbf{B})) \subset \mathbf{B}$;
(3) $f_\varphi^{-1}(\bigcup_{j\in J} \mathbf{B}_j = \bigcup_{j\in J} f_\varphi^{-1}(\mathbf{B}_j)$;
(4) $f_\varphi^{-1}(\bigcap_{j\in J} \mathbf{B}_j = \bigcap_{j\in J} f_\varphi^{-1}(\mathbf{B}_j)$;
(5) *If* $\mathbf{B}_1 \subset \mathbf{B}_2$, *then* $f_\varphi^{-1}(\mathbf{B}_1) \subset f_\varphi^{-1}(\mathbf{B}_2)$;
(6) $f_\varphi(f_\varphi^{-1}(\mathbf{B}^c)) = (f_\varphi(\mathbf{B}))^c$;
(7) $f_\varphi^{-1}(\widetilde{\varnothing}_{E'}) = \widetilde{\varnothing}_E$.

Remark 3. *(1) In Proposition 7 (1), if* f_φ *is injective, then the equality holds.*
(2) In Proposition 7 (2), if f_φ *is surjective, then the equality holds.*

Proposition 8. *If* $f_\varphi : IVSS_E(X) \to IVSS_{E'}(Y)$ *and* $g_\phi : IVSS_{E'}(Y) \to IVSS_{E''}(Z)$ *are two interval-valued soft mappings, then* $(g \circ f)_{\phi\circ\varphi} : IVSS_E(X) \to IVSS_{E''}(Z)$ *is an interval-valued soft mapping. In fact, for each* $\mathbf{A} \in IVSS_E(X)$,

$$(g \circ f)_{\phi\circ\varphi}(\mathbf{A}) = (g_\phi \circ f_\varphi)(\mathbf{A}) = g_\phi(f_\varphi(\mathbf{A})).$$

Furthermore, $(g \circ f)^{-1}_{\phi\circ\varphi} = f_\varphi^{-1} \circ g_\phi^{-1}$.

Remark 4. *Let* $id_X : X \to X$ *and* $id_E : E \to E$ *be the identity mappings on* X *and* E*, respectively. Then clearly, by Definition 15,* $id_{X_{id_E}} : IVSS_E(X) \to IVSS_E(X)$ *is a bijective interval-valued soft mapping. In this case,* $id_{X_{id_E}} : IVSS_E(X) \to IVSS_E(X)$ *will be called the interval-valued soft identity mapping.*

4. Interval-Valued Soft Continuities

In this section, we propose the continuity and pointwise continuity of an interval-valued soft mapping and obtain a characterization of them (see Theorem 1). Also, we define an interval-valued soft open and closed mapping and obtain a characterization of each concept (see Theorems 3 and 4). Moreover, we introduce the notion of interval-valued soft quotient topologies and study some of their properties.

Definition 16. *Let* (X, τ) *and* (Y, δ) *be IVSTSs and* $f_\varphi : IVSS_E(X) \to IVSS_{E'}(Y)$ *an interval-valued soft mapping. Then,* f *is said to be an interval-valued soft continuous mapping (briefly, IVSCM), if* $f^{-1}(\mathbf{V}) \in \tau$ *for each* $\mathbf{V} \in \delta$.

Proposition 9. *Let* X, Y, Z *be IVSTSs and* $f_\varphi : IVSS_E(X) \to IVSS_{E'}(Y)$ *and* $g_\phi : IVSS_{E'}(Y) \to IVSS_{E''}(Z)$ *two IVSCMs. We have the following:*
(1) *The identity mapping* $id_{\varphi_{id_E}} : IVSS_E(X) \to IVSS_E(X)$ *is an IVSCM.*
(2) *If* f_φ *and* g_ϕ *are IVSCMs, then* $(g \circ f)_{\phi\circ\varphi}$ *is an IVSCM.*

Proof. The proofs follow from Definition 16, Remark 4, and Proposition 8. □

Remark 5. *Let* $\mathbf{IVS_{Top}}$ *be the collection of all IVSTSs and all IVSMs between them. Then, we can easily see that* $\mathbf{IVS_{Top}}$ *forms a concrete category from Proposition 9.*

Definition 17 ([28])**.** *Let* (X, τ, E) *be an IVSTS and* $\mathbf{N} \in IVSS_E(X)$*. Then, we have the following:*

(i) $\mathbf{N}$ *is called an interval-valued soft neighborhood (briefly, IVSN) of* $e_{a_1} \in \widetilde{X}_E$ *if there exists a* $\mathbf{U} \in \tau$ *such that*

$$e_{a_1} \in \mathbf{U} \subset \mathbf{N}, \text{ i.e., } a \in U^-(e) \subset N^-(e),$$

(ii) $\mathbf{N}$ *is called an interval-valued soft vanishing neighborhood (briefly, IVSVN) of* $e_{a_0} \in \widetilde{X}_E$ *if there exists a* $\mathbf{U} \in \tau$ *such that*

$$e_{a_0} \in \mathbf{U} \subset \mathbf{N}, \text{ i.e., } a \in U^+(e) \subset N^+(e).$$

We will denote the set of all IVSNs [resp. IVSVNs] of e_{a_1} *[resp.* e_{a_0}*] by* $\mathbf{N}(e_{a_1})$ *[resp.* $\mathbf{N}(e_{a_0})$*]. It is obvious that* $\mathbf{N}_\tau(e_{a_1})(e) = N_{\tau_e}(a_1)$ *[resp.* $\mathbf{N}_\tau(e_{a_0})(e) = N_{\tau_e}(a_0)$*].*

Definition 18. *Let X and Y be IVSTSs,* $a \in X$*, and* $f_\varphi : IVSS_E(X) \to IVSS_{E'}(Y)$ *be an interval-valued soft mapping. Then,* f_φ *is called the following:*

(i) An interval-valued soft continuous mapping (briefly, IVSCM) at e_{a_1} *if* $f_\varphi^{-1}(V) \in \mathbf{N}(e_{a_1})$ *for each* $V \in \mathbf{N}(f_\varphi(e_{a_1})) = \mathbf{N}(e_{f(a)_1})$*;*

(ii) An interval-valued vanishing continuous mapping (briefly, IVVSCM) at e_{a_0} *if* $f_\varphi^{-1}(V) \in \mathbf{N}(e_{a_0})$ *for each* $V \in \mathbf{N}(f_\varphi(e_{a_0})) = \mathbf{N}(e_{f(a)_0})$*.*

Theorem 1. *Let* (X, τ) *and* (Y, δ) *be two IVSTSs; let* $f_\varphi : IVSS_E(X) \to IVSS_{E'}(Y)$ *an interval-valued soft mapping. Then,* f_φ *is an IVSCM if and only if it is both IVSCM at each* e_{a_1} *and IVVSCM at each* e_{a_0}*.*

Proof. Suppose f_φ is an IVSCM and let $\mathbf{V} \in \mathbf{N}(f_\varphi(e_{a_1}))$ for any $a \in X$. Then there is $\mathbf{U} \in \delta$ such that $f_\varphi(e_{a_1}) \in \mathbf{U} \subset \mathbf{V}$. Thus, by Proposition 7 (5), we have

$$e_{a_1} \in f_\varphi^{-1}(\mathbf{U}) \subset f_\varphi^{-1}(\mathbf{V}) \text{ and } f_\varphi^{-1}(\mathbf{U}) \in \tau.$$

So, f is an IVSCM at e_{a_1}. Similarly, the second part is proved.

Conversely, suppose the necessary condition holds and let $V \in \delta$ such that $f_\varphi(e_{a_1}) \in V$ and $f_\varphi(e_{a_0}) \in V$ for any $a \in X$. Then by the hypotheses and Proposition 3.27 in [28], there are $\mathbf{U}_1$, $\mathbf{U}_0 \in \tau$ such that $f_\varphi(e_{a_1}) \in \mathbf{U}_1 \subset \mathbf{V}_1$, $f_\varphi(e_{a_0}) \in \mathbf{U}_0 \subset \mathbf{V}_0$ with $\mathbf{U} = \mathbf{U}_1 \cup \mathbf{U}_0$ and $\mathbf{V} = \mathbf{V}_1 \cup \mathbf{V}_0$. Thus, by Proposition 7 (5), we obtain

$$e_{a_1} \in f_\varphi^{-1}(\mathbf{U}_1) \subset f_\varphi^{-1}(\mathbf{V}_1) \text{ and } e_{a_0} \in f_\varphi^{-1}(\mathbf{U}_0) \subset f_\varphi^{-1}(\mathbf{V}_0).$$

So, by Proposition 7 (3), we have

$$\begin{aligned} f_\varphi^{-1}(\mathbf{V}) &= f_\varphi^{-1}(\mathbf{V}_1) \cup f_\varphi^{-1}(\mathbf{V}_0) \\ &= \left(\bigcup_{e_{a_1} \in f_\varphi^{-1}(\mathbf{V}_1)} f_\varphi^{-1}(\mathbf{U}_1) \right) \cup \left(\bigcup_{e_{a_0} \in f_\varphi^{-1}(\mathbf{V}_0)} f_\varphi^{-1}(\mathbf{U}_0) \right). \end{aligned}$$

Hence, $f^{-1}(\mathbf{V}) \in \tau$. Therefore, f is an IVSCM. □

Definition 19 ([28])**.** *Let* (X, τ, E) *be an IVSTS and* $\mathbf{A} \in IVS(X)_E$*. Then, we have the following:*

(i) The interval-valued soft closure of $\mathbf{A}$ *with respect to* τ*, denoted by* $IVScl(\mathbf{A})$*, is an IVSS over* X *defined as*

$$IVScl(\mathbf{A}) = \bigcap\{\mathbf{K} \in \tau^c : \mathbf{A} \subset \mathbf{K}\}.$$

(ii) The interval-valued soft interior of $\mathbf{A}$ *with respect to* τ*, denoted by* $IVSint(\mathbf{A})$*, is an IVSS over* X *defined as*

$$IVSint(\mathbf{A}) = \bigcup\{\mathbf{U} \in \tau : \mathbf{U} \subset \mathbf{A}\}.$$

Definition 20 ([28])**.** *Let* (X, τ, E) *be an IVSTS and* β*,* $\sigma \subset \tau$*. Then, we have the following:*

(i) β *is called an interval-valued soft base (briefly, IVSB) for* τ *if* $\mathbf{U} = \widetilde{\varnothing}_E$ *or there is* $\beta' \subset \beta$ *such that* $\mathbf{U} = \bigcup\{\mathbf{B} : \mathbf{B} \in \beta'\}$ *for any* $\mathbf{U} \in \tau$*.*

(ii) σ *is called an interval-valued soft subbase (briefly, IVSSB) for* τ *if the family of all finite intersections of members of* σ *is an IVSB for* τ*.*

Theorem 2. *Let* (X, τ) *and* (Y, δ) *be IVTSs,* $f_\varphi : IVSS_E(X) \to IVSS_{E'}(Y)$ *be an interval-valued mapping, and* β *and* σ *be a base and subbase for* τ*, respectively. Then, the following are equivalent:*

(1) f_φ *is an IVSCM;*
(2) $f_\varphi^{-1}(\mathbf{C}) \in \tau^c$ *for each* $\mathbf{C} \in \delta^c$*;*
(3) $f_\varphi(IVScl(\mathbf{A})) \subset IVScl(f_\varphi(\mathbf{A}))$ *for each* $\mathbf{A} \in IVSS_E(X)$*;*
(4) $IVScl(f_\varphi^{-1}(\mathbf{B}) \subset f_\varphi^{-1}(IVcl(\mathbf{B}))$ *for each* $\mathbf{B} \in IVSS_{E'}(Y)$*;*
(5) $f_\varphi^{-1}(\mathbf{B}) \in \tau$ *for each* $\mathbf{B} \in \beta$*;*
(6) $f_\varphi^{-1}(\mathbf{S}) \in \tau$ *for each* $\mathbf{S} \in \sigma$*.*

Definition 21. *Let* (X, τ) *and* (Y, δ) *be IVSTSs and* $f_\varphi : IVSS_E(X) \to IVSS_{E'}(Y)$ *be an interval-valued mapping. Then,* f_φ *is said to be interval-valued soft open [resp. closed] if* $f_\varphi(\mathbf{A}) \in \delta$ *for each* $\mathbf{A} \in \tau$ *[resp.* $f_\varphi(\mathbf{C}) \in \delta^c$ *for each* $\mathbf{C} \in \tau^c$*].*

From Proposition 8 and Definition 21, we have the following.

Proposition 10. *Let* X*,* Y*, and* Z *be IVSTSs and* $f_\varphi : IVSS_E(X) \to IVSS_{E'}(Y)$ *and* $g_\phi : IVSS_{E'}(Y) \to IVSS_{E''}(Z)$ *be two interval-valued mappings. If* f_φ *and* g_ϕ *are interval-valued soft open [resp. closed], then so is* $(g \circ f)_{\phi\circ\varphi}$*.*

We provide a necessary and sufficient condition for a mapping to be interval-valued soft open.

Theorem 3. *Let* (X, τ) *and* (Y, δ) *be IVSTSs and* $f_\varphi : IVSS_E(X) \to IVSS_{E'}(Y)$ *be interval-valued soft. Then, the following are equivalent:*

(1) f_φ *is interval-valued soft open;*
(2) $f_\varphi(IVSint(\mathbf{A})) \subset IVSint(f_\varphi(\mathbf{A}))$ *for each* $\mathbf{A} \in IVSS_E(X)$*.*

Proof. (1)⇒(2): Suppose f_φ is interval-valued soft open and let $\mathbf{A} \in IVSS_E(X)$. Since $IVSint(\mathbf{A}) \in \tau$, $f_\varphi(IVint(\mathbf{A})) \in \delta$. Since $IVSint(\mathbf{A}) \subset \mathbf{A}$, by Proposition 5 (5), $f_\varphi(IVSint(\mathbf{A})) \subset f_\varphi(\mathbf{A})$. On the other hand, $IVSint(f_\varphi(\mathbf{A}))$ is the largest IVSOS in X contained in $f_\varphi(\mathbf{A})$. Then, $f_\varphi(IVSint(\mathbf{A})) \subset IVSint(f_\varphi(\mathbf{A}))$.

(2)⇒(1): Suppose (2) holds and let $\mathbf{U} \in \tau$. Then, by Theorem 5.22 (2) in [28], $\mathbf{U} = IVSint(\mathbf{U})$. Thus, by the hypothesis, $f_\varphi(\mathbf{U}) = f_\varphi(IVSint(\mathbf{U})) \subset IVSint(f_\varphi(\mathbf{U}))$. On the other hand, it is obvious that $IVSint(f_\varphi(\mathbf{U})) \subset f_\varphi(\mathbf{U})$. So, $f_\varphi(\mathbf{U}) = IVSint(f_\varphi(\mathbf{U}))$. Hence, $f_\varphi(\mathbf{U}) \in \delta$. Therefore, f_φ is interval-valued soft open. □

Proposition 11. *Let* (X, τ)*,* (Y, δ) *be IVSTSs and* $f_\varphi : IVSS_E(X) \to IVSS_{E'}(Y)$ *be an interval-valued soft mapping. If* f_φ *is an IVSCM and injection, then* $IVSint(f_\varphi(\mathbf{A})) \subset f_\varphi(IVint(\mathbf{A}))$ *for each* $\mathbf{A} \in IVSS_E(X)$*.*

Proof. Suppose f_φ is an IVSCM and injection, and let $\mathbf{A} \in IVSS_E(X)$. Since $f_\varphi(IVSint(\mathbf{A})) \in \delta$, $f_\varphi^{-1}(f_\varphi(IVSint(\mathbf{A}))) \in \tau$ by the hypothesis. By the fact that f_φ is injective, from Remark 3 (1), we have

$$f_\varphi^{-1}(f_\varphi(IVSint(\mathbf{A}))) \subset f_\varphi^{-1}(f_\varphi(\mathbf{A})) = \mathbf{A}.$$

On the other hand, $IVSint(\mathbf{A})$ is the largest IVSOS in X contained in $\mathbf{A}$. Then, $f_\varphi^{-1}(IVSint(f_\varphi(\mathbf{A}))) \subset IVSint(\mathbf{A})$. Thus, $IVSint(f_\varphi(\mathbf{A})) \subset f_\varphi(IVSint(\mathbf{A}))$. □

The following is the immediate consequence of Theorem 3 and Proposition 11.

Corollary 1. *Let X and Y be IVSTSs and $f_\varphi : IVSS_E(X) \to IVSS_{E'}(Y)$ be an interval-valued soft mapping. If f_φ is interval-valued soft continuous, open, and injective, then $f_\varphi(IVSint(\mathbf{A})) = IVSint(f_\varphi(\mathbf{A}))$ for each $\mathbf{A} \in IVSS_E(X)$.*

The following provides a necessary and sufficient condition for a mapping to be interval-valued soft closed.

Theorem 4. *Let (X, τ), (Y, δ) be IVSTSs and $f_\varphi : IVSS_E(X) \to IVSS_{E'}(Y)$ be an interval-valued soft mapping. Then, f_φ is interval-valued soft closed if and only if $IVScl(f_\varphi(\mathbf{A})) \subset f_\varphi(IVScl(\mathbf{A}))$ for each $\mathbf{A} \in IVSS_E(X)$.*

Proof. Suppose f_φ is interval-valued soft closed and let $\mathbf{A} \in IVSS_E(X)$. Then clearly, $\mathbf{A} \subset IVScl(\mathbf{A})$. Since $IVScl(\mathbf{A}) \in \tau^c$, $f_\varphi(IVScl(\mathbf{A})) \in \delta^c$ by the hypothesis. Thus, $IVScl(f_\varphi(\mathbf{A})) \subset f_\varphi(IVScl(\mathbf{A}))$.

Conversely, suppose the necessary condition holds and let $\mathbf{C} \in \tau^c$. Since $\mathbf{C} = IVScl(\mathbf{C})$, we have

$$IVScl(f_\varphi(\mathbf{C})) \subset f_\varphi(IVScl(\mathbf{C})) = f_\varphi(\mathbf{C}) \subset IVScl(f_\varphi(\mathbf{C})).$$

Then, $f_\varphi(\mathbf{C}) = IVScl(f_\varphi(\mathbf{C}))$. Thus, $f_\varphi(\mathbf{C}) \in \delta^c$. So, f_φ is interval-valued soft closed. □

Theorem 5. *Let X and Y be IVSTSs and $f_\varphi : IVSS_E(X) \to IVSS_{E'}(Y)$ be an interval-valued soft mapping. Then, f_φ is interval-valued soft continuous and closed if and only if $f_\varphi(IVScl(\mathbf{A})) = IVScl(f_\varphi(\mathbf{A}))$ for each $\mathbf{A} \in IVSS_E(X)$.*

Proof. Let $\mathbf{A} \in IVSS_E(X)$. Then, from Theorem 2 (3), we have

f_φ is interval-valued soft continuous if and only $f_\varphi(IVScl(\mathbf{A})) \subset IVScl(f_\varphi(\mathbf{A}))$.

Also, by Theorem 4, $IVScl(f_\varphi(\mathbf{A})) \subset f_\varphi(IVScl(\mathbf{A}))$. Thus, the result holds. □

Definition 22. *Let X and Y be IVTSs and $f_\varphi : IVSS_E(X) \to IVSS_{E'}(Y)$ be an interval-valued soft mapping. Then, f_φ is called an interval-valued soft homeomorphism if it is bijective, interval-valued continuous, and open.*

Definition 23 ([28])**.** *Let τ_1, $\tau_2 \in IVST_E(X)$. Then, we say the following:*
(i) τ_1 is coarser than τ_2 or τ_2 is finer than τ_1 if $\tau_1 \subset \tau_2$;
(ii) τ_1 is strictly coarser than τ_2 or τ_2 is strictly finer than τ_1 if $\tau_1 \subset \tau_2$ and $\tau_1 \neq \tau_2$;
(iii) τ_1 is comparable with τ_2 if either $\tau_1 \subset \tau_2$ or $\tau_2 \subset \tau_1$.

It is obvious that $\widetilde{\tau}_0 \subset \tau \subset \widetilde{\tau}_1$ for each $\tau \in IVST_E(X)$, and $(IVST_E(X), \subset)$ forms a meet lattice with the smallest element $\widetilde{\tau}_0$ and $\widetilde{\tau}_1$ from Corollary 4.9 in [28].

We would like to see if there is an IVST on a set X such that an interval-valued soft mapping or a family of interval-valued soft mappings of an $IVSS_E(X)$ into an $IVSS_{E'}(Y)$ is interval-valued soft continuous. The following propositions answer this question.

Proposition 12. *Let X be a set, (Y,δ) an IVSTS, and $f_\varphi : IVSS_E(X) \to IVSS_{E'}(Y)$ an interval-valued soft mapping. Then, there is the coarsest IVST τ on X such that f_φ is an IVSCM.*

Proof. Let $\tau = \{f_\varphi^{-1}(\mathbf{V}) \in IVSS_E(X) : \mathbf{V} \in \delta\}$. Then, we can easily check that τ satisfies conditions (IVSO$_1$), (IVSO$_2$), and (IVSO$_3$). Thus, τ is an IVST on X. By the definition of τ, it is clear that $f_\varphi : IVSS_E(X,\tau) \to IVSS_{E'}(Y,\delta)$ is an IVSCM. It is easy to prove that τ is the coarsest IVST on X such that $f_\varphi : IVSS_E(X,\tau) \to IVSS_{E'}(Y,\delta)$ is an IVSCM. □

Proposition 13. *Let X be a set, (Y,δ) an IVTS, and $f_\varphi : IVSS_E(X) \to IVSS_{E'}(Y)$ an interval-valued soft mapping for each $\varphi \in \Phi$, where Φ is an index set. Then, there is the coarsest IVST τ on X such that f_φ is an IVSCM for each $\varphi \in \Phi$.*

Proof. Let $\sigma = \{f_\varphi^{-1}(\mathbf{V}) \in IVSS_E(X) : \mathbf{V} \in \delta,\ \varphi \in \Phi\}$. Then, we can easily check that τ is the IVST on X with σ as its IVSB. Thus, τ is the coarsest IVST on X such that $f_\varphi : IVSS_E(X,\tau) \to IVSS_{E'}(Y,\delta)$ is an IVSCM for each $\varphi \in \Phi$. □

Proposition 14. (The dual of Proposition 12) *Let (X,τ) be an IVSTS, Y a set, and $f_\varphi : IVSS_E(X) \to IVSS_{E'}(Y)$ an interval-valued soft mapping. Then, there is the finest IVST δ on Y such that f_φ is an IVSCM.*

Proof. Let $\delta = \{\mathbf{V} \in IVSS_{E'}(Y) : f_\varphi^{-1}(\mathbf{V}) \in \tau\}$. Then, we can easily check that δ is the finest IVST on Y such that $f_\varphi : IVSS_E(X,\tau) \to IVSS_{E'}(Y,\delta)$ is an IVSCM. □

Definition 24. *Let (X,τ) be an IVSTS, Y a set, and $f_\varphi : IVSS_E(X) \to IVSS_{E'}(Y)$ an interval-valued soft surjective mapping. Then,*

$$\delta = \{\mathbf{V} \in IVSS_{E'}(Y) : f_\varphi^{-1}(\mathbf{V}) \in \tau\}$$

is called the interval-valued soft quotient topology (briefly, IVSQT) on Y induced by f_φ. The pair (Y,δ) is called an interval-valued soft quotient space (briefly, IVSQS), and f_φ is called an interval-valued soft quotient mapping (briefly, IVSQM).

From Proposition 14, it is obvious that $\delta \in IVST_{E'}(Y)$. Moreover, it is easy to see that if (Y,δ) is an IVSQS of (X,τ) with IVSQM f_φ. Then, for an IVSS $\mathbf{C}$ over Y, $\mathbf{C} \in \delta^c$ if and only if $f_\varphi^{-1}(\mathbf{C}) \in \tau^c$.

Let (X,τ) and (Y,η) be IVSTSs and let $f_\varphi : IVSS_E(X) \to IVSS_{E'}(Y)$ be an interval-valued soft surjective mapping. Then, the following provides conditions on f_φ such that $\eta = \delta$, where δ is the IVSQT on Y induced by f_φ.

Proposition 15. *Let (X,τ) and (Y,η) be IVTSs, $f_\varphi : IVSS_E(X,\tau) \to IVSS_{E'}(Y,\eta)$ an interval-valued soft continuous surjective mapping, and δ the IVSQT on Y induced by f_φ. If f_φ is interval-valued soft open or closed, then $\eta = \delta$.*

Proof. Suppose f_φ is interval-valued soft open and let δ be the IVSQT on Y induced by f_φ. Then clearly, by Proposition 14, δ is the finest IVST on Y for which f_φ is interval-valued soft continuous. Thus, $\eta \subset \delta$. Let $\mathbf{U} \in \delta$. Then clearly, $f_\varphi^{-1}(\mathbf{U}) \in \delta$ by the definition of δ. Since f_φ is interval-valued soft open and surjective, $\mathbf{U} = f_\varphi(f_\varphi^{-1}(\mathbf{U})) \in \eta$. Thus, $\delta \subset \eta$. So, $\eta = \delta$.

When f is interval-valued soft closed, the proof is similar. □

Proposition 16. *The composition of two IVSQMs is an IVSQM.*

Proof. Let $f_\varphi : IVSS_E(X,\tau) \to IVSS_{E'}(Y,\delta)$ and $g_\phi : IVSS_{E'}(Y,\delta) \to IVSS_{E''}(Z,\gamma)$ be two IVQMs. Let η be the IVSQM on Z induced by $(g \circ f)_\phi \circ \varphi$ and let $\mathbf{V} \in \gamma$. Since

$g_\phi : IVSS_{E'}(Y,\delta) \to IVSS_{E''}(Z,\gamma)$ is an IVSQM, $g_\phi^{-1}(\mathbf{V}) \in \delta$. Since $f_\varphi : IVSS_E(X,\tau) \to IVSS_{E'}(Y,\delta)$ is an IVSQM, $(g \circ f)^{-1}_{\phi\circ\varphi}(\mathbf{V}) = f_\varphi^{-1}(g_\phi^{-1}(\mathbf{V})) \in \tau$. Then, $\mathbf{V} \in \eta$. Thus, $\gamma \subset \eta$. Moreover, we can easily show that $\eta \subset \gamma$. Thus, $\eta = \gamma$. So, $(g \circ f)_{\phi\circ\varphi}$ is an IVSQM. □

Theorem 6. *Let (X,τ) and (Z,η) be two IVSTSs, Y a set, $f_\varphi : IVSS_E(X) \to IVSS_{E'}(Y)$ an interval-valued soft surjective mapping, and δ the IVSQT on Y induced by f_φ. Then, $g_\phi : IVSS_E(X,\tau) \to IVSS_{E''}(Z,\eta)$ is an IVSCM if and only if $(g \circ f)_{\phi\circ\varphi} : IVSS_E(X,\tau) \to IVSS_{E''}(Z,\eta)$ is an IVSCM.*

Proof. Suppose g_ϕ is an IVSCM. Since $f_\varphi : IVSS_E(X,\tau) \to IVSS_{E'}(Y,\delta)$ is an IVSCM, by Proposition 9 (2), $(g \circ f)_{\phi\circ\varphi} : IVSS_E(X,\tau) \to IVSS_{E''}(Z,\eta)$ is an IVSCM.

Suppose $(g \circ f)_{\phi\circ\varphi}$ is an IVSCM and let $\mathbf{V} \in \eta$. Then clearly, $(g \circ f)^{-1}_{\phi\circ\varphi}(\mathbf{V}) \in \tau$ and $(g \circ f)^{-1}_{\phi\circ\varphi}(\mathbf{V}) = f_\varphi^{-1}(g_\phi^{-1}(\mathbf{V}))$. Thus, by the definition of δ, $g_\varphi^{-1}(\mathbf{V}) \in \delta$. So, g_ϕ is an IVSCM. □

Proposition 17. *Let (X,τ_1) and (Y,τ_2) be two IVSTSs and $\beta = \{\mathbf{U} \times \mathbf{V} : \mathbf{U} \in \tau_1,\ \mathbf{V} \in \tau_2\}$. Then, β is an IVSB for an IVST τ on $X \times Y$.*

In this case, τ is called the interval-valued soft product topology (briefly, IVSPT) on $X \times Y$, and the pair $(X \times Y, \tau)$ is called an interval-valued soft product space (briefly, IVSPS) of X and Y.

Proof. It is obvious that $\widetilde{X}_E \in \tau_1$ and $\widetilde{Y}_{E'} \in \tau_2$. Then, $\widetilde{X \times Y}_{E\times E'} = \widetilde{X}_E \times \widetilde{Y}_{E'} \in \beta$. Thus, $\widetilde{X \times Y} = \bigcup \beta$. So, [Theorem 4.25 (1), [28]] holds.

Now, suppose $\mathbf{B}_1 = \mathbf{U}_1 \times \mathbf{V}_1$, $\mathbf{B}_2 = \mathbf{U}_2 \times \mathbf{V}_2 \in \beta$, where $\mathbf{U}_1,\ \mathbf{U}_2 \in \tau_1$ and $\mathbf{V}_1,\ \mathbf{V}_2 \in \tau_2$. For any $(a,b) \in X \times Y$, let $e_{(a,b)_1},\ e_{(a,b)_0} \in \mathbf{B}_1 \cap \mathbf{B}_2$. Then, we have

$$\mathbf{B}_1 \cap \mathbf{B}_2 = (\mathbf{U}_1 \times \mathbf{V}_1) \cap (\mathbf{U}_2 \times \mathbf{V}_2) = (\mathbf{U}_1 \times \mathbf{U}_2) \cap (\mathbf{V}_1 \times \mathbf{V}_2). \tag{3}$$

Since $\mathbf{U}_1,\ \mathbf{U}_2 \in \tau_1$ and $\mathbf{V}_1,\ \mathbf{V}_2 \in \tau_2$, $\mathbf{U}_1 \times \mathbf{U}_2 \in \tau_1$ and $\mathbf{V}_1 \times \mathbf{V}_2 \in \tau_2$. Thus, $\mathbf{B}_1 \cap \mathbf{B}_2 \in \beta$. So, [Theorem 4.25 (2), [28]] holds. Hence, β is an IVSB for an IVST τ on $X \times Y$. □

Remark 6. *Let $\pi_X : X \times Y \to X$, $\pi_Y : X \times Y \to Y$, $\pi_E : E \times E' \to E$, and $\pi_{E'} : E \times E' \to E'$ be the usual projections. Then, we can easily see that the following are interval-valued soft mappings:*

$$\pi_{X_{\pi_E}} : IVSS_{E\times E'}(X \times Y) \to IVSS_E(X),$$

$$\pi_{Y_{\pi_{E'}}} : IVSS_{E\times E'}(X \times Y) \to IVSS_{E'}(Y).$$

In this case, we will call $\pi_{X_{\pi_E}}$ and $\pi_{Y_{\pi_{E'}}}$ interval-valued soft projections.

5. Partial Interval-Valued Soft Separation Axioms

In this section, first, we recall separation axioms in an IVSTS proposed by Baek (See [33]). Next, we introduce new separation axioms in interval-valued soft topological spaces using belong and nonbelong relations and study some of their properties and some relationships among them.

Definition 25 ([33]). *An IVSTS (X,τ,E) is called the following:*

(i) An interval-valued soft T_0(i)-space (briefly, IVST$_0$(i)-space) if for any $x,\ y \in X$ with $x \neq y$, there is $\mathbf{U},\ \mathbf{V} \in \tau$ such that either $x_1 \in \mathbf{U}$, $y_1 \notin \mathbf{U}$ or $y_1 \in \mathbf{V}$, $x_1 \notin \mathbf{V}$;

(ii) An interval-valued soft T_0(ii)-space (briefly, IVST$_0$(ii)-space) if for any $x,\ y \in X$ with $x \neq y$, there is $\mathbf{U},\ \mathbf{V} \in \tau$ such that either $x_0 \in \mathbf{U}$, $y_0 \notin \mathbf{U}$ or $y_0 \in \mathbf{V}$, $x_0 \notin \mathbf{V}$;

(iii) An interval-valued soft T_1(i)-space (briefly, IVST$_1$(i)-space) if for any $x,\ y \in X$ with $x \neq y$, there are $\mathbf{U},\ \mathbf{V} \in \tau$ such that $x_1 \in \mathbf{U}$, $y_1 \notin \mathbf{U}$ and $y_1 \in \mathbf{V}$, $x_1 \notin \mathbf{V}$;

(iv) An interval-valued soft T_1(ii)*-space (briefly, IVST*$_1$*(ii)-space) if for any* $x,\ y \in X$ *with* $x \neq y$*, there are* $\mathbf{U},\ \mathbf{V} \in \tau$ *such that* $x_{_0} \in \mathbf{U}$*,* $y_{_0} \notin \mathbf{U}$ *and* $y_{_0} \in \mathbf{V}$*,* $x_{_0} \notin \mathbf{V}$*;*

(v) An interval-valued soft T_2(i)*-space (briefly, IVST*$_2$(i)*-space) if for any* $x,\ y \in X$ *with* $x \neq y$*, there are* $\mathbf{U},\ \mathbf{V} \in \tau$ *such that* $x_{_1} \in \mathbf{U}$*,* $y_{_1} \in \mathbf{V}$ *and* $\mathbf{U} \cap \mathbf{V} = \widetilde{\varnothing}_E$*;*

(vi) An interval-valued soft T_2(ii)*-space (briefly, IVST*$_2$(ii)*-space) if for any* $x,\ y \in X$ *with* $x \neq y$*, there are* $\mathbf{U},\ \mathbf{V} \in \tau$ *such that* $x_{_0} \in \mathbf{U}$*,* $y_{_0} \in \mathbf{V}$ *and* $\mathbf{U} \cap \mathbf{V} = \widetilde{\varnothing}_E$*;*

(vii) An interval-valued soft regular (i)*-space (briefly, IVSR(i)-space) if for each* $x \in X$ *with* $x_{_1} \notin \mathbf{A}$*, there are* $\mathbf{U},\ \mathbf{V} \in \tau$ *such that* $x_{_1} \in \mathbf{U}$*,* $\mathbf{A} \subset \mathbf{V}$ *and* $\mathbf{U} \cap \mathbf{V} = \widetilde{\varnothing}_E$*;*

(viii) An interval-valued soft regular (ii)*-space (briefly, IVSR(ii)-space) if for each* $x \in X$ *with* $x_{_0} \notin \mathbf{A}$*, there are* $\mathbf{U},\ \mathbf{V} \in \tau$ *such that* $x_{_0} \in \mathbf{U}$*,* $\mathbf{A} \subset \mathbf{V}$ *and* $\mathbf{U} \cap \mathbf{V} = \widetilde{\varnothing}_E$*;*

(xi) An interval-valued soft T_3(i)*-space (briefly, IVST*$_3$*(i)-space) if it is an IVSR(i) and IVST*$_1$*(i)-space;*

(x) An interval-valued soft T_3(ii)*-space (briefly, IVST*$_3$*(ii)-space) if it is an IVSR(ii) and IVST*$_1$*(ii)-space;*

(xi) An interval-valued soft normal space (briefly, IVSNS) if for any IVSCSs $\mathbf{F}_1$ *and* $\mathbf{F}_2$ *in X with* $\mathbf{F}_1 \cap \mathbf{F}_2 = \widetilde{\varnothing}_E$*, there are* $\mathbf{U},\ \mathbf{V} \in \tau$ *such that* $\mathbf{F}_1 \subset \mathbf{U}$*,* $\mathbf{F}_2 \subset \mathbf{V}$ *and* $\mathbf{U} \cap \mathbf{V} = \widetilde{\varnothing}_E$*;*

(xii) An interval-valued soft T_4(i)*-space (briefly, IVST*$_4$*(i)-space) if it is an* T_1(i)*-space and an IVSNS;*

(xiii) An interval-valued soft T_4(ii)*-space (briefly, IVST*$_4$*(ii)-space) if it is an* T_1(ii)*-space and an IVSNS.*

Definition 26. *An IVSTS* (X, τ, E) *is called the following:*

(i) A partial interval-valued soft T_0(i)*-space (briefly, PIVST*$_0$*(i)-space) if for any* $x \neq y \in X$*, there is* $\mathbf{U} \in \tau$ *such that either* $x_{_1} \in \mathbf{U}$*,* $y_{_1} \notin_T \mathbf{U}$ *or* $y_{_1} \in \mathbf{U}$*,* $x_{_1} \notin_T \mathbf{U}$*;*

(ii) A partial interval-valued soft T_0(ii)*-space (briefly, PIVST*$_0$*(ii)-space) if for any* $x \neq y \in X$*, there is* $\mathbf{U} \in \tau$ *such that either* $x_{_0} \in \mathbf{U}$*,* $y_{_0} \notin_T \mathbf{U}$ *or* $y_{_0} \in \mathbf{U}$*,* $x_{_0} \notin_T \mathbf{U}$*;*

(iii) A partial interval-valued soft T_1(i)*-space (briefly, PIVST*$_1$*(i)-space) if for any* $x \neq y \in X$*, there are* $\mathbf{U},\ \mathbf{V} \in \tau$ *such that* $x_{_1} \in \mathbf{U}$*,* $y_{_1} \notin_T \mathbf{U}$*,* $y_{_1} \in \mathbf{V}$*, and* $x_{_1} \notin_T \mathbf{V}$*;*

(iv) A partial interval-valued soft T_1(ii)*-space (briefly, PIVST*$_1$*(ii)-space) if for any* $x \neq y \in X$*, there is* $\mathbf{U} \in \tau$ *such that* $x_{_0} \in \mathbf{U}$*,* $y_{_0} \notin_T \mathbf{U}$*,* $y_{_0} \in \mathbf{V}$*, and* $x_{_0} \notin_T \mathbf{V}$*;*

(v) A partial interval-valued soft T_2(i)*-space (briefly, PIVST*$_2$*(i)-space) if for any* $x \neq y \in X$*, there are* $\mathbf{U},\ \mathbf{V} \in \tau$ *such that* $x_{_1} \in \mathbf{U}$*,* $y_{_1} \notin_T \mathbf{U}$*,* $y_{_1} \in \mathbf{V}$*,* $x_{_1} \notin_T \mathbf{V}$*, and* $\mathbf{U} \cap \mathbf{V} = \widetilde{\varnothing}_E$*;*

(vi) A partial interval-valued soft T_2(ii)*-space (briefly, PIVST*$_2$*(ii)-space) if for any* $x \neq y \in X$*, there is* $\mathbf{U} \in \tau$ *such that* $x_{_0} \in \mathbf{U}$*,* $y_{_0} \notin_T \mathbf{U}$*,* $y_{_0} \in \mathbf{V}$*,* $x_{_0} \notin_T \mathbf{V}$*, and* $\mathbf{U} \cap \mathbf{V} = \widetilde{\varnothing}_E$*.*

Remark 7. *(1) From the definitions of PIVST*$_2$(i) [resp. PIVST$_2$(ii)]*-space and IVST*$_2$(i) [resp. IVST$_2$(ii)]*-space* (see [33]), *we can easily check that the notions of PIVST*$_2$(i) [resp. PIVST$_2$(ii)]*-spaces and IVST*$_2$(i) [resp. IVST$_2$(ii)]*-spaces coincide.*

(2) If an IVSTS (X, τ, E) *is a PIVST*$_j$(i) [resp. PIVST$_j$(ii)]*-space, then* (X, τ^-, E) *and* (X, τ^+, E) *are p-soft* T_j*-spaces [resp.* (X, τ^+, E) *is a p-soft* T_j*-space] for* $j = 0,\ 1,\ 2$ *in the sense of El-Shafei et al. (see [34]).*

Proposition 18. *Every PIVST*$_j$(i) [resp. PIVST$_2$(ii)]*-space is an IVST*$_j$(i) [resp. IVST$_2$(ii)]*-space, where* $j = 0,\ 1$*. But the converses are not true in general* (see Example 4).

Proof. The proofs follow from relationships $\notin_T$ and $\notin$. □

Example 4. *Let* $X = \{x, y\}$ *and* $E = \{e, f\}$*. Consider the IVST* τ *on X given by*

$$\tau = \{\widetilde{\varnothing}_E, \mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_4, \mathbf{A}_5, \mathbf{A}_6, \widetilde{X}_E\},$$

where $\mathbf{A}_1(e) = [X, X],\ \mathbf{A}_1(f) = [\{x\}, \{x\}],\ \mathbf{A}_2(e) = [X, X],\ \mathbf{A}_2(f) = [\{y\}, \{y\}],$
$\mathbf{A}_3(e) = [\varnothing, \varnothing],\ \mathbf{A}_3(f) = [\{y\}, \{y\}],\ \mathbf{A}_4(e) = [\varnothing, \varnothing],\ \mathbf{A}_4(f) = [\{x\}, \{x\}],$
$\mathbf{A}_5(e) = [\varnothing, \varnothing],\ \mathbf{A}_5(f) = [X, X],\ \mathbf{A}_6(e) = [X, X]$*, and* $\mathbf{A}_6(f) = [\varnothing, \varnothing]$*.*

Then clearly, X is an $IVST_1$(i)-space. But there is no $\mathbf{U} \in \tau$ *such that* $x_{_1} \in \mathbf{U}$ *and* $y_{_1} \notin_T \mathbf{U}$. *Thus, X is not a $PIVST_1$(i)-space.*

Lemma 3. *Let* (X, τ, E) *be an IVSTS,* $\mathbf{A} \in IVSS_E(X)$*, and* $x \in X$*. Then,* $x_{_1} \notin_T IVScl(\mathbf{A})$ [resp. $x_{_0} \notin_T IVScl(\mathbf{A})$] *if and only if there is* $\mathbf{U} \in \tau$ *such that* $x_{_1} \in \mathbf{U}$ [resp. $x_{_0} \in \mathbf{U}$] *and* $\mathbf{A} \cap \mathbf{U} = \widetilde{\varnothing}_E$.

Proof. Suppose $x_{_1} \notin_T IVScl(\mathbf{A})$. Then, by Proposition 1 (2), $x_{_1} \in (IVScl(\mathbf{A})^c)$. Let $\mathbf{U} = (IVScl(\mathbf{A})^c)$. Then clearly, $x_{_1} \in \mathbf{U} \in \tau$. Moreover, $\mathbf{A} \cap \mathbf{U} = \widetilde{\varnothing}_E$. Conversely, suppose the necessary condition holds. Then, $\mathbf{A} \subset \mathbf{U}^c$. Since $\mathbf{U}^c \in \tau^c$, $IVScl(\mathbf{A}) \subset \mathbf{U}^c$. Since $x_{_1} \in \mathbf{A}$, by Proposition 1 (2), $x_{_1} \notin_T \mathbf{U}^c$. Thus, $x_{_1} \notin_T IVScl(\mathbf{A})$. The proof of the second part is analogous. □

Proposition 19. *If* (X, τ, E) *is a $PIVST_0$*(i)*-space* [resp. PIVST_0(ii)-space]*, then* $IVScl(\mathbf{x}_{_1}) \neq IVScl(\mathbf{y}_{_1})$ [resp. $IVScl(\mathbf{x}_{_0}) \neq IVScl(\mathbf{y}_{_0})$] *for any* $x \neq y \in X$*. However, the converse is not true in general.*

Proof. Suppose (X, τ, E) is a PIVST_0(i)-space and let $x \neq y \in X$. Then, there is $\mathbf{U} \in \tau$ such that either $x_{_1} \in \mathbf{U}$, $y_{_1} \notin_T \mathbf{U}$ or $y_{_1} \in \mathbf{U}$, $x_{_1} \notin_T \mathbf{U}$. Say $x_{_1} \in \mathbf{U}$ and $y_{_1} \notin_T \mathbf{U}$. Thus, $y_{_1} \notin \mathbf{U}(e)$ for each $e \in E$. So, $\mathbf{y}_{_1} \cap \mathbf{U} = \widetilde{\varnothing}_E$. Hence, by Lemma 3, $x_{_1} \notin_T IVScl(\mathbf{y}_{_1})$ but $x_{_1} \in IVScl(\mathbf{x}_{_1})$. Therefore, $IVScl(\mathbf{x}_{_1}) \neq IVScl(\mathbf{y}_{_1})$. See Example 5 for the proof of the converse

The second part is similarly proved. □

Example 5. *Let* (X, τ, E) *be the IVSTS given in Example 4. Then clearly, X is not a $PIVST_0$(i)-space but* $IVScl(\mathbf{x}_{_1}) \neq IVScl(\mathbf{y}_{_1})$.

We have an immediate consequence of Proposition 19.

Proposition 20. *If* (X, τ, E) *is a $PIVST_0$*(i)*-space* [resp. PIVST_0(ii)-space]*, then* $IVScl(e_{x_1}) \neq IVScl(f_{y_1})$ [resp. $IVScl(e_{x_0}) \neq IVScl(f_{y_0})$] *for any* $x \neq y \in X$ *and any* $e,\ f \in E$.

We have a characterization of a PIVST_1(i)-space [resp. PIVST_1(ii)-space].

Theorem 7. *Let* (X, τ, E) *be an IVSTS. Then, X is a $PIVST_1$*(i)*-space* [resp. PIVST_1(ii)-space] *if and only if* $\mathbf{x}_{_1} \in \tau^c$ [resp. $\mathbf{x}_{_0} \in \tau^c$] *for each* $x \in X$.

Proof. Suppose X is a PIVST_1(i)-space and let $y_j \in X \setminus \{x\}$ for each $j \in J$, where J is an index set. Then, there is $\mathbf{U}_j \in \tau$ such that $y_{j_1} \in \mathbf{U}_j$ and $x_{_1} \notin_T \mathbf{U}_j$. Thus, we have the following: for each $e \in E$,

$$\mathbf{x}_{_1}^c(e) = (\widetilde{X}_E \setminus \mathbf{x}_{_1})(e) = [X \setminus \{x\}, X \setminus \{x\}] = \bigcup_{j \in J} \mathbf{U}_j(e) \text{ and } x_{_1} \notin \bigcup_{j \in J} \mathbf{U}_j(e).$$

Since $\bigcup_{j \in J} \mathbf{U}_j \in \tau$ and $\mathbf{x}_{_1}^c \in \tau$. So, $\mathbf{x}_{_1} \in \tau^c$.

Conversely, suppose the necessary condition holds and let $x \neq y \in X$. Then clearly, $\mathbf{x}_{_1},\ \mathbf{y}_{_1} \in \tau^c$. Thus, $\mathbf{x}_{_1}^c,\ \mathbf{y}_{_1}^c \in \tau$ and $y_{_1} \in \mathbf{x}_{_1}^c$, $x_{_1} \in \mathbf{y}_{_1}^c$. Moreover, $x_{_1} \notin_T \mathbf{x}_{_1}^c$ and $y_{_1} \notin_T \mathbf{y}_{_1}^c$. So, X is a PIVST_1(i)-space. The second part is similarly proved. □

Also, we obtain another characterization of a PIVST_1(i)-space [resp. PIVST_1(ii)-space].

Theorem 8. *Let* (X, τ, E) *be an IVSTS and E be finite. Then, X is a $PIVST_1$*(i)*-space* [resp. PIVST_1(ii)-space] *if and only if* $\mathbf{x}_{_1} = \bigcap \{\mathbf{U} \in \tau : \mathbf{x}_{_1} \in \mathbf{U}\}$ [resp. $\bigcap \{\mathbf{U} \in \tau : \mathbf{x}_{_0} \in \mathbf{U}\}$] *for each* $x \in X$.

Proof. Suppose X is a PIVST$_1$(i)-space and let $y \in X$. Then, for each $x \in X \setminus \{y\}$, there is $\mathbf{U} \in \tau$ such that $x_{_1} \in \mathbf{U}$ and $y_{_1} \notin_T \mathbf{U}$. Thus, $y_{_1} \notin \mathbf{U}(e)$, i.e., $y_{_1} \notin \bigcap_{x_{_1} \in \mathbf{U} \in \tau} \mathbf{U}(e)$ for each $e \in E$. Since y is arbitrary, $\mathbf{x}_{_1} = \bigcap\{\mathbf{U} \in \tau : \mathbf{x}_{_1} \in \mathbf{U}\}$.

Conversely, suppose the necessary condition holds and let $x \neq y \in X$. Since $y_{_1} \notin_T \mathbf{x}_{_1}$ and E is finite, say $\mid E \mid = m$, there are at most $\mathbf{U}_i \in \tau$ such that $x_{_1} \in \mathbf{U}_i$ and $y_{_1} \notin \mathbf{U}_i(e_i)$ for each $i \in \{1, 2, \cdots, m\}$. Then, $\bigcap_{i=1}^m \mathbf{U}_i \in \tau$ such that $y_{_1} \notin_T \bigcap_{i=1}^m \mathbf{U}_i$ and $x_{_1} \in \bigcap_{i=1}^m \mathbf{U}_i$. Thus, X is a PIVST$_1$(i)-space.

Also, the second part is similarly proved. □

We obtain an immediate consequence of Theorem 8.

Corollary 2. *Let (X, τ, E) be an IVSTS. If X is a PIVST$_1$*(i)*-space* [resp. PIVST$_1$(ii)-space]*, then* $\mathbf{x}_{_1} = \bigcap_{x_{_1} \in \mathbf{U} \in \tau} \mathbf{U}$ [resp. $\mathbf{x}_{_1} = \bigcap_{x_{_0} \in \mathbf{U} \in \tau} \mathbf{U}$] *for each $x \in X$.*

We have a relationship of a PIVST$_1$(i)-space [resp. PIVST$_1$(ii)-space] and a PIVST$_2$(i)-space [resp. PIVST$_2$(ii)-space].

Theorem 9. *Let (X, τ, E) be a finite IVSTS. Then, X is a PIVST$_1$*(i)*-space* [resp. PIVST$_1$(ii)-space] *if and only if it is a PIVST$_2$*(i)*-space* [resp. PIVST$_2$(ii)-space].

Proof. Suppose X is a PIVST$_1$(i)-space and let $y \in X \setminus \{x\}$, $y \in X \setminus \{y\}$. Then, by Theorem 7, $\mathbf{y}_{_1}$, $\mathbf{x}_{_1} \in \tau^c$. Since X is finite, $\bigcup_{y \in X \setminus \{x\}} \mathbf{y}_{_1}$ and $\bigcup_{x \in X \setminus \{y\}} \mathbf{x}_{_1} \in \tau^c$. Thus, $(\bigcup_{y \in X \setminus \{x\}} \mathbf{y}_{_1})^c = \mathbf{x}_{_1}$, $(\bigcup_{x \in X \setminus \{y\}} \mathbf{x}_{_1})^c = \mathbf{y}_{_1} \in \tau$. Moreover, $\mathbf{x}_{_1} \cap \mathbf{y}_{_1} = \widetilde{\varnothing}_E$, where $x_{_1} \in \mathbf{x}_{_1}$, $y_{_1} \notin_T \mathbf{x}_{_1}$ and $y_{_1} \in \mathbf{y}_{_1}$, $x_{_1} \notin_T \mathbf{y}_{_1}$. So, X is a PIVST$_2$(i)-space. The proof of the converse follows from Definition 26.

The second part can be similarly proved. □

Remark 8. *In Theorem 8, if X is infinite, then an IVSS $\mathbf{x}_{_1}$ in a PIVST$_1$*(i)*-space [resp. PIVST$_1$*(ii)*-space] need not be an IVSOS in X (see Example 9).*

Example 6. *Let E be the set of natural numbers $\mathbb{N}$ and consider the family τ of IVSSs over the set of real numbers $\mathbb{R}$ given by*

$$\tau = \{\widetilde{\varnothing}_E\} \bigcup \{\mathbf{U} \in IVSS_E(\mathbb{R}) : \mathbf{U} \textit{ is finite}\}.$$

Then, we can easily check that $(\mathbb{R}, \tau, E)$ is an IVSTS. But $\mathbf{x}_{_1} \notin \tau$ for each $x \in \mathbb{R}$.

Definition 27. *An IVSTS (X, τ, E) is said to be the following:*

(i) Partial interval-valued soft regular (i) *(briefly, PIVSR(i)) if for each $\mathbf{A} \in \tau^c$ and each $x \in X$ with $x_{_1} \notin_T \mathbf{A}$, there are $\mathbf{U}$, $\mathbf{V} \in \tau$ such that $\mathbf{A} \subset \mathbf{U}$, $x_{_1} \in \mathbf{V}$, and $\mathbf{U} \cap \mathbf{V} = \widetilde{\varnothing}_E$;*

(ii) Partial interval-valued soft regular (ii) *(briefly, PIVSR(ii)) if for each $\mathbf{A} \in \tau^c$ and each $x \in X$ with $x_{_0} \notin_T \mathbf{A}$, there are $\mathbf{U}$, $\mathbf{V} \in \tau$ such that $\mathbf{A} \subset \mathbf{U}$, $x_{_0} \in \mathbf{V}$, and $\mathbf{U} \cap \mathbf{V} = \widetilde{\varnothing}_E$.*

Proposition 21. *Every* IVSR(i) [resp. IVSR(ii)]*-space is* PIVSR(i) [resp. PIVSR(ii)]. *But the converse is not true in general.*

Proof. The proof follows from Definition 8 and Proposition 11. See Example 27 for the converse. □

Example 7. *Let $X = \{x, y\}$ and let $E = \{e, f, g\}$. Consider the IVST τ on X defined by*

$$\tau = \{\widetilde{\varnothing}_E, \mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_4, \mathbf{A}_5, \mathbf{A}_6, \mathbf{A}_7, \widetilde{X}_E\},$$

where $\mathbf{A}_1(e) = \mathbf{A}_1(f) = \mathbf{A}_1(g) = [\{x\}, \{x\}]$, $\mathbf{A}_2(e) = \mathbf{A}_2(f) = \mathbf{A}_2(g) = [\{y\}, \{y\}]$,
$\mathbf{A}_3(e) = [\varnothing, \varnothing]$, $\mathbf{A}_3(f) = \mathbf{A}_3(g) = [\{x\}, \{x\}]$,
$\mathbf{A}_4(e) = [\varnothing, \varnothing]$, $\mathbf{A}_4(f) = \mathbf{A}_4(g) = [\{y\}, \{y\}]$,

$\mathbf{A}_5(e) = [\{x\}, \{x\}],\ \mathbf{A}_5(f) = \mathbf{A}_5(g) = [X, X],$
$\mathbf{A}_6(e) = [\{y\}, \{y\}],\ \mathbf{A}_6(f) = \mathbf{A}_6(g) = [X, X],$
$\mathbf{A}_7(e) = [\varnothing, \varnothing]$, *and* $\mathbf{A}_7(f) = \mathbf{A}_7(g) = [X, X].$
Then, we can see that X *is PIVSR(i). On the other hand, we have*

$$\tau^c = \{\widetilde{\varnothing}_E, \mathbf{A}_1^c, \mathbf{A}_2^c, \mathbf{A}_3^c, \mathbf{A}_4^c, \mathbf{A}_5^c, \mathbf{A}_6^c, \mathbf{A}_7^c, \widetilde{X}_E\},$$

where $\mathbf{A}_1^c(e) = \mathbf{A}_1^c(f) = \mathbf{A}_1^c(g) = [\{y\}, \{y\}],\ \mathbf{A}_2^c(e) = \mathbf{A}_2^c(f) = \mathbf{A}_2^c(g) = [\{x\}, \{x\}],$
$\mathbf{A}_3^c(e) = [X, X],\ \mathbf{A}_3^c(f) = \mathbf{A}_3^c(g) = [\{y\}, \{y\}],$
$\mathbf{A}_4^c(e) = [X, X],\ \mathbf{A}_4^c(f) = \mathbf{A}_4^c(g) = [\{x\}, \{x\}],$
$\mathbf{A}_5^c(e) = [\{y\}, \{y\}],\ \mathbf{A}_5^c(f) = \mathbf{A}_5^c(g) = [\varnothing, \varnothing],$
$\mathbf{A}_6^c(e) = [\{x\}, \{x\}],\ \mathbf{A}_6^c(f) = \mathbf{A}_6^c(g) = [\varnothing, \varnothing],$
$\mathbf{A}_7^c(e) = [X, X]$, *and* $\mathbf{A}_7^c(f) = \mathbf{A}_7^c(g) = [\varnothing, \varnothing].$

Then clearly, $\mathbf{A}_3^c \in \tau^c$ *such that* $x_1 \notin \mathbf{A}_3^c$*. But we cannot find* $\mathbf{U},\ \mathbf{V} \in \tau$ *such that* $x_1 \in \mathbf{U}$*,* $\mathbf{A}_3^c \subset \mathbf{V}$ *and* $\mathbf{U} \cap \mathbf{V} = \widetilde{\varnothing}_E$*. Thus,* X *is not an IVSR(i)-space.*

We obtain a characterization of a PIVSR(i) [resp. PIVSR(ii)]-space.

Theorem 10. *An IVSTS* (X, τ, E) *is a PIVSR*(i) [resp. PIVSR(ii)]*-space if and only if for each* $x \in X$ *and each* $\mathbf{U} \in \tau$ *with* $x_1 \in \mathbf{U}$ [resp. $x_0 \in \mathbf{U}$]*, there is* $\mathbf{V} \in \tau$ *such that* $x_1 \in \mathbf{V} \subset IVScl(\mathbf{V}) \subset \mathbf{U}$ [resp. $x_0 \in \mathbf{V} \subset IVScl(\mathbf{V}) \subset \mathbf{U}$].

Proof. Suppose an IVSTS (X, τ, E) is PIVSR(i) and let $x \in X$ and $\mathbf{U} \in \tau$ with $x_1 \in \mathbf{U}$. Then clearly, $\mathbf{U}^c \in \tau^c$ and $\mathbf{x}_1 \cap \mathbf{U}^c = \widetilde{\varnothing}_E$. Thus, $x_1 \notin_T \mathbf{U}^c$. By the hypothesis, there are $\mathbf{A},\ \mathbf{V} \in \tau$ such that $\mathbf{U}^c \subset \mathbf{A}$, $x_1 \in \mathbf{V}$, and $\mathbf{A} \cap \mathbf{V} = \widetilde{\varnothing}_E$. So, $\mathbf{V} \subset \mathbf{A}^c \subset \mathbf{U}$. Since $\mathbf{A} \in \tau$, $\mathbf{A}^c \in \tau^c$. Hence, $\mathbf{V} \subset IVScl(\mathbf{V}) \subset \mathbf{U}$.

Conversely, suppose the necessary condition holds and let $\mathbf{U}^c \in \tau^c$ with $x_1 \notin_T \mathbf{U}^c$. Then clearly, $x_1 \in \mathbf{U}$. Thus, by the hypothesis, there is $\mathbf{U} \in \tau$ such that $x_1 \in \mathbf{V} \subset IVScl(\mathbf{V}) \subset \mathbf{U}$. So, $\mathbf{U}^c \subset (IVScl(\mathbf{V}))^c$ and $\mathbf{V} \cap (IVScl(\mathbf{V}))^c = \widetilde{\varnothing}_E$. Hence, X is PIVSR(i).

The proof of the second part is similar. □

We provide a sufficient condition for PIVST$_0$(i) [resp. PIVST$_0$(ii)], PIVST$_1$(i) [resp. PIVST$_1$(ii)], and PIVST$_2$(i) [resp. PIVST$_2$(ii)] to be equivalent.

Theorem 11. *Let* (X, τ, E) *be an IVSTS. If* X *is PIVSR*(i) [resp. PIVSR(i)]*, then the following are equivalent:*

(1) *X is a PIVST$_2$*(i) [resp. PIVST$_2$(ii)]*-space;*
(2) *X is a PIVST$_1$*(i) [resp. PIVST$_1$(ii)]*-space;*
(3) *X is a PIVST$_0$*(i) [resp. PIVST$_0$(ii)]*-space.*

Proof. (1)⇒(2)⇒(3): The proofs follow from Defintion 26.

(3)⇒(1): Suppose X is a PIVST$_0$(i)-space and let $x \neq y \in X$. Then, there is $\mathbf{U} \in \tau$ such that either $x_1 \in \mathbf{U}$, $y_1 \notin_T \mathbf{U}$, or $y_1 \in \mathbf{U}$, $x_1 \notin_T \mathbf{U}$, say $x_1 \in \mathbf{U}$ and $y_1 \notin_T \mathbf{U}$. Thus, by Proposition 1 (2), $x_1 \notin_T \mathbf{U}^c$ and $y_1 \in \mathbf{U}^c$. Since $\mathbf{U}^c \in \tau^c$, by the hypothesis, there are $\mathbf{A},\ \mathbf{B} \in \tau$ such that $x_1 \in \mathbf{A}$ and $y_1 \in \mathbf{U}^c \subset \mathbf{B}$. So, X is a a PIVST$_2$(i)-space.

The proofs of the second parts are similar. □

The following provide a sufficient condition for PIVST$_1$(i) [resp. PIVST$_1$(ii)] and IVST$_2$(i) [resp. IVST$_2$(ii)] to be equivalent.

Definition 28. *An IVSTS* (X, τ, E) *is called the following:*

(i) A partial interval-valued soft T_3(i)*-space (briefly, PIVST$_3$(i)-space) if it is both PIVSR(i) and a PIVST$_1$(i)-space;*

(ii) A partial interval-valued soft T_3(ii)-space (briefly, PIVST$_3$(ii)-space) if it is both PIVSR(ii) and a PIVST$_1$(ii)-space;

(i) A partial interval-valued soft T_4(i)-space (briefly, PIVST$_4$(i)-space) if it is both IVSN and a PIVST$_1$(i)-space;

(ii) A partial interval-valued soft T_4(ii)-space (briefly, PIVST$_4$(ii)-space) if it is both IVSN and a PIVST$_1$(ii)-space.

Proposition 22. *Every IVST$_3$(i) [resp. IVST$_3$(ii)]-space is a PIVST$_3$(i) [resp. PIVST$_3$(ii)]-space, but the converse is not true in general.*

Proof. The proof follows from Proposition 21 and Theorem 11. See Example 22 for the converse. □

Example 8. *Let X be the IVSTS given in Example 7. Then, we can easily check that X is a PIVST$_3$(i)-space but not an IVST$_3$(i)-space.*

Proposition 23. *Every PIVST$_4$(i) [resp. PIVST$_4$(ii)]-space is an IVST$_4$(i) [resp. IVST$_4$(ii)]-space, but the converse is not true in general.*

Proof. The proof is straightforward. See Example 9 for the converse. □

Example 9. *Let $X = \{x, y\}$ and let $E = \{e, f, g\}$. Consider the IVST τ on X defined by*

$$\tau = \{\widetilde{\varnothing}_E, \mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_4, \mathbf{A}_5, \mathbf{A}_6, \widetilde{X}_E\},$$

where $\mathbf{A}_1(e) = [X, X],\ \mathbf{A}_1(f) = [\{x\}, \{x\}],\ \mathbf{A}_1(g) = [X, X],$
$\mathbf{A}_2(e) = [X, X],\ \mathbf{A}_2(f) = [\{y\}, \{y\}],\ \mathbf{A}_2(g) = [X, X],$
$\mathbf{A}_3(e) = [X, X],\ \mathbf{A}_3(f) = [\varnothing, \varnothing],\ \mathbf{A}_3(g) = [X, X],$
$\mathbf{A}_4(e) = [\varnothing, \varnothing],\ \mathbf{A}_4(f) = [\{x\}, \{x\}],\ \mathbf{A}_4(g) = [\varnothing, \varnothing],$
$\mathbf{A}_5(e) = [\varnothing, \varnothing],\ \mathbf{A}_5(f) = [\{y\}, \{y\}],\ \mathbf{A}_5(g) = [\varnothing, \varnothing],$
$\mathbf{A}_6(e) = [\varnothing, \varnothing],\ \mathbf{A}_6(f) = [X, X]$, *and* $\mathbf{A}_6(g) = [\varnothing, \varnothing]$.

Then, we can easily see that X is an IVST$_4$(i)-space. On the other hand, we cannot find $\mathbf{U} \in \tau$ such that $y_{_1} \in \mathbf{U}$ and $x_{_1} \notin_T \mathbf{U}$. Then, X is not a PIVST$_1$(i)-space. Thus, X is not a PIVST$_4$(i)-space.

Proposition 24. *Every PIVST$_j$(i) [resp. PIVST$_j$(ii)]-space is a PIVST$_{j-1}$(i) [resp. IVST$_{j-1}$(ii)]-space for $j = 0,\ 1,\ 2,\ 3,\ 4$.*

Proof. Let (X, τ, E) be a PIVST$_3$(i)-space and let $x \neq y \in X$. Since X is a PIVST$_1$(i)-space, by Theorem 7, $\mathbf{x}_{_1} \in \tau^c$. Then clearly, $y_{_1} \notin_T \mathbf{x}_{_1}$. Since X is PIVSR(i), there are $\mathbf{U},\ \mathbf{V} \in \tau$ such that $\mathbf{x}_{_1} \subset \mathbf{U}$, $y_{_1} \in \mathbf{V}$, and $\mathbf{U} \cap \mathbf{V} = \widetilde{\varnothing}_E$. Thus, X is a PIVST$_2$(i)-space.

Now, let (X, τ, E) be a PIVST$_4$(i)-space. Let $x \in X$ and let $\mathbf{A} \in \tau^c$ with $x_{_1} \notin_T \mathbf{A}$. Since X is a PIVST$_1$(i)-space, by Theorem 7, $\mathbf{x}_{_1} \in \tau^c$. Then, $\mathbf{x}_{_1} \cap \mathbf{A} = \widetilde{\varnothing}_E$. Since X is IVSN(i), there are $\mathbf{U},\ \mathbf{U} \in \tau$ such that $\mathbf{A} \subset \mathbf{U}$, $\mathbf{x}_{_1} \subset \mathbf{V}$, and $\mathbf{U} \cap \mathbf{V} = \widetilde{\varnothing}_E$. Thus, X is a PIVST$_3$(i)-space.

The rest of the proof follows from similar arguments. Also, the proofs of the second parts can be completed by the same token. □

Definition 29 ([33])**.** *Let Y be a nonempty subset of X and $\mathbf{A} \in IVSS_E(X)$. Then, we have the following:*

(i) The interval valued soft set (Y, E) over X, denoted by $\widetilde{Y}_E$, is defined as

$$\widetilde{Y}_E(e) = [Y, Y] \text{ for each } e \in E,$$

(ii) The interval-valued soft subset of $\mathbf{A}$ *over* Y*, denoted by* $\mathbf{A}_Y$*, is defined as*

$$\mathbf{A}_Y = \widetilde{Y}_E \cap \mathbf{A}, \text{ i.e., } \mathbf{A}_Y(e) = [Y \cap A^-(e), Y \cap A^+(e)] \text{ for each } e \in E.$$

Result 2 (See Proposition 4.3, [33]). *Let* (X, τ, E) *be an IVSTS and* Y *a nonempty subset of* X*. Then,* $\tau_Y = \{\mathbf{A}_Y : \mathbf{A} \in \tau\}$ *is an IVST on* Y*.*

In this case, τ_Y is called the *interval-valued soft relative topology* on Y, and (Y, τ_Y, E) is called an *interval-valued soft subspace* (briefly, IVS-subspace) of (X, τ, E). Each member of τ_Y is called an *interval-valued soft open set* (briefly, IVSOS) in Y, and an IVSS $\mathbf{A}$ over X is called an *interval-valued soft closed set* (briefly, IVSCS) in Y if $[Y, Y] \setminus \mathbf{A} = [Y \setminus A^+, Y \setminus A^-] \in \tau_Y$.

Proposition 25. *Every IVS-subspace* (Y, τ_Y, E) *of a PIVST*$_j$(i) [resp. PIVST$_j$(ii)]-*space* (X, τ, E) *is a PIVST*$_j$(i) [resp. IVST$_j$(ii)]-*space for* $j = 0,\ 1,\ 2,\ 3$.

Proof. Let X be a PIVST$_3$(i)-space and let $x \neq y \in Y$. Since X is a PIVST$_1$(i)-space, there are $\mathbf{U},\ \mathbf{V} \in \tau$ such that $x_1 \in \mathbf{U}$, $y_1 \notin_T \mathbf{U}$, and $y_1 \in \mathbf{V}$, $x_1 \notin_T \mathbf{V}$. Thus, $x_1 \in \mathbf{U}_Y$, $y_1 \notin_T \mathbf{U}_Y$ and $y_1 \in \mathbf{V}_Y$, $x_1 \notin_T \mathbf{V}_Y$, where $\mathbf{U}_Y = \widetilde{Y}_E \cap \mathbf{U}$ and $\mathbf{V}_Y = \widetilde{Y}_E \cap \mathbf{V}$. Note that $\mathbf{U}_Y,\ \mathbf{V}_Y \in \tau_Y$ by Result 2. So, (Y, τ_Y, E) is a PIVST$_1$(i)-space.

Now, let $y \in Y$ and let $\mathbf{A} \in \tau_Y$ with $y_1 \notin_T \mathbf{A}$. Then, by Theorem 4.9 (2) in [33], there is $\mathbf{A} \in \tau^c$ such that $\mathbf{A} = \widetilde{Y}_E \cap \mathbf{B}$, and $y_1 \notin_T \mathbf{B}$. Since X is PIVSR(i), there are $\mathbf{U},\ \mathbf{V} \in \tau$ such that $\mathbf{B} \subset \mathbf{U}$, $y_1 \in \mathbf{V}$, and $\mathbf{U} \cap \mathbf{V} = \widetilde{\varnothing}_E$. Thus, $\mathbf{A} \subset \widetilde{Y}_E \cap \mathbf{U}$, $y_1 \in \widetilde{Y}_E \cap \mathbf{V}$, and $(\widetilde{Y}_E \cap \mathbf{U}) \cap (\widetilde{Y}_E \cap \mathbf{V}) = \widetilde{\varnothing}_E$. So, (Y, τ_Y, E) is PIVSR(i). Hence, (Y, τ_Y, E) is PIVST$_3$(i)-space.

The proofs for the cases of $j = 0,\ 1,\ 2$ and the second parts are similar. □

Proposition 26. *Let* $f_\varphi : (X, \tau, E) \to (Y, \delta, E')$ *be an interval-valued soft continuous mapping. If* f *is injective and* (Y, δ, E') *is a PIVST*$_j$(i) [resp. PIVST$_j$(ii)]-*space, then* (X, τ, E) *is a PIVST*$_j$(i) [resp. IVST$_j$(ii)]-*space for* $j = 0,\ 1,\ 2$.

Proof. Suppose f is injective and (Y, δ, E') is a PIVST$_2$(i)-space, and let $a \neq b \in X$. Since f is injective, there are distinct x and y in Y such that $x = f(a)$ and $y = f(b)$. Since Y is a PIVST$_2$(i)-space, there are $\mathbf{U},\ \mathbf{V} \in \delta$ such that $x_1 \in \mathbf{U}$, $y_1 \in \mathbf{V}$, and $\mathbf{U} \cap \mathbf{V} = \widetilde{\varnothing}_{E'}$. Then, by Proposition 4 (3) and Proposition 7 (4) and (7), we have

$$a_1 \in f_\varphi^{-1}(\mathbf{U}),\ b_1 \in f_\varphi^{-1}(\mathbf{V}) \text{ and } f_\varphi^{-1}(\mathbf{U}) \cap f_\varphi^{-1}(\mathbf{V}) = \widetilde{\varnothing}_E.$$

Since f_φ is continuous, $f_\varphi^{-1}(\mathbf{U})$ and $f_\varphi^{-1}(\mathbf{V}) \in \tau$. Thus, X is a PIVST$_2$(i)-space.

The proofs for the cases of $j = 0,\ 1$ and the second parts are similar. □

Proposition 27. *Let* $f_\varphi : (X, \tau, E) \to (Y, \delta, E')$ *be an interval-valued soft bijective open mapping. If* X *is a PIVST*$_j$(i) [resp. PIVST$_j$(ii)]-*space, then* Y *is a PIVST*$_j$(i) [resp. IVST$_j$(ii)]-*space for* $j = 0,\ 1,\ 2,\ 3,\ 4$.

Proof. Suppose X is a PIVST$_4$(i)-space and let $x \neq y \in Y$, $e' \in E'$. Since f_φ is bijective, there are unique $a \neq b \in Y$ and $e \in E$ such that $x = f(a)$, $y = f(b)$, and $e' = \varphi(e)$. Since X is a PIVST$_1$(i)-space, there are $\mathbf{U},\ \mathbf{V} \in \tau$ such that

$$a_1 \in \mathbf{U},\ b_1 \notin_T \mathbf{U} \text{ and } b_1 \in \mathbf{V},\ a_1 \notin_T \mathbf{V}.$$

Since f_φ is open, $f_\varphi(\mathbf{U}),\ f_\varphi(\mathbf{V}) \in \delta$. Moreover, we obtain

$$x_1 \in f_\varphi(\mathbf{U}),\ y_1 \notin_T f_\varphi(\mathbf{U}) \text{ and } y_1 \in f_\varphi(\mathbf{V}),\ x_1 \notin_T f_\varphi(\mathbf{V}).$$

Then, Y is a PIVST$_1$(i)-space.

Now, let $\mathbf{A}, \mathbf{B} \in \delta^c$ such that $\mathbf{A} \cap \mathbf{B} = \widetilde{\emptyset}_{E'}$. Then, by Proposition 7 (4) and (7), we have

$$f_\varphi^{-1}(\mathbf{A}) \cap f_\varphi^{-1}(\mathbf{B}) = f_\varphi^{-1}(\mathbf{A} \cap \mathbf{B}) = f_\varphi^{-1}(\widetilde{\emptyset}_{E'}) = \widetilde{\emptyset}_E.$$

By Theorem 2 (2), $f_\varphi^{-1}(\mathbf{A})$ and $f_\varphi^{-1}(\mathbf{B}) \in \tau^c$. Thus, Y is IVSN. So, Y is a PIVST$_4$(i)-space. The proofs for the cases of $j = 0, 1, 2, 3$ and the second parts are similar. □

6. Partial Total Interval-Valued Soft α-Separation Axioms

In this section, first, we recall the concepts of interval-valued soft α-open sets and interval-valued soft α-separation axioms and some of their properties. Next, we define a new class of interval-valued soft separation axioms using partial belong and total nonbelong relations and study some of their properties and some relationships between them.

Definition 30. *(i) Let (X, τ, E) be a soft topological space and $A \in SS_E(X)$. Then, A is called a soft α-open set in X [27] if $A \subset int(cl(int(A)))$. The complement of a soft α-open set is called a soft α-closed set in X.*

(ii) Let (X, τ, E) be an IVSTS and $\mathbf{A} \in IVSS_E(X)$. Then, $\mathbf{A}$ is called an interval-valued soft α-open set (briefly, IVSαOS) in X [33] if it satisfies the following condition:

$$\mathbf{A} \subset IVSint(IVScl(IVSint(\mathbf{A}))).$$

The complement of an IVSαOS is called an interval-valued soft α-closed set (briefly, IVSαCS) in X.

(iii) Let (X, τ) be an IVTS and let $A \in IVS(X)$. Then, A is called an interval-valued α-open set (briefly, IVαOS) in X [33] if $A \subset IVint(IVcl(IVint(A)))$, where $IVint(A)$ and $IVcl(A)$ denote the interval-valued interior and the interval-valued closure of A (see [36]). The complement of an IVαOS is called an interval-valued α-closed set (briefly, IVαCS) in X.

The set of all soft α-open [resp. closed] sets in a soft topological space (X, τ, E) will be denoted by SαOS(X) [resp. SαCS(X)]. We will denote the set of all IVSαOSs [resp. IVSαCS] by IVSαOS(X) [resp. IVSαCS(X)]. Also, we will denote the set of all IVαOSs [resp. IVαCS] by IVαOS(X) [resp. IVαCS(X)].

Definition 31 ([33]). *An IVSTS (X, τ, E) is called the following:*

(i) An interval-valued soft αT_0(i)-space (briefly, IVSαT_0(i)-space) if for any $x \neq y \in X$, there are $\mathbf{U}, \mathbf{V} \in IVS\alpha OS(X)$ such that either $x_1 \in \mathbf{U}, y_1 \notin \mathbf{U}$ or $y_1 \in \mathbf{V}, x_1 \notin \mathbf{V}$;

(ii) An interval-valued soft αT_0(ii)-space (briefly, IVSαT_0(ii)-space) if for any $x \neq y \in X$, there are $\mathbf{U}, \mathbf{V} \in IVS\alpha OS(X)$ such that either $x_0 \in \mathbf{U}, y_0 \notin \mathbf{U}$ or $y_0 \in \mathbf{V}, x_0 \notin \mathbf{V}$;

(iii) An interval-valued soft αT_1(i)-space (briefly, IVSαT_1(i)-space) if for any $x \neq y \in X$, there are $\mathbf{U}, \mathbf{V} \in IVS\alpha OS(X)$ such that $x_1 \in \mathbf{U}, y_1 \notin \mathbf{U}$ and $y_1 \in \mathbf{V}, x_1 \notin \mathbf{V}$;

(iv) An interval-valued soft αT_1(ii)-space (briefly, IVSαT_1(ii)-space) if for any $x \neq y \in X$, there are $\mathbf{U}, \mathbf{V} \in IVS\alpha OS(X)$ such that $x_0 \in \mathbf{U}, y_0 \notin \mathbf{U}$ and $y_0 \in \mathbf{V}, x_0 \notin \mathbf{V}$;

(v) An interval-valued soft αT_2(i)-space (briefly, IVSαT_2(i)-space) if for any $x \neq y \in X$, there are $\mathbf{U}, \mathbf{V} \in IVS\alpha OS(X)$ such that $x_1 \in \mathbf{U}, y_1 \in \mathbf{V}$, and $\mathbf{U} \cap \mathbf{V} = \widetilde{\emptyset}_E$;

(vi) An interval-valued soft αT_2(ii)-space (briefly, IVSαT_2(ii)-space) if for any $x \neq y \in X$, there are $\mathbf{U}, \mathbf{V} \in IVS\alpha OS(X)$ such that $x_0 \in \mathbf{U}, y_0 \in \mathbf{V}$, and $\mathbf{U} \cap \mathbf{V} = \widetilde{\emptyset}_E$;

(vii) An interval-valued soft α-regular(i)-space (briefly, IVSαR(i)-space) if for each $\mathbf{A} \in IVS\alpha CS(X)$ and each $x \in X$ with $x_1 \notin \mathbf{A}$, there are $\mathbf{U}, \mathbf{V} \in IVS\alpha OS(X)$ such that $x_1 \in \mathbf{U}, \mathbf{A} \subset \mathbf{V}$, and $\mathbf{U} \cap \mathbf{V} = \widetilde{\emptyset}_E$;

(viii) An interval-valued soft α-regular(ii)-space (briefly, IVSαR(ii)-space) if for each $\mathbf{A} \in IVS\alpha CS(X)$ and each $x \in X$ with $x_0 \notin \mathbf{A}$, there are $\mathbf{U}, \mathbf{V} \in IVS\alpha OS(X)$ such that $x_0 \in \mathbf{U}, \mathbf{A} \subset \mathbf{V}$, and $\mathbf{U} \cap \mathbf{V} = \widetilde{\emptyset}_E$;

(ix) An interval-valued soft αT_3(i)-space (briefly, IVSαT_3(i)-space) if it is an IVSαT_1(i)-space and an IVSαR(i)-space;

(x) An interval-valued soft αT_3(ii)-space (briefly, IVSαT_3(ii)-space) if it is an IVSαT_1(ii)-space and an IVSαR(ii)-space;

(xi) An interval-valued soft α-normal-space (briefly, IVSαN-space), if for each $\mathbf{A}$, $\mathbf{B} \in IVS\alpha CS(X)$ *with* $\mathbf{A} \cap \mathbf{B} = \widetilde{\varnothing}_E$, *there are* $\mathbf{U}$, $\mathbf{V} \in IVS\alpha OS(X)$ *such that* $x_1 \in \mathbf{U}$, $\mathbf{A} \subset \mathbf{V}$, *and* $\mathbf{U} \cap \mathbf{V} = \widetilde{\varnothing}_E$;

(xii) An interval-valued soft αT_4(i)-*space (briefly, IVS*αT_4*(i)-space) if it is an IVS*αT_1*(i)-space and an IVSαN-space;*

(xiii) An interval-valued soft αT_4(ii)-*space (briefly, IVS*αT_4*(ii)-space) if it is an IVS*αT_1*(ii)-space and an IVSαN-space.*

Definition 32. *An IVSTS* (X, τ, E) *is said to be the following:*

(i) Partial total interval-valued soft αT_0(i) *(briefly, PTIVS*αT_0*(i)) if for any* $x \neq y \in X$, *there is* $\mathbf{U} \in IVS\alpha OS(X)$ *such that either* $x_1 \in_P \mathbf{U}$, $y_1 \notin_T \mathbf{U}$ *or* $y_1 \in_P \mathbf{U}$, $x_1 \notin_T \mathbf{U}$;

(ii) Partial total interval-valued soft αT_0(ii) *(briefly, PTIVS*αT_0*(ii)) if for any* $x \neq y \in X$, *there is* $\mathbf{U} \in IVS\alpha OS(X)$ *such that either* $x_0 \in_P \mathbf{U}$, $y_0 \notin_T \mathbf{U}$ *or* $y_0 \in_P \mathbf{U}$, $x_0 \notin_T \mathbf{U}$;

(iii) Partial total interval-valued soft αT_1(i) *(briefly, PTIVS*αT_1*(i)) if for any* $x \neq y \in X$, *there is* $\mathbf{U} \in IVS\alpha OS(X)$ *such that* $x_1 \in_P \mathbf{U}$, $y_1 \notin_T \mathbf{U}$ *and* $y_1 \in_P \mathbf{U}$, $x_1 \notin_T \mathbf{U}$;

(iv) Partial total interval-valued soft αT_1(ii) *(briefly, PTIVS*αT_1*(ii)) if for any* $x \neq y \in X$, *there is* $\mathbf{U} \in IVS\alpha OS(X)$ *such that* $x_0 \in_P \mathbf{U}$, $y_0 \notin_T \mathbf{U}$ *and* $y_0 \in_P \mathbf{V}$, $x_0 \notin_T \mathbf{V}$;

(v) Partial total interval-valued soft αT_2(i) *(briefly, PTIVS*αT_2*(i)) if for any* $x \neq y \in X$, *there are* $\mathbf{U}$, $\mathbf{V} \in IVS\alpha OS(X)$ *such that* $x_1 \in_P \mathbf{U}$, $y_1 \notin_T \mathbf{U}$ *and* $y_1 \in_P \mathbf{V}$, $x_1 \notin_T \mathbf{V}$ *and* $\mathbf{U} \cap \mathbf{V} = \widetilde{\varnothing}_E$;

(vi) Partial total interval-valued soft αT_2(ii) *(briefly, PTIVS*αT_2*(ii)) if for any* $x \neq y \in X$, *there are* $\mathbf{U}$, $\mathbf{V} \in IVS\alpha OS(X)$ *such that* $x_0 \in_P \mathbf{U}$, $y_0 \notin_T \mathbf{U}$ *and* $y_0 \in_P \mathbf{V}$, $x_0 \notin_T \mathbf{V}$ *and* $\mathbf{U} \cap \mathbf{V} = \widetilde{\varnothing}_E$;

(vii) Partial total interval-valued soft α regular(i) *(briefly, PTIVSαR(i)) if for any* $x \in\in X$ *and any* $\mathbf{A} \in IVS\alpha CS(X)$ *with* $x_1 \notin \mathbf{A}$, *there are* $\mathbf{U}$, $\mathbf{V} \in IVS\alpha OS(X)$ *such that* $\mathbf{A} \subset \mathbf{U}$, $x_1 \in_P \mathbf{V}$, *and* $\mathbf{U} \cap \mathbf{V} = \widetilde{\varnothing}_E$;

(viii) Partial total interval-valued soft α regular(ii) *(briefly, PTIVSαR(ii)) if for any* $x \in\in X$ *and any* $\mathbf{A} \in IVS\alpha CS(X)$ *with* $x_0 \notin \mathbf{A}$, *there are* $\mathbf{U}$, $\mathbf{V} \in IVS\alpha OS(X)$ *such that* $\mathbf{A} \subset \mathbf{U}$, $x_0 \in_P \mathbf{V}$, *and* $\mathbf{U} \cap \mathbf{V} = \widetilde{\varnothing}_E$;

(ix) Partial total interval-valued soft αT_3(i) *(briefly, PTIVS*αT_3*(i)) if it is both PTIVSαR(i) and PTIVS*αT_1*(i);*

(x) Partial total interval-valued soft αT_3(ii) *(briefly, PTIVS*αT_3*(ii)) if it is both PTIVSαR(ii) and PTIVS*αT_1*(ii);*

(xi) Partial total interval-valued soft αT_4(i) *(briefly, PTIVS*αT_4*(i)) if it is both IVSαN and PTIVS*αT_1*(i);*

(xii) Partial total interval-valuedsoft αT_4(ii), *(briefly, PTIVS*αT_4*(ii)) if it is both IVSαN and PTIVS*αT_1*(ii).*

Proposition 28. (1) *Every PTIVS*αT_j(i)-*space* [resp. PTIVSαT$_j$(ii)-space] *is a PTIVS*αT_{j-1}(i)-*space* [resp. PTIVSαT$_{j-1}$(ii)-space] *for* $j = 1,\ 2,\ 3$. *However, the converse is not true in general.*

(2) *Every IVS*αT_2(i)-*space* [resp. IVSαT$_j$(ii)-space] *is a PTIVS*αT_2(i)-*space* [resp. PTIVSαT$_j$(ii)-space]. *However, the converse is not true in general.*

Proof. (1) The proofs of PTIVSαT$_2$(i)⇒PTIVSαT$_1$(i)⇒PTIVSαT$_0$(i) are obvious from Definition 32.

Let (X, τ, E) be PTIVSαT$_3$(i) and $x \neq y \in X$. Since X is PTIVSαT$_1$(i), there are $\mathbf{U}$, $\mathbf{V} \in IVS\alpha OS(X)$ such that $x_1 \in_P \mathbf{U}$, $y_1 \notin_T \mathbf{U}$ and $y_1 \in_P \mathbf{V}$, $x_1 \notin_T \mathbf{U}$. It is clear that $\mathbf{U}^c$, $\mathbf{V}^c \in IVS\alpha CS(X)$ such that $x_1 \notin \mathbf{U}^c$ and $y_1 \notin \mathbf{V}^c$. Since X is PTIVSR(i), we have the following.

For $\mathbf{U}^c \in IVS\alpha CS(X)$ such that $x_1 \notin \mathbf{U}^c$, there are $\mathbf{U}_1$, $\mathbf{V}_1 \in IVS\alpha OS(X)$ such that $\mathbf{U}^c \subset \mathbf{U}_1$, $x_1 \in_P \mathbf{V}_1$, and $\mathbf{U}_1 \cap \mathbf{V}_1 = \widetilde{\varnothing}_E$. Since $y_1 \notin_T \mathbf{U}$, by Proposition 1 (2), $y_1 \in \mathbf{U}_1$, i.e., $y_1 \in_P \mathbf{U}_1$. Since $\mathbf{U}_1 \cap \mathbf{V}_1 = \widetilde{\varnothing}_E$, $y_1 \notin_T \mathbf{V}_1$. Then, we obtain that there are $\mathbf{U}_1$, $\mathbf{V}_1 \in IVS\alpha OS(X)$ such that

$$\mathbf{U}^c \subset \mathbf{U}_1,\ x_1 \in_P \mathbf{V}_1,\ y_1 \in_P \mathbf{U}_1,\ y_1 \notin_T \mathbf{V}_1. \tag{4}$$

For $\mathbf{V}^c \in IVS\alpha CS(X)$ such that $y_1 \notin \mathbf{V}^c$, by arguments similar to those above, we obtain that there are $\mathbf{U}_2, \mathbf{V}_2 \in IVS\alpha OS(X)$ such that

$$\mathbf{V}^c \subset \mathbf{U}_2,\ y_1 \in_P \mathbf{V}_2,\ x_1 \in_P \mathbf{U}_2,\ y_1 \notin_T \mathbf{V}_2. \tag{5}$$

Thus, from (4) and (5), we have

$$x_1 \in_P \mathbf{V}_1 \cap \mathbf{U}_2,\ y_1 \notin_T \mathbf{V}_1 \cap \mathbf{U}_2 \text{ and } y_1 \in_P \mathbf{U}_1 \cap \mathbf{V}_2,\ x_1 \notin_T \mathbf{U}_1 \cap \mathbf{V}_2.$$

By Proposition 5.8 (1) in [33], $\mathbf{V}_1 \cap \mathbf{U}_2,\ \mathbf{U}_1 \cap \mathbf{V}_2 \in IVS\alpha OS(X)$. It is clear that $(\mathbf{V}_1 \cap \mathbf{U}_2) \cap (\mathbf{U}_1 \cap \mathbf{V}_2) = \widetilde{\emptyset}_E$. So, X is PTIVSαT$_2$(i).

The proofs of the second parts are similar. See Example 10 for the converse.

(2) Let (X, τ, E) be an IVSαT$_2$(i)-space and let $x \neq y \in X$. Then, there are $\mathbf{U}, \mathbf{V} \in IVS\alpha OS(X)$ such that $x_1 \in \mathbf{U}$, $y_1 \notin \mathbf{U}$ and $y_1 \in \mathbf{V}$, $x_1 \notin \mathbf{V}$ and $\mathbf{U} \cap \mathbf{V} = \widetilde{\emptyset}_E$. Thus, $y_1 \notin_T \mathbf{U}$ and $x_1 \notin_T \mathbf{V}$. So, X is a PTIVSαT$_2$(i)-space.

The proof of the second part is similar. See Example 10 (3) for the converse. □

Example 10. *(1) Let $X = \{x, y\}$ and $E = \{e, f\}$ Consider the IVST τ on X given by*

$$\tau = \{\widetilde{\emptyset}_E, \mathbf{A}, \widetilde{C}_E\},$$

where $\mathbf{A}(e) = [\{x\}, X],\ \mathbf{A}(f) = [\emptyset, \{y\}]$.

Then, we can easily check that (X, τ, E) is a PTIVSαT$_0$(i)-space but not a PTIVSαT$_1$(i)-space.

(2) Let E be a set of parameters and τ the families of IVSSs over $\mathbb{N}$, defined as follows:

$$\tau = \{\widetilde{\emptyset}_E\} \bigcup \{\mathbf{A} \in IVSS_E(X) : \mathbf{A}^c \textit{ is finite}\}.$$

Then clearly, τ is an IVST on X. Moreover, $\tau = IVS\alpha OS(X)$. Let $x \neq y \in \mathbb{N}$ and let $[\mathbb{N} \setminus \{y\}, \mathbb{N} \setminus \{y\}] = \widetilde{\mathbb{N}} \setminus y_1$. Then, $\widetilde{\mathbb{N}} \setminus y_1,\ \widetilde{\mathbb{N}} \setminus x_1 \in IVS\alpha OS(X)$ such that

$$x_1 \in_P \widetilde{\mathbb{N}} \setminus y_1,\ y_1 \notin_T \widetilde{\mathbb{N}} \setminus y_1 \text{ and } y_1 \in_P \widetilde{\mathbb{N}} \setminus x_1,\ x_1 \notin_T \widetilde{\mathbb{N}} \setminus x_1.$$

Thus, $(\mathbb{N}, \tau, E)$ is a PTIVSαT$_1$(i)-space. On the other hand, we cannot find two disjoint IVSαOSs over $\mathbb{N}$ except $\widetilde{\emptyset}_E$ and $\widetilde{\mathbb{N}}_E$. So, $(\mathbb{N}, \tau, E)$ is not a PTIVSαT$_2$(i)-space.

(3) Let $X = \{x, y\}$, $E = \{e, f\}$ and consider the IVST τ on X given by

$$\tau = \{\widetilde{\emptyset}_E, \mathbf{A}, \mathbf{B}, \mathbf{C}, \widetilde{X}_E\},$$

where $\mathbf{A}(e) = [\{x\}, \{x\}],\ \mathbf{A}(f) = [\emptyset, \emptyset]$,
$\mathbf{B}(e) = [\emptyset, \emptyset],\ \mathbf{B}(f) = [\{y\}, \{y\}]$,
$\mathbf{C}(e) = [\{x\}, \{x\}]$, *and* $\mathbf{C}(f) = [\{y\}, \{y\}]$.

Then clearly, $x_1 \in_P \mathbf{A}$, $y_1 \in_P \mathbf{A}$ and $y_1 \in_P \mathbf{B}$, $x_1 \in_P \mathbf{B}$ and $\mathbf{A} \cap \mathbf{B} = \widetilde{\emptyset}_E$. Thus, X is a PTIVSαT$_2$(i)-space. On the other hand, $\mathbf{C}^c \in IVS\alpha CS(X)$ such that $x_1 \notin \mathbf{C}^c$. But $\widetilde{X}_E$ is the only IVSαOS containing $\mathbf{C}^c$. So, X is not PTIVSαR(i). Hence, X is not a PTIVSαT$_3$(i)-space. Furthermore, we cannot have $\mathbf{U}, \mathbf{V} \in IVS\alpha OS(X)$ such that $\mathbf{U}, \mathbf{V} \neq \widetilde{X}_E$ and $x_1 \in \mathbf{U}$, $y_1 \in \mathbf{V}$. Therefore, X is not an IVSαT$_2$(i)-space.

Proposition 29. *Let (X, τ, E) be an IVSTS. If $\mathbf{x}_1 \in IVS\alpha CS(X)$ [resp. $\mathbf{x}_0 \in IVS\alpha CS(X)$] for each $x \in X$, then X is a PTIVSαT$_1$(i) [resp. PTIVSαT$_1$(ii)]-space*

Proof. Suppose $\mathbf{x}_1 \in IVS\alpha CS(X)$ for each $x \in X$ and let $x \neq y \in X$. Then clearly, $\mathbf{x}_1^c,\ \mathbf{y}_1^c \in IVS\alpha OS(X)$ such that $y_1 \in \mathbf{x}_1^c$ and $x_1 \in \mathbf{y}_1^c$. Thus, $x_1 \in \mathbf{y}_1^c$, $y_1 \notin_T \mathbf{y}_1^c$ and $y_1 \in \mathbf{x}_1^c$, $x_1 \notin_T \mathbf{x}_1^c$. So, X is a PTIVSαT$_1$(i)-space. The proof of the second part is similar. □

Proposition 30. *Let (X, τ, E) be an IVSTS and β the set of all interval-valued soft α-clopen sets in X. If β is a base for τ, then X is IVSαR*(i) *and IVSαR*(ii)*.*

Proof. Let $x \in X$ and let $\mathbf{A} \in IVS\alpha CS(X)$ with $x_1 \notin \mathbf{A}$. Then clearly, $\mathbf{A}^c \in IVS\alpha OS(X)$ such that $x_1 \in_P \mathbf{A}^c$. Thus, by the hypothesis, there is $\mathbf{B} \in \beta$ such that $x_1 \in_P \mathbf{B} \subset \mathbf{A}^c$. Since $\mathbf{A} \subset \mathbf{B}^c$, $\mathbf{B} \cap \mathbf{B}^c = \widetilde{\emptyset}_E$. Moreover, $\mathbf{B},\ \mathbf{B}^c \in IVS\alpha OS(X)$. So, X is IVSαR(i). Similarly, we prove that X is IVSαR(ii). □

Lemma 4 (See Proposition 2.11, [22]). *Let (X, τ, E) be an IVSTS and $\tau^* = \{\mathbf{A} \in IVSS_E(X) : \mathbf{A}(e) \in \tau_e \text{ for each } e \in E\}$. Then, τ^* is an IVST on X such that $\tau^*_e = \tau_e$ for each $e \in E$.*

Proof. The proof is similar to Proposition 2.11 in [22]. □

Remark 9. *In Proposition 4, $\tau \neq \tau^*$ in general (see Example 11).*

Example 11. *Let $X = \{x, y\}$ and $E = \{e, f\}$, and consider the IVST τ on X defined as follows:*

$$\tau = \{\widetilde{\emptyset}_E, \mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \widetilde{X}_E\},$$

where $\mathbf{A}_1(e) = [\emptyset, \emptyset],\ \mathbf{A}_1(f) = [\{y\}, \{y\}]$,
$\mathbf{A}_2(e) = [\{x\}, \{x\}],\ \mathbf{A}_2(f) = [\{y\}, \{y\}]$,
$\mathbf{A}_3(e) = [\{y\}, \{y\}]$, *and* $\mathbf{A}_4(f) = [X, X]$.

Then, $\tau_e = \{\emptyset_E, [\{x\}, \{x\}], [\{y\}, \{y\}], X_E\}$ and $\tau_f = \{\emptyset_E, [\{y\}, \{y\}], X_E\}$. Thus, we have

$$\tau^* = \{\widetilde{\emptyset}_E, \mathbf{A}_1, \cdots, \mathbf{A}_{14}, \widetilde{X}_E\},$$

where $\mathbf{A}_4(e) = [\emptyset, \emptyset],\ \mathbf{A}_4(f) = [\{x\}, \{x\}]$,
$\mathbf{A}_5(e) = [\emptyset, \emptyset],\ \mathbf{A}_5(f) = [X, X]$,
$\mathbf{A}_6(e) = [\{x\}, \{x\}],\ \mathbf{A}_6(f) = [\emptyset, \emptyset]$,
$\mathbf{A}_7(e) = \mathbf{A}_7(f) = [\{x\}, \{x\}]$,
$\mathbf{A}_8(e) = [\{x\}, \{x\}],\ \mathbf{A}_8(f) = [X, X]$,
$\mathbf{A}_9(e) = [\{y\}, \{y\}],\ \mathbf{A}_9(f) = [\emptyset, \emptyset]$,
$\mathbf{A}_{10}(e) = [\{y\}, \{y\}],\ \mathbf{A}_{10}(f) = [\{x\}, \{x\}]$,
$\mathbf{A}_{11}(e) = \mathbf{A}_{11}(f) = [\{y\}, \{y\}]$,
$\mathbf{A}_{12}(e) = [X, X],\ \mathbf{A}_{12}(f) = [\emptyset, \emptyset]$,
$\mathbf{A}_{13}(e) = [X, X],\ \mathbf{A}_{13}(f) = [\{x\}, \{x\}]$,
$\mathbf{A}_{14}(e) = [X, X]$, *and* $\mathbf{A}_{14}(f) = [\{y\}, \{y\}]$.

Moreover, we can confirm that $\tau \neq \tau^*$ but $\tau^*_e = \tau$ for each $e \in E$.

From Remark 9, we obtain the following concept.

Definition 33. *(i) A soft topological space (X, τ, E) is said to be extended if $\tau = \tau^*$ (see [22]).*
(ii) An IVSTS (X, τ, E) is said to be extended if $\tau = \tau^*$.

Lemma 5 (See Corollary 1, [42]). *Let (X, τ, E) be an extended IVSTS and $\mathbf{A} \in IVSS_E(X)$. Then, $\mathbf{A} \in IVS\alpha OS(X)$ if and only if $\mathbf{A}(e)$ is an IVαOS in (X, τ_e) for each $e \in E$.*

Proof. The proof is almost similar to Corollary 1 in [42]. □

Theorem 12. *Let (X, τ, E) be an IVSTS. If X is extended, then the notions of PTIVSαT$_j$*(i) [resp. PTIVSαT$_j$(ii)] *and IVSαT$_j$*(i) [resp. IVSαT$_j$(ii)] *are equivalent for $j = 0,\ 1$.*

Proof. Suppose X is extended and let X be a PTIVSαT$_0$(i)-space, $x \neq y \in X$. Then, there is $\mathbf{U} \in IVS\alpha OS(X)$ such that either $x_1 \in_P \mathbf{U}$, $y_1 \notin_T \mathbf{U}$ or $y_1 \in_P \mathbf{U}$, $x_1 \notin_T \mathbf{U}$, say $x_1 \in_P \mathbf{U}$ and $y_1 \notin_T \mathbf{U}$. Since $x_1 \in \mathbf{U}(e)$, $x \in U^-(e)$ for some $e \in E$. Suppose $x \in U^-(e)$ for each $e \in E$. Then, the proof is obvious. Thus, without loss of generality, there is $e \in E$ such that $x \in U^-(e)$ and $x \notin U^-(e^{'})$ for each $e^{'} \in E \setminus \{e\}$. Since (X, τ, E) is extended, there is $\mathbf{V} \in IVS\alpha OS(X)$ such that $\mathbf{V}(e) = \mathbf{U}(e)$, i.e., $V^-(e) = U^-(e)$ and $\mathbf{V}(e^{'}) = \widetilde{X}$, i.e., $V^-(e^{'}) = X$ for each $x^{'} \in E \setminus \{e\}$. Thus, $x_1 \in \mathbf{V}$ and $y_1 \notin \mathbf{V}$. So, X is an IVSαT$_0$(i)-space.

Conversely, suppose X is an IVSαT$_0$(i)-space and let $x \neq y \in X$. Then, there are $\mathbf{U}$, $\mathbf{V} \in IVS\alpha OS(X)$ such that either $x_1 \in \mathbf{U}$, $y_1 \notin \mathbf{U}$ or $y_1 \in \mathbf{V}$, $x_1 \notin \mathbf{V}$, say $x_1 \in \mathbf{U}$ and $y_1 \notin \mathbf{U}$. Since $y_1 \notin \mathbf{U}$, $y_1 \notin \mathbf{U}(e)$, i.e., $y \notin U^-(e)$ for some $e \in E$. Suppose $y \notin U^-(e)$ for each $e \in E$. Then, the proof is clear. Thus, without loss of generality, there is $e \in E$ such that $y \notin U^-(e)$ and $y \in U^-(e^{'})$ for each $e^{'} \in E \setminus \{e\}$. Since (X, τ, E) is extended, $\mathbf{U}(e)$ is an IVαOS in (X, τ_e). So, by Lemma 5, there is $\mathbf{V} \in IVS\alpha OS(X)$ such that $\mathbf{V}(e) = \mathbf{U}(e)$, i.e., $V^-(e) = U^-(e)$ and $\mathbf{V}(e^{'}) = \mathbf{U}(e) = \widetilde{\varnothing}$, i.e., $V^-(e^{'}) = X$ for each $e^{'} \in X \setminus \{e\}$. Moreover, $x_1 \in_P \in \mathbf{V}$ and $y_1 \notin_T \in \mathbf{V}$. Hence, X is a PTIVSαT$_0$(i)-space.

The proof of the second part is similar. □

From Theorem 12 and Definition 32, we have the following.

Corollary 3. *Let (X, τ, E) be an IVSTS. If X is extended, then the notions of PTIVSαT$_4$*(i) [resp. PTIVSαT$_4$(ii)] *and IVSαT$_4$*(i) [resp. IVSαT$_4$(ii)] *are equivalent.*

Proposition 31. *The property of being a PTIVSαT$_j$*(i) [resp. PTIVSαT$_j$(ii)] *is hereditary for $j = 0,\ 1,\ 2,\ 3$.*

Proof. The proof follows from Result 2 and Definition 32. □

7. Conclusions

First, we defined the relationships between interval-valued points and interval-valued soft sets, defined interval-valued soft continuous mappings, and obtained their various properties. Second, we defined new separation axioms in interval-valued soft topological spaces called *partial interval-valued soft T_i*(j)*-spaces* (i = 0, 1, 2, 3, 4; j = i, ii) and dealt with some of their properties and some relationships among them. Finally, we defined another new separation axioms in interval-valued soft topological spaces called *partial total interval-valued soft T_i*(j)*-spaces* (i = 0, 1, 2, 3, 4; j = i, ii) and dealt with some of their properties and some relationships among them.

In the future, we plan to apply the decision-making problems presented by Al-Shami and El-Shafe [35] and Al-Shami [43] to interval-valued soft separation axioms. Furthermore, we will try to study the structures of the Vietoris topology based on soft topology or interval-valued topology. Also, we will study whether all the properties of our study are still valid in interval-valued supra soft topological spaces.

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References

1. Molodtsov, D. Soft set theory–First results. *Comput. Math. Appl.* **1999**, *37*, 19–31. [CrossRef]
2. Maji, P.K.; Biswas, R.; Roy, A.R. Soft set theory. *Comput. Math. Appl.* **2003**, *45*, 555–562. [CrossRef]
3. Ali, M.I.; Feng, F.; Liu, X.; Min, W.K. Shabir, M. On some new operations in soft set theory. *Comput. Math. Appl.* **2009**, *57*, 1547–1553. [CrossRef]
4. Babitha, K.V.; Sunil, J.J. Soft set relations and functions. *Comput. Math. Appl.* **2010**, *60*, 1840–1849. [CrossRef]
5. Kharal, A.; Ahmad, B. Mappings on soft classes. *New Math. Nat. Comput.* **2011**, *7*, 471–481. [CrossRef]
6. Şenel, G.; Lee, J.-G.; Hur, K. Advanced soft relation and soft mapping. *Int. J. Comput. Intell. Syst.* **2021**, *14*, 461–470. [CrossRef]
7. Majumdar, P.; Samanta, S.K. Similarity measure of soft sets. *New Math. Nat. Comput.* **2008**, *4*, 1–12. [CrossRef]
8. Çağman, N.; Enginoglu, S. Soft set theory and uni-int decision making. *Eur. J. Oper. Res.* **2010**, *207*, 848–855. [CrossRef]
9. Çağman, N.; Enginoglu, S. Soft matrix theory and its decision making. *Comput. Math. Appl.* **2010**, *59*, 3308–3314. [CrossRef]
10. Nazmul, S.; Samanta, S.K. Soft topological soft groups. *Math. Sci.* **2012**, *6*, 66. [CrossRef]
11. Nazmul, S.; Samanta, S.K. Group soft topology. *J. Fuzzy Math.* **2014**, *22*, 435–450.
12. Hida, T. Soft topological group. *Ann. Fuzzy Math. Inform.* **2014**, *8*, 1001–1025.
13. Tahat, M.K.; Sidky, F.; Abo-Elhamayel, M. Soft topological soft groups and soft rings. *Soft Comput.* **2018**, *22*, 7143–7156. [CrossRef]
14. Tahat, M.K.; Sidky, F.; Abo-Elhamayel, M. Soft topological rings. *J. King Saud Univ. Sci.* **2019**, *31*, 1127–1136. [CrossRef]
15. Sidky, F.; El-Shafei, M.E.; Tahat, M.K. Soft topological soft modules. *Ann. Fuzzy Math. Inform.* **2020**, *20*, 257–272.
16. Shabir, M.; Naz, M. On soft topological spaces. *Comput. Math. Appl.* **2011**, *61*, 1786–1799. [CrossRef]
17. Çağman, N.; Karataş, S.; Enginoglu, S. Soft topology. *Comput. Math. Appl.* **2011**, *62*, 351–358. [CrossRef]
18. Min, W.K. A note on soft topological spaces. *Comput. Math. Appl*. **2011**, *62*, 524–3528. [CrossRef]
19. Hussain, S.; Ahmad, B. Some properties of soft topological spaces. *Comput. Math. Appl.* **2011**, *62*, 4058–4067. [CrossRef]
20. Zorlutuna, I.; Akdag, M.; Min, W.K.; Atmaca, S. Remarks on soft topological spaces. *Ann. Fuzzy Math. Inform.* **2012**, *3*, 171–185.
21. Aygunoglu, A.; Aygun, H. Some notes on soft topological spaces. *Neural. Comput. Appl.* **2012**, *21*, 113–119. [CrossRef]
22. Nazmul, S.; Samanta, S.K. Neighborhood properties of soft topological spaces. *Ann. Fuzzy Math. Inform.* **2013**, *6*, 1–15.
23. Georgiou, D.N.; Megaritis, A.C. Soft set theory and topology. *Appl. Gen. Topol.* **2013**, *14*, 93–100. [CrossRef]
24. Georgiou, D.N.; Megaritis, A.C.; Petropoulos, V. I. On soft topological spaces. *Appl. Math. Inf. Sci.* **2013**, *7*, 1889–1901. [CrossRef]
25. Debnath, P.; Tripathy, B.C. On separation axioms in soft bitopological spaces. *Songklanakarin J. Sci. Technol.* **2020**, *42*, 830–835.
26. Akdag, M.; Ozkan, A. On soft α-separation axioms. *J. Adv. Stud. Topol.* **2014**, *5*, 16–24. [CrossRef]
27. Akdag, M.; Ozkan, A. Soft α-pen sets and soft α-continuous functions. *Abstr. Appl. Anal.* **2014**, *2014*, 891341. [CrossRef]
28. Lee, J.G.; Şenel, G.; Jun, Y. B.; Abbas, F.; Hur, K. Topological structures via interval-valued soft sets. *Ann. Fuzzy Math. Inform.* **2021**, *22*, 133–169.
29. Alcantud, J.C.R. The relationship between fuzzy soft and soft Topologies. *Int. J. Fuzzy Syst.* **2022**, *24*, 1653–1668. [CrossRef]
30. Ghour, S.A.; Ameen, Z.A. Maximal soft compact and maximal soft connected topologies. *Appl. Comput. Intell. Soft Comput.* **2022**, *2022*, 9860015. [CrossRef]
31. Garg, H.; Perveen P A, F.; John, S.J.; Perez-Dominguez, L. Spherical fuzzy soft topology and its application in group decision-making problems. *Math. Probl. Eng.* **2022**, *2022*, 1007133. [CrossRef]
32. Alajlan, A.I.; Alghamdi, A.M. Innovative strategy for constructing soft topology. *Axioms* **2023**, *12*, 967. [CrossRef]
33. Baek, J.I.; Senel, G.; Jafari, S.; Han, S.H.; Hur, K. Separation axioms in interval-valued soft topological spaces. 2024, *to be submitted*.
34. El-Shafei, M.E.; Abo-Elhamayel, M.; Al-shami, T.M. Partial soft separation axioms and soft compact spaces. *Filomat* **2018**, *32*, 4755–4771. [CrossRef]
35. Al-Shami, T.M.; El-Shafe, M.E. Partial belong relation on soft separation axioms and decision-making problem, two birds with one stone. *Soft Comput.* **2020**, *24*, 5377–5387. [CrossRef]
36. Kim, J.; Jun, Y.B.; Lee, J.G.; Hur, K. Topological structures based on interval-valued sets. *Ann. Fuzzy Math. Inform.* **2020**, *20*, 273–295.
37. Yao, Y. Interval sets and interval set algebras. In Proceedings of the 8th IEEE International Conference on Cognitive Informatics, ICCI 2009, Hong Kong, China, 15–17 June 2009; pp. 307–314.
38. Zadeh, L.A. The concept of a linguistic variable and its application to approximate reasoning-I. *Inform. Sci.* **1975**, *8*, 199–249. [CrossRef]
39. Yang, X.; Lin, T.Y.; Yang, J.; Li, Y.; Yu, D. Combination of interval-valued fuzzy set and soft set. *Comput. Math. Appl.* **2009**, *58*, 521–527. [CrossRef]
40. Ali, M.; KİlİÇman, A.; Khameneh, A.Z. Separation axioms interval-valued fuzzy soft topology via quasi-neighborhood structure. *Mathematics* **2020**, *8*, 178. [CrossRef]
41. Kelly, J.C. Bitopological spaces. *Proc. London Math. Soc.* **1963**, *13*, 71–89. [CrossRef]

42. Al-Shami, T.M.; Kočinac, L.D.R. The equivalence between the enriched and extended soft topologies. *Appl. Comput. Math.* **2019**, *18*, 149–162.
43. Al-shami, T.M. On soft separation axioms and their applications on decision-making problem. *Math. Probl. Eng.* **2021**, 2021, 8876978. [CrossRef]