Spectral Radius Inequalities for Accretive-Dissipative Matrices

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Abstract

In this paper, we prove new spectral radius inequalities for sums, differences and commutators involving accretive-dissipative matrices of Hilbert space. Earlier well-known results used the spectral radius for its importance for general matrices. In our paper, we focus on some results related to spectral radius for special kind of matrices which are accretive-dissipative. A particular example is also presented in this work.

Keywords: Spectral radius; Commutators; Accretive-Dissipative Matrices

1 Introduction

The concept of spectral radius has many applications in different fields of science, so there are many researchers who are interested with spectral radius inequalities started with Hou-Do, 1995 who established a nice inequality related to positive operators [4]. This inequality was then used by Kittaneh when he found the spectral radius for block matrices. Moreover, Abu-Omar and Kittaneh (2015) proved a general spectral radius inequality in [1].
In the current work, we present bounds of sums, differences, and commutators for accretive-dissipative matrices depending only on the spectral radius of both the real and imaginary parts of these kind on matrices.
Let $\mathcal{M}_n(\mathbb{C})$ be the algebra of all $n \times n$ complex matrices. For $\Upsilon \in \mathcal{M}_n(\mathbb{C})$, let $r(\Upsilon)$ and $\| \Upsilon \|$ denote the spectral radius and the usual norm of Υ, where

$$r(\Upsilon) = max\left\{|\lambda|, \lambda \in \sigma(\Upsilon)\right\}, \tag{1}$$

provided that $\sigma(\Upsilon)$ is the set of all eigenvalues of the matrix Υ.

It is known that for $\Upsilon \in \mathcal{M}_n(\mathbb{C})$, we have

$$r(\Upsilon) \leq \|\Upsilon\|, \tag{2}$$

if Υ is positive semidefinite, then

$$r(\Upsilon) = \|\Upsilon\|, \tag{3}$$

and for any integer k,

$$r(\Upsilon^k) = r^k(\Upsilon). \tag{4}$$

Also, for $\zeta \in \mathbb{C}, r(\zeta\Upsilon) = |\zeta| r(\Upsilon)$. Note that for non commutating matrices, the spectral radius is neither subadditive nor submultiplicative but for $\Upsilon, \Psi \in \mathcal{M}_n(\mathbb{C})$ such that $\Upsilon\Psi = \Psi\Upsilon$, then

$$r(\Upsilon + \Psi) \leq r(\Upsilon) + r(\Psi), \tag{5}$$

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and

$$r(\Upsilon\Psi) \leq r(\Upsilon)r(\Psi). \tag{6}$$

A matrix $\Upsilon \in \mathcal{M}_n(\mathbb{C})$ is called accretive-dissipative if in its Cartesian decomposition $\Upsilon = \Upsilon_1 + i\Upsilon_2$, the matrices $\Upsilon_1 = \Re(\Upsilon) = \frac{\Upsilon+\Upsilon^*}{2}$ and $\Upsilon_2 = \mathfrak{Im}(\Upsilon) = \frac{\Upsilon-\Upsilon^*}{2i}$ are positive semidefinite. For Additional properties, the reader can see [3].
The cummutator of Υ and Ψ is the operator $\Upsilon\Psi - \Psi\Upsilon$. It plays vital role in operator theory.

2 Basic Lemmas

In this section, we present some important lemmas that are important in proving our main results.

Lemma 2.1. [2] *If $\Upsilon, \Psi \in \mathcal{M}_n(\mathbb{C})$ are positive semidifinite, then*

$$\|\Upsilon + i\Psi\| \leq \|\Upsilon + \Psi\|.$$

Lemma 2.2. [5] *If $\Upsilon, \Psi \in \mathcal{M}_n(\mathbb{C})$ are positive semidifinite, then*

$$\|\Upsilon + \Psi\| \leq max(\|\Upsilon\|, \|\Psi\|) + \|\Upsilon^{\frac{1}{2}}\Psi^{\frac{1}{2}}\|.$$

Lemma 2.3. [5] *If $\Upsilon, \Psi \in \mathcal{M}_n(\mathbb{C})$ are positive semidifinite, then*

$$\|\Upsilon - \Psi\| \leq max(\|\Upsilon\|, \|\Psi\|).$$

Lemma 2.4. [5] *If $\Upsilon, \Psi \in \mathcal{M}_n(\mathbb{C})$ are positive semidifinite, then*

$$max(\|\Upsilon\|, \|\Psi\|) - \|\Upsilon^{\frac{1}{2}}\Psi^{\frac{1}{2}}\| \leq \|\Upsilon - \Psi\| \leq \|\Upsilon + \Psi\| \leq max(\|\Upsilon\|, \|\Psi\|) + \|\Upsilon^{\frac{1}{2}}\Psi^{\frac{1}{2}}\|.$$

Lemma 2.5. [7] *If $\Upsilon, \Psi \in \mathcal{M}_n(\mathbb{C})$ are positive semidifinite, then*

$$\|\Upsilon\Psi - \Psi\Upsilon\| \leq \frac{1}{2}\|\Upsilon\|\|\Psi\|.$$

3 Main Results

In this section, we establish spectral radius inequalities for sums, differences and commutators of accretive-dissipative matrices. In 2005, Kittaneh [6], proved many spectral radius inequalities in general for any matrix. In this paper, we are making special mention using accretive-dissipative matrices.
Many researchers have studied these type of accretive-dissipative matrices. For example, Kittaneh and Sakkijha introduced norm inequality for them in [8] and in [9], the authors presented new bounds for determinant inequalities involving them.

Theorem 3.1. *Let $\Upsilon, \Psi \in \mathcal{M}_n(\mathbb{C})$ be accretive-dissipative with Cartesian decomposition $\Upsilon = \Upsilon_1 + i\Upsilon_2$ and $\Psi = \Psi_1 + i\Psi_2$. Then*

$$r(\Upsilon + \Psi) \leq max(r(\Upsilon_1 + \Upsilon_2), r(\Psi_1 + \Psi_2)) + \sqrt{r(\Upsilon_1 + \Upsilon_2)r(\Psi_1 + \Psi_2)}.$$

Proof.

$$\begin{aligned}
\text{Consider} \quad r(\Upsilon + \Psi) &= r((\Upsilon_1 + \Psi_1) + i(\Upsilon_2 + \Psi_2)) \\
&\leq \|(\Upsilon_1 + \Psi_1) + i(\Upsilon_2 + \Psi_2)\| \qquad \text{(by Inequality 2)} \\
&\leq \|\Upsilon_1 + \Psi_1 + \Upsilon_2 + \Psi_2\| \qquad \text{(by Lemma 2.1)} \\
&= \|(\Upsilon_1 + \Upsilon_2) + (\Psi_1 + \Psi_2)\| \\
&\leq max(\|\Upsilon_1 + \Upsilon_2\|, \|\Psi_1 + \Psi_2\|) + \|(\Upsilon_1 + \Upsilon_2)^{\frac{1}{2}}(\Psi_1 + \Psi_2)^{\frac{1}{2}}\| \qquad \text{(by Lemma 2.2)} \\
&\leq max(\|\Upsilon_1 + \Upsilon_2\|, \|\Psi_1 + \Psi_2\|) + \|(\Upsilon_1 + \Upsilon_2)^{\frac{1}{2}}\|\|(\Psi_1 + \Psi_2)^{\frac{1}{2}}\| \\
&= max(r(\Upsilon_1 + \Upsilon_2), r(\Psi_1 + \Psi_2)) + \sqrt{r(\Upsilon_1 + \Upsilon_2)r(\Psi_1 + \Psi_2)}.
\end{aligned}$$

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The result follows by using (3) since the matrices $\Upsilon_1, \Upsilon_2, \Psi_1, \Psi_2$ are positive semidefinite and since for any positive semidefinite matrix $\mathfrak{X}$, $\|\mathfrak{X}^{\frac{1}{2}}\| = \|\mathfrak{X}\|^{\frac{1}{2}}$. □

Theorem 3.2. *Let $\Upsilon, \Psi \in \mathcal{M}_n(\mathbb{C})$ be accretive-dissipative with Cartesian decomposition $\Upsilon = \Upsilon_1 + i\Upsilon_2$ and $\Psi = \Psi_1 + i\Psi_2$. Then*

$$r(\Upsilon + \Psi) \leq max(r(\Upsilon_1), r(\Upsilon_2)) + max(r(\Psi_1), r(\Psi_2)) + \sqrt{r(\Upsilon_1)r(\Upsilon_2)} + \sqrt{r(\Psi_1)r(\Psi_2)}$$

Proof.

$$\begin{aligned} \text{Consider} \quad r(\Upsilon + \Psi) &\leq \|\Upsilon + \Psi\| \leq \|\Upsilon\| + \|\Psi\| \qquad \text{(by 2)} \\ &= \|\Upsilon_1 + i\Upsilon_2\| + \|\Psi_1 + i\Psi_2\| \\ &\leq \|\Upsilon_1 + \Upsilon_2\| + \|\Psi_1 + \Psi_2\| \qquad \text{(by Lemma 2.1)} \\ &\leq max(\|\Upsilon_1\|, \|\Upsilon_2\|) + \|\Upsilon_1^{\frac{1}{2}}\Upsilon_2^{\frac{1}{2}}\| + max(\|\Psi_1\|, \|\Psi_2\|) + \|\Psi_1^{\frac{1}{2}}\Psi_2^{\frac{1}{2}}\|. \end{aligned}$$

Thus the result is obvious. □

Corollary 3.3. *If $\Upsilon, \Psi \in \mathcal{M}_n(\mathbb{C})$ are accretive-dissipative with Cartesian decomposition $\Upsilon = \Upsilon_1 + i\Upsilon_2$ and $\Psi = \Psi_1 + i\Psi_2$, and if $\Upsilon_1\Upsilon_2 = 0$ and $\Psi_1\Psi_2 = 0$, then*

$$r(\Upsilon + \Psi) \leq max(r(\Upsilon_1), r(\Upsilon_2)) + max(r(\Psi_1), r(\Psi_2)).$$

Proof. The result follows using Theorem 3.2 and Lemma 2.4, noticing that when $\Upsilon_1\Upsilon_2 = 0$ and $\Psi_1\Psi_2 = 0$, then $\|\Upsilon_1 - \Upsilon_2\| = \|\Upsilon_1 + \Upsilon_2\| = max(\|\Upsilon_1\|, \|\Upsilon_2\|)$ □

Theorem 3.4. *Let $\Upsilon, \Psi \in \mathcal{M}_n(\mathbb{C})$ be accretive-dissipative with Cartesian decomposition $\Upsilon = \Upsilon_1 + i\Upsilon_2$ and $\Psi = \Psi_1 + i\Psi_2$. Then*

$$r(\Upsilon - \Psi) \leq max(r(\Upsilon_1), r(\Psi_1)) + max(r(\Upsilon_2), r(\Psi_2)).$$

Proof.

$$\begin{aligned} \text{Consider} \quad r(\Upsilon - \Psi) &\leq \|\Upsilon - \Psi\| = \|(\Upsilon_1 - \Psi_1) + i(\Upsilon_2 - \Psi_2)\| \\ &\leq \|\Upsilon_1 - \Psi_1\| + \|\Upsilon_2 - \Psi_2\| \\ &\leq max(\|\Upsilon_1\|, \|\Psi_1\|) + max(\|\Upsilon_2\|, \|\Psi_2\|) \qquad \text{(by Lemma 2.3).} \end{aligned}$$

Thus the result follows using (3). □

Theorem 3.5. *Let $\Upsilon, \Psi \in \mathcal{M}_n(\mathbb{C})$ be accretive-dissipative with Cartesian decomposition $\Upsilon = \Upsilon_1 + i\Upsilon_2$ and $\Psi = \Psi_1 + i\Psi_2$. Then*

$$r(\Upsilon\Psi - \Psi\Upsilon) \leq \frac{1}{2}(r(\Upsilon_1) + r(\Upsilon_2))(r(\Psi_1) + r(\Psi_2)).$$

Proof.

$$\begin{aligned} \text{Consider} \quad \Upsilon\Psi - \Psi\Upsilon &= (\Upsilon_1 + i\Upsilon_2)(\Psi_1 + i\Psi_2) - (\Psi_1 + i\Psi_2)(\Upsilon_1 + i\Upsilon_2) \\ &= (\Upsilon_1\Psi_1 - \Psi_1\Upsilon_1) + (\Psi_2\Upsilon_2 - \Upsilon_2\Psi_2) + i(\Upsilon_2\Psi_1 - \Psi_1\Upsilon_2) + i(\Upsilon_1\Psi_2 - \Psi_2\Upsilon_1). \end{aligned}$$

$$\begin{aligned} \text{Now} \quad r(\Upsilon\Psi - \Psi\Upsilon) &\leq \|\Upsilon\Psi - \Psi\Upsilon\| \\ &\leq \|\Upsilon_1\Psi_1 - \Psi_1\Upsilon_1\| + \|\Psi_2\Upsilon_2 - \Upsilon_2\Psi_2\| + \|\Upsilon_2\Psi_1 - \Psi_1\Upsilon_2\| + \|\Upsilon_1\Psi_2 - \Psi_2\Upsilon_1\| \\ &\leq \frac{1}{2}\|\Upsilon_1\|\|\Psi_1\| + \frac{1}{2}\|\Psi_2\|\|\Upsilon_2\| + \frac{1}{2}\|\Upsilon_2\|\|\Psi_1\| + \frac{1}{2}\|\Upsilon_1\|\|\Psi_2\| \qquad \text{(by Lemma 2.5)} \\ &= \frac{1}{2}(\|\Psi_1\| + \|\Psi_2\|)\|\Upsilon_1\| + \frac{1}{2}(\|\Psi_1\| + \|\Psi_2\|)\|\Upsilon_2\| \\ &= \frac{1}{2}(\|\Psi_1\| + \|\Psi_2\|)(\|\Upsilon_1\| + \|\Upsilon_2\|). \end{aligned}$$

□

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Theorem 3.6. *Let $\Upsilon \in \mathcal{M}_n(\mathbb{C})$ be accretive-dissipative with Cartesian decomposition $\Upsilon = \Upsilon_1 + i\Upsilon_2$. Then*

$$r(\Upsilon\Upsilon^* - \Upsilon^*\Upsilon) \leq r(\Upsilon_1)r(\Upsilon_2)$$

Proof. Consider $\Upsilon^* = \Upsilon_1 - i\Upsilon_2$, then $\Upsilon\Upsilon^* - \Upsilon^*\Upsilon = 2i\Upsilon_2\Upsilon_1 - 2i\Upsilon_1\Upsilon_2$.

$$\begin{aligned} \text{Now} \quad r(\Upsilon\Upsilon^* - \Upsilon^*\Upsilon) &= r(2i\Upsilon_2\Upsilon_1 - 2i\Upsilon_1\Upsilon_2) \\ &\leq \|2i\Upsilon_2\Upsilon_1 - 2i\Upsilon_1\Upsilon_2\| \qquad \text{(by 2)} \\ &= 2\|\Upsilon_1\Upsilon_2 - \Upsilon_2\Upsilon_1\| \\ &\leq 2(\frac{1}{2})\|\Upsilon_1\|\|\Upsilon_2\| \qquad \text{(by Lemma 2.5)} \\ &= r(\Upsilon_1)r(\Upsilon_2). \end{aligned}$$

□

Example Let $\Upsilon = \begin{pmatrix} 1+i & -1+i \\ -1+i & 1+i \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} i.$

Here, $\Upsilon_1 = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$, $\Upsilon_2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, and $\Upsilon^* = \begin{pmatrix} 1-i & -1-i \\ -1-i & 1-i \end{pmatrix}$. Then

$$r(\Upsilon\Upsilon^* - \Upsilon^*\Upsilon) \leq r\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} r\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Now, since $\sigma(\Upsilon_1) = \{0, 2\}$, then $r(\Upsilon_1) = 2$. Also, $\sigma(\Upsilon_2) = \{0, 2\}$, then $r(\Upsilon_2) = 2$. Thus,

$$r(\Upsilon\Upsilon^* - \Upsilon^*\Upsilon) \leq (2)(2) = 4.$$

4 Conclusion

In this paper, some results related to spectral radius for special kind of matrices which are accretive-dissipative were given with their proofs. A particular example was presented to illustrate the results. The importance of our results is that it makes it easy to find bounds for sums, differences, and commutators of these matrices using only the spectral radius of the real and imaginary parts since these matrices are positive semidefinite and their spectrum is positive.
For future works, one may try to find the spectral radius for 2×2 block matrices involving accretive-dissipative matrices.

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