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## Solutions behavior of mechanical oscillator equations with impulsive effects under Power Caputo fractional operator and its symmetric cases

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This study investigates the qualitative behavior of mechanical oscillator equations with impulsive effects using the generalized Power Caputo fractional operator, encompassing several known derivatives (such as Caputo–Fabrizio and Atangana–Baleanu) as special cases. The operator’s parameter ‘ $p$ ’ offers enhanced flexibility in modeling memory effects. Key findings, derived through integral equation formulations and fixed-point theory (including Banach’s contraction principle), establish rigorous conditions for the existence, uniqueness, and Ulam–Hyers stability of solutions under specific assumptions on the nonlinear and impulsive terms. The numerical scheme is developed by the Lagrange interpolation polynomial to obtain approximate solutions, and the analysis of symmetric cases connects the model to established fractional derivatives. This work offers a rigorous mathematical framework and numerical tools for analyzing systems, like mechanical oscillators, that exhibit both fractional-order memory and impulsive behavior, significantly enhancing modeling accuracy in engineering and control systems.

**Keywords** Power Caputo fractional operator, Impulsive solutions behavior, Fixed point theorem, Symmetric solutions, Simulation

Fractional calculus has emerged as a powerful mathematical tool in recent years, offering significant advantages over traditional integer-order calculus for modeling and analyzing complex real-world phenomena<sup>1–6</sup>. By extending the principles of calculus to non-integer orders, fractional calculus provides a more nuanced framework for representing physical systems characterized by anomalous diffusion, long-term memory, and fractional-order dynamics. The versatility of fractional calculus has allowed scientists to deepen their understanding of complex systems, improve predictive capabilities, and improve design, control, and optimization strategies<sup>7–11</sup>.

Among the various fractional operators, the power Caputo fractional derivative, recently introduced by Ref.<sup>12</sup>, represents a novel and more general framework for fractional operators featuring non-singular kernels. This framework encompasses well-established concepts, each offering distinct advantages for specific physical scenarios. For instance, the Caputo–Fabrizio (CF) derivative<sup>13</sup>, known for its non-singular exponential kernel, is valuable for modeling systems with fading memory that decays exponentially, like certain heat transfer processes. The Atangana–Baleanu (AB) derivative<sup>14</sup>, employing a non-singular Mittag-Leffler kernel, excels at capturing more complex memory effects, including crossover behaviors observed in viscoelasticity and anomalous diffusion. Weighted versions, such as the weighted Atangana–Baleanu<sup>15</sup> and the generalized weighted Hattaf derivatives<sup>16</sup>, incorporate weighting functions, allowing for the modeling of heterogeneous systems where memory properties vary spatially or temporally. The power Caputo operator, grounded in a generalized Mittag-

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Leffler function, not only unifies these diverse operators but also introduces a parameter ‘ $p$ ’, offering enhanced flexibility in adjusting the memory decay characteristics to precisely match the system under study. This adaptability is crucial for developing accurate mathematical models that can better predict and explain complex behaviors, with applications ranging from disease modeling (impulses as vaccinations) to control systems (impulses as sudden control actions). While ‘ $p$ ’ adds complexity, it significantly refines the modeling capabilities.

Impulsive fractional differential equations are a powerful tool for modeling systems that exhibit both continuous evolution and sudden, abrupt changes<sup>17–19</sup>. These equations are essential in capturing real-world phenomena where smooth transitions are interrupted by external shocks or impulses. For instance, in pharmacology, they can model the sudden spike in drug concentration following a pill’s ingestion, while in engineering, they can represent systems subjected to sudden forces or failures. By incorporating memory and hereditary properties, fractional differential equations already provide a nuanced view of system behavior, and the addition of impulses allows for an even more accurate representation of systems with sudden changes.

However, the study of impulsive behavior in fractional differential equations remains a challenging area due to the inherent complexities of non-integer order derivatives and discontinuities. These factors complicate the analysis, solution methods, and the establishment of the existence, uniqueness, and stability of the solution. Previous research has made significant strides to address these challenges<sup>20–25</sup>. Despite these advancements, there remains a critical gap in the literature: the complexities associated with impulsive behavior in FDEs involving the generalized power fractional derivative have not yet been addressed.

This study aims to bridge this gap by exploring the criteria for the existence, uniqueness and stability of solutions for the following impulsive FDEs using the power fractional derivative

$$\begin{cases} {}_a^{\mathbb{P}-\mathbb{C}}D_{[\eta],w}^{\sigma,\mu,p}\mathbb{V}(\eta) = \mathbb{Y}\left(\eta, \mathbb{V}(\eta), {}_a^{\mathbb{P}-\mathbb{C}}D_{[\eta],w}^{\sigma,\mu,p}\mathbb{V}(\eta)\right), \eta \in \mathcal{I}_c = \mathcal{I} - \{\eta_1, \eta_2, \dots, \eta_m\}, \mathcal{I} := [a, T] \\ \Delta\mathbb{V}(\eta_k) = \mathbb{V}(\eta_k^+) - \mathbb{V}(\eta_k^-) = \mathbb{A}_k\mathbb{V}(\eta_k), k = 1, \dots, m, \\ \mathbb{V}(a) = \mathbb{V}_a, \end{cases} \quad (1.1)$$

where

- ${}_a^{\mathbb{P}-\mathbb{C}}D_{[\eta],w}^{\sigma,\mu,p}$  is a Power-caputo fractional derivative of order  $\sigma \in (0, 1)$ , with the power  $p$  and  $\min(\mu, p) > 0$ , w.r.t nondecreasing function  $w(\eta)$ .
- $[\eta] = \eta_k$  if  $\eta \in (\eta_k, \eta_{k+1}]$ ,  $k = 0, 1, \dots, m, \eta_0 = a$ .
- $\mathbb{Y} : I \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  is a nonlinear continuous function satisfies some conditions described later in the hypotheses. This function represents the restoring force (e.g., nonlinear spring), damping force, and external forcing.
- $\mathbb{A}_k : \mathbb{R} \rightarrow \mathbb{R}$ ,  $k = 1, 2, \dots, m$ , are functions satisfying conditions described later in the hypotheses.
- $\Delta\mathbb{V}(\eta_k)$  represents impulsive effects at points  $\eta_k$  such that  $a = \eta_0 < \eta_1 < \dots < \eta_k < \eta_{k+1} = T$ , and  $\Delta\mathbb{V}(\eta_k) = \mathbb{V}(\eta_k^+) - \mathbb{V}(\eta_k^-) = \mathbb{V}(\eta_k^+) - \mathbb{V}(\eta_k)$ ,  $\mathbb{V}(\eta_k^+) = \lim_{h \rightarrow 0^+} \mathbb{V}(\eta_k + h)$ ,  $\mathbb{V}(\eta_k^-) = \lim_{h \rightarrow 0^-} \mathbb{V}(\eta_k + h)$  represent the right and left limits of  $\mathbb{V}(\eta)$  at  $\eta \in (\eta_k, \eta_{k+1}]$ ,  $k = 0, 1, \dots, m$ .

This model describes a mechanical oscillator subject to fractional damping and impulsive effects. The displacement  $\mathbb{V}(\eta)$  is governed by a fractional-order differential equation involving the Power-Caputo derivative, which accounts for the system’s memory effects. The external forces acting on the system are represented by the nonlinear function  $\mathbb{Y}$ , which includes both exponentially decaying and sinusoidal components, modulated by the system’s displacement. Impulsive effects at specific times  $\eta_k, k = 0, 1, 2, \dots, m$  introduce sudden changes in displacement, proportional to the current displacement and scaled by exponential functions of time.

A key feature of model (1.1) is its generality. By selecting specific values for the parameters  $\sigma, \mu, p$ , and the weighting function  $w(\eta)$ , the power Caputo operator reduces to several well-known fractional derivatives, making (1.1) a unifying framework. These symmetric cases include:

- If  $p = e$ , the model (1.1) reduced to the following weighted generalized Hattaf fractional model.

$$\begin{cases} {}_a^{\mathbb{P}-\mathbb{C}}D_{[\eta],w}^{\sigma,\mu,e}\mathbb{V}(\eta) = \mathbb{Y}\left(\eta, \mathbb{V}(\eta), {}_a^{\mathbb{P}-\mathbb{C}}D_{[\eta],w}^{\sigma,\mu,e}\mathbb{V}(\eta)\right), \eta \in \mathcal{I}_c = \mathcal{I} - \{\eta_1, \eta_2, \dots, \eta_m\}, \mathcal{I} := [a, T] \\ \Delta\mathbb{V}(\eta_k) = \mathbb{V}(\eta_k^+) - \mathbb{V}(\eta_k^-) = \mathbb{A}_k\mathbb{V}(\eta_k), k = 1, \dots, m, \\ \mathbb{V}(a) = \mathbb{V}_a. \end{cases} \quad (1.2)$$

- If  $\mu = \sigma, p = e$  and  $w(\eta) = 1$ , the model (1.1) reduced to the following Atangana–Baleanu fractional model.

$$\begin{cases} {}_a^{\mathbb{P}-\mathbb{C}}D_{[\eta],1}^{\sigma,e}\mathbb{V}(\eta) = \mathbb{Y}\left(\eta, \mathbb{V}(\eta), {}_a^{\mathbb{P}-\mathbb{C}}D_{[\eta],1}^{\sigma,e}\mathbb{V}(\eta)\right), \eta \in \mathcal{I}_c = \mathcal{I} - \{\eta_1, \eta_2, \dots, \eta_m\}, \mathcal{I} := [a, T] \\ \Delta\mathbb{V}(\eta_k) = \mathbb{V}(\eta_k^+) - \mathbb{V}(\eta_k^-) = \mathbb{A}_k\mathbb{V}(\eta_k), k = 1, \dots, m, \\ \mathbb{V}(a) = \mathbb{V}_a. \end{cases} \quad (1.3)$$

This model is widely applied in complex systems modeling due to its Mittag–Leffler kernel. See Ref.<sup>30</sup>.

- If  $\mu = \sigma, p = e$ , the model (1.1) reduced to the following weighted Atangana–Baleanu fractional model.

$$\begin{cases} {}^{\mathbb{P}-\mathbb{C}}D_{[\eta],w}^{\sigma,e}\mathbb{V}(\eta) = \mathbb{Y}\left(\eta, \mathbb{V}(\eta), {}^{\mathbb{P}-\mathbb{C}}D_{[\eta],w}^{\sigma,e}\mathbb{V}(\eta)\right), \eta \in \mathcal{I}_c = \mathcal{I} - \{\eta_1, \eta_2, \dots, \eta_m\}, \mathcal{I} := [a, T] \\ \Delta\mathbb{V}(\eta_k) = \mathbb{V}(\eta_k^+) - \mathbb{V}(\eta_k^-) = \mathbb{A}_k\mathbb{V}(\eta_k), k = 1, \dots, m, \\ \mathbb{V}(a) = \mathbb{V}_a. \end{cases} \quad (1.4)$$

This type (weighted Atangana–Baleanu) is suitable for heterogeneous systems with AB-type memory. For more information about weighted fractional operator, we refer to<sup>31</sup>.

- If  $\mu = 1$ ,  $p = e$  and  $w(\eta) = 1$ , the model (1.1) reduced to the following Caputo–Fabrizio fractional model.

$$\begin{cases} {}^{\mathbb{P}-\mathbb{C}}D_{[\eta],1}^{\sigma,1,e}\mathbb{V}(\eta) = \mathbb{Y}\left(\eta, \mathbb{V}(\eta), {}^{\mathbb{P}-\mathbb{C}}D_{[\eta],1}^{\sigma,1,e}\mathbb{V}(\eta)\right), \eta \in \mathcal{I}_c = \mathcal{I} - \{\eta_1, \eta_2, \dots, \eta_m\}, \mathcal{I} := [a, T] \\ \Delta\mathbb{V}(\eta_k) = \mathbb{V}(\eta_k^+) - \mathbb{V}(\eta_k^-) = \mathbb{A}_k\mathbb{V}(\eta_k), k = 1, \dots, m, \\ \mathbb{V}(a) = \mathbb{V}_a. \end{cases} \quad (1.5)$$

This model is recognized for its non-singular exponential kernel, which is useful in diffusion and relaxation processes.

This study represents a significant advancement by being the first to explore impulsive behavior in FDEs using the power Caputo fractional derivative, thereby expanding the toolkit available for scientists and engineers to model complex systems more accurately. Also, the analysis of impulsive FDEs presents challenges due to the complexities of non-integer order derivatives and discontinuities, necessitating novel approaches to establish solution existence, uniqueness, and stability. By addressing this gap, the research enhances the modeling accuracy and predictive capabilities in various fields.

The remainder of this paper is structured to systematically develop and validate our approach. “Basic concepts” section establishes the foundational definitions and lemmas related to the power Caputo operator. “Primary findings” section constitutes the core theoretical contribution, where we transform the impulsive FDE into an equivalent integral equation and employ fixed-point theorems to rigorously prove the existence, uniqueness, and Ulam–Hyers stability of solutions. “Numerical scheme” section details the construction of a numerical scheme for approximating solutions, demonstrating practical applicability. “An application with simulation results” section provides illustrative examples and simulations, including explorations of the significant symmetric cases, thereby showcasing the versatility and unifying nature of the power Caputo framework for modeling impulsive systems with diverse memory characteristics. Finally, concluding remarks summarize the key findings and potential avenues for future research.

## Basic concepts

**Definition 1**<sup>12</sup> Let  $\sigma \in [0, 1)$ , with  $\min(\mu, p) > 0$ , and  $\mathbb{V} \in H^1(a, b)$ , where  $H^1(a, b)$  is Sobolev space. The power Caputo fractional derivative of order  $\sigma$ , of a function  $\mathbb{V}$  with respect to the weight function  $w$ ,  $0 < w \in C^1([a, b])$ , is defined by

$${}^{\mathbb{P}-\mathbb{C}}D_{\eta,w}^{\sigma,\mu,p}\mathbb{V}(\eta) = \frac{\mathbb{P}(\sigma)}{1-\sigma} \frac{1}{w(\eta)} \int_a^\eta {}^p\mathbb{E}_{\mu,1} \left( -\frac{\sigma}{1-\sigma} (\eta-s)^\mu \right) (w\mathbb{V})'(s) ds, \quad (2.1)$$

where

- ${}^p\mathbb{E}_{\mu,1}$  represents the PML function given by

$${}^p\mathbb{E}_{\mu,l}(s) = \sum_{n=0}^{+\infty} \frac{(s \ln p)^n}{\Gamma(kn+l)}, s \in \mathbb{C}, \text{ and } k, l, p > 0.$$

- $\mathbb{P}(\sigma)$  represents a normalization positive function obeying  $\mathbb{P}(0) = \mathbb{P}(1) = 1$ .

According to Theorem 1 of<sup>12</sup>, the PML function  ${}^p\mathbb{E}_{\mu,l}(s)$  is locally uniformly convergent for any  $s \in \mathbb{C}$ . This definition provides a generalized way to measure the rate of change of a function  $\mathbb{V}$  considering its past history, weighted by the function  $w$  and characterized by the parameters  $\sigma, \mu, p$ . The power Mittag–Leffler (PML) function  ${}^p\mathbb{E}_{\mu,1}$  acts as a non-singular kernel, enabling the modeling of complex memory effects, while the weighting function  $w$  allows for incorporating system heterogeneities. The parameter  $p$  adds further flexibility in tuning the memory characteristics. Its importance lies in unifying various fractional operators and offering enhanced modeling capabilities for complex systems.

**Remark 1** The power Caputo fractional derivative, as presented in Definition 1, serves as a generalization of numerous fractional derivatives found in the literature, as follows:

- (1) If  $w(\eta) = 1$ ,  $p = e$ ,  $\sigma = \mu$ . The, Definition 1 reduce to the Atangana–Baleanu fractional derivative given by

$${}^{\mathbb{P}-\mathbb{C}}D_{\eta,1}^{\sigma,\sigma,e}\mathbb{V}(\eta) = \frac{\mathbb{P}(\sigma)}{1-\sigma} \int_a^\eta {}^e\mathbb{E}_{\sigma,1} \left( -\frac{\sigma}{1-\sigma} (\eta-s)^\sigma \right) \mathbb{V}'(s) ds.$$

(2) If  $w(\eta) = 1, p = e, \mu = 1$ . The, Definition 1 reduce to the Caputo–Fabrizio fractional derivative given by

$${}^{\mathbb{P}\text{-C}}D_{\eta,1}^{\sigma,1,e}\mathbb{V}(\eta) = \frac{\mathbb{P}(\sigma)}{1-\sigma} \int_a^\eta \exp\left(-\frac{\sigma}{1-\sigma}(\eta-s)\right) \mathbb{V}'(s) ds.$$

(3) If  $p = e, \sigma = \mu$ . The, Definition 1 reduce to the weighted Atangana–Baleanu fractional derivative given by

$${}^{\mathbb{P}\text{-C}}D_{\eta,w}^{\sigma,\sigma,e}\mathbb{V}(\eta) = \frac{\mathbb{P}(\sigma)}{1-\sigma} \frac{1}{w(\eta)} \int_a^\eta {}^e\mathbb{E}_{\sigma,1}\left(-\frac{\sigma}{1-\sigma}(\eta-s)\right) (w\mathbb{V})'(s) ds.$$

(4) If  $p = e$ . The, Definition 1 reduce to the weighted generalized Hattaf fractional derivative given by

$${}^{\mathbb{P}\text{-C}}D_{\eta,w}^{\sigma,\mu,e}\mathbb{V}(\eta) = \frac{\mathbb{P}(\sigma)}{1-\sigma} \frac{1}{w(\eta)} \int_a^\eta {}^e\mathbb{E}_{\mu,1}\left(-\frac{\sigma}{1-\sigma}(\eta-s)^\mu\right) (w\mathbb{V})'(s) ds.$$

This Remark highlights a crucial aspect of the power Caputo fractional derivative: its ability to encapsulate several established fractional operators as specific instances.

This unification is significant because it positions the power Caputo operator as a comprehensive tool. Results obtained for this general operator can potentially be specialized to these well-known cases, streamlining analysis and allowing for comparisons across different fractional modeling approaches. It underscores the operator’s flexibility and its foundation in existing fractional calculus concepts.

**Definition 2** <sup>12</sup> Let  $\sigma \in [0, 1)$ , with  $\min(\mu, p) > 0$ . The power caputo fractional integral of order  $\sigma$ , of a function  $\mathbb{V}$  with respect to the weight function  $w, 0 < w \in C^1([a, b])$ , is defined by

$${}^{\mathbb{P}\text{-C}}I_{\eta,w}^{\sigma,\mu,p}\mathbb{V}(\eta) = \frac{1-\sigma}{\mathbb{P}(\sigma)} \mathbb{V}(\eta) + \ln p \frac{\sigma}{\mathbb{P}(\sigma)} {}^{\text{RL}}I_{a,w}^\mu \mathbb{V}(\eta).$$

•  ${}^{\text{RL}}I_{a,w}^\mu \mathbb{V}(\eta)$  denotes the standard weighted Riemann–Liouville fractional integral of order  $\mu$  given by

$${}^{\text{RL}}I_{a,w}^\mu \mathbb{V}(\eta) = \frac{1}{\Gamma(\mu)} \frac{1}{w(\eta)} \int_a^\eta (\eta-s)^{\mu-1} (w\mathbb{V})(s) ds.$$

This definition is fundamental for reformulating fractional differential equations into equivalent Volterra-type integral equations. This transformation is a cornerstone of the analytical methods used later in this paper, particularly for applying fixed-point theorems.

**Theorem 1** (Theorem 1<sup>26</sup>) Let  $\sigma \in [0, 1)$ , with  $\mu, p > 0$ , and  $\mathbb{V} \in H^1(a, b)$ . Then, the PFD and PFI are commutative operators as follows:

- (i)  $\frac{{}^{\mathbb{P}\text{-C}}D_{\eta,w}^{\sigma,\mu,p} \left( \frac{{}^{\mathbb{P}\text{-C}}I_{\eta,w}^{\sigma,\mu,p}\mathbb{V}(\eta)}{w(\eta)} \right)}{{}^{\mathbb{P}\text{-C}}I_{\eta,w}^{\sigma,\mu,p} \left( \frac{{}^{\mathbb{P}\text{-C}}D_{\eta,w}^{\sigma,\mu,p}\mathbb{V}(\eta)}{w(\eta)} \right)} (\eta) = \mathbb{V}(\eta) - \frac{w\mathbb{V}(a)}{w(\eta)}$ ;
- (ii)  $\frac{{}^{\mathbb{P}\text{-C}}I_{\eta,w}^{\sigma,\mu,p} \left( \frac{{}^{\mathbb{P}\text{-C}}D_{\eta,w}^{\sigma,\mu,p}\mathbb{V}(\eta)}{w(\eta)} \right)}{{}^{\mathbb{P}\text{-C}}D_{\eta,w}^{\sigma,\mu,p} \left( \frac{{}^{\mathbb{P}\text{-C}}I_{\eta,w}^{\sigma,\mu,p}\mathbb{V}(\eta)}{w(\eta)} \right)} (\eta) = \mathbb{V}(\eta) - \frac{w\mathbb{V}(a)}{w(\eta)}$ ;
- (iii)  $\frac{{}^{\mathbb{P}\text{-C}}D_{\eta,1}^{\sigma,\mu,p}\mathbb{V}(\eta)}{{}^{\mathbb{P}\text{-C}}I_{\eta,1}^{\sigma,\mu,p}\mathbb{V}(\eta)} = 0$ , for all constant function  $\mathbb{V}(\eta)$ .

These properties establish the fundamental relationship between the power Caputo derivative and its corresponding integral and are essential for analytical manipulations.

If we put  $p = e$ , then we obtain the results of generalized Hattaf fractional operators<sup>27</sup>.

**Lemma 2.1** The PFD and PFI satisfy the Newton–Leibniz formula

$${}^{\mathbb{P}\text{-C}}D_{\eta,w}^{\sigma,\mu,p} \left( \frac{{}^{\mathbb{P}\text{-C}}I_{\eta,w}^{\sigma,\mu,p}\mathbb{V}(\eta)}{w(\eta)} \right) (\eta) = \frac{{}^{\mathbb{P}\text{-C}}I_{\eta,w}^{\sigma,\mu,p} \left( \frac{{}^{\mathbb{P}\text{-C}}D_{\eta,w}^{\sigma,\mu,p}\mathbb{V}(\eta)}{w(\eta)} \right) (\eta)}{w(\eta)} = \mathbb{V}(\eta) - \mathbb{V}(a).$$

This lemma provides the power Caputo analogue of the Newton–Leibniz formula, directly linking the function’s values at the endpoints via its fractional derivative and integral, which is crucial for solving initial value problems.

**Lemma 2.2** <sup>26</sup> Let  $\mathbb{K} : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$  be a continuous nonlinear function such that  $\mathbb{K}(a, \mathbb{V}(a)) = 0$ . Then, the function  $\mathbb{V} \in C([a, b])$  is a solution of the following problem

$$\begin{cases} \frac{{}^{\mathbb{P}\text{-C}}D_{\eta,w}^{\sigma,\mu,p}\mathbb{V}(\eta)}{w(\eta)} = \mathbb{K}(\eta, \mathbb{V}(\eta)), \\ \mathbb{V}(a) = \mathbb{V}_a \in \mathbb{R}, \end{cases}$$

if and only if  $\mathbb{V}$  satisfies the following integral equation

$$\mathbb{V}(\eta) = \frac{w(a)}{w(\eta)} \mathbb{V}_a + {}^{\mathbb{P}-\mathbb{C}}\mathbb{I}_{\eta,w}^{\sigma,\mu,p} \mathbb{K}(\eta, \mathbb{V}(\eta)).$$

This lemma establishes the equivalence between the initial value problem involving the power Caputo derivative and a Volterra integral equation. This equivalence is the foundation for applying fixed-point theorems to prove the existence and uniqueness of solutions, transforming the differential problem into finding a fixed point of an integral operator.

**Definition 3** <sup>28</sup> Assume  $\mathbb{G} : \mathcal{N} \subset \mathbb{Q} \rightarrow \mathbb{Q}$  is a continuous and bounded operator. Then, if

$$\|\mathbb{G}(u) - \mathbb{G}(\widehat{u})\| \leq \Upsilon \|u - \widehat{u}\|, \Upsilon > 0,$$

for all  $u, \widehat{u} \in \mathcal{N}$ . Then, the operator  $\mathbb{G}$  is Lipschitz with a specific constant  $\Upsilon$ . Furthermore, if  $\Upsilon < 1$ , the operator  $\mathbb{G}$  is classified as a strict contraction.

These concepts are crucial for applying fixed-point theorems. Lipschitz continuity provides a bound on how much the operator stretches distances between points, while a strict contraction (Lipschitz constant) ensures the operator shrinks distances, guaranteeing a unique fixed point via the Banach Contraction Principle.

**Hypothesis**

In our analysis of existence, uniqueness, and stability, we adopt the following assumptions.

(H<sub>1</sub>) The function  $\mathbb{Y} : I \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  is continuous, and there exist positive constants  $\varrho_f$  and  $\varrho'_f$ , such that for each  $\mathbb{V}, \widehat{\mathbb{V}} \in \mathbb{R}$ , the following inequality holds:

$$\left| \mathbb{Y}(\eta, \mathbb{V}(\eta), \vartheta(\eta)) - \mathbb{Y}(\eta, \widehat{\mathbb{V}}(\eta), \widehat{\vartheta}(\eta)) \right| \leq \varrho_{\mathbb{Y}} \left| \mathbb{V}(\eta) - \widehat{\mathbb{V}}(\eta) \right| + \varrho'_{\mathbb{Y}} \left| \vartheta(\eta) - \widehat{\vartheta}(\eta) \right|.$$

(H<sub>2</sub>) For  $k = 0, 1, \dots, m$ , the functions  $\mathbb{A}_k : \mathbb{R} \rightarrow \mathbb{R}$  are continuous, and there exist a positive constant  $L_{\mathbb{A}}$ , such that

$$|\mathbb{A}_k(\Theta) - \mathbb{A}_k(\Theta^*)| \leq L_{\mathbb{A}} |\Theta - \Theta^*|, k = 1, \dots, m, \Theta, \Theta^* \in \mathbb{R}.$$

**Primary findings**

In this section, we present the key achievements of our research, highlighting the significant outcomes and discoveries that have been made. These findings are central to our study and provide a comprehensive understanding of the topic at hand. To analyze the impulsive fractional differential equation (1.1), we convert it to equivalent integral equations. This conversion is advantageous because integral operators often behave better than differential operators, especially when dealing with discontinuities introduced by impulses, and it directly sets the stage for applying fixed-point theorems. The following theorem formally establishes this equivalence.

**Equivalent integral formulation of problem (1.1)**

In this subsection, we convert the problem (1.1) into an equivalent integral formulation. This conversion offers a versatile and powerful approach to solving complex problems, providing advantages in solvability, boundary condition handling, numerical methods, physical interpretation, and theoretical analysis. Define the piecewise space  $\mathcal{PC}(I)$  as follows

$$\mathcal{PC}(I) = \{ \mathbb{V} : I \rightarrow \mathbb{R}, \mathbb{V} \in \mathcal{C}(I, \mathbb{R}) \text{ and } \mathbb{V}(\eta_k^+), \mathbb{V}(\eta_k^-) \text{ exist so that } \Delta \mathbb{V}(\eta_k) = \mathbb{V}(\eta_k^+) - \mathbb{V}(\eta_k^-), k = 1, \dots, m \},$$

with the norm

$$\|\mathbb{V}\|_{\mathcal{PC}} = \max_{\eta \in I} |\mathbb{V}(\eta)|.$$

**Theorem 2** Let  $\sigma \in (0, 1)$ ,  $\min(\mu, p) > 0$ , and  $\varphi : I' \rightarrow \mathbb{R}$  be a continuous functions. Then  $\mathbb{V} \in \mathcal{PC}(I)$  satisfies

$$\begin{cases} {}^{\mathbb{P}-\mathbb{C}}\mathbb{D}_{[\eta],w}^{\sigma,\mu,p} \mathbb{V}(\eta) = \varphi(\eta), \eta \in I := [a, T], \eta \neq \eta_k, k = 1, \dots, m, \\ \Delta \mathbb{V}|_{\eta=\eta_k} = \mathbb{A}_k \mathbb{V}(\eta_k^-), k = 1, \dots, m \\ \mathbb{V}(a) = \mathbb{V}_a. \end{cases} \tag{3.1}$$

if and only if  $\mathbb{V}$  satisfy the following integral equations

$$\mathbb{V}(\eta) = \begin{cases} \frac{w(a)}{w(\eta)} \mathbb{V}_a + {}^{\mathbb{P}-\mathbb{C}}\mathbb{I}_{a,w}^{\sigma,\mu,p} \varphi(s)(\eta), \eta \in [a, \eta_1], \\ \frac{w(a)}{w(\eta)} \mathbb{V}_a + \sum_{i=1}^k \frac{w(\eta_i)}{w(\eta)} {}^{\mathbb{P}-\mathbb{C}}\mathbb{I}_{\eta_{i-1},w}^{\sigma,\mu,p} \varphi(s)(\eta_i) \\ + {}^{\mathbb{P}-\mathbb{C}}\mathbb{I}_{\eta_k,w}^{\sigma,\mu,p} \varphi(s)(\eta) + \sum_{i=1}^k \frac{w(\eta_i)}{w(\eta)} \mathbb{A}_i \mathbb{V}(\eta_i^-), \eta \in (\eta_k, \eta_{k+1}]. \end{cases} \tag{3.2}$$

**Proof** First, let  $\mathbb{V} \in \mathcal{PC}^\gamma(I)$  be a solution of model (3.1). We prove that  $\mathbb{V}$  is a solution of (3.2). If  $\eta \in [a, \eta_1]$ , then  ${}^{\mathbb{P}-\mathbb{C}}D_{[\eta],w}^{\sigma,\mu,p}\mathbb{V}(\eta) = \varphi(\eta)$ ,  $[\eta] = a$ . Taking the operator  ${}^{\mathbb{P}-\mathbb{C}}I_{\eta,w}^{\sigma,\mu,p}$  on both sides of first equation (3.1) with using Definition 2 and Theorem 1, we have

$$\mathbb{V}(\eta) = \frac{w(a)}{w(\eta)}\mathbb{V}(a) + {}^{\mathbb{P}-\mathbb{C}}I_{a,w}^{\sigma,\mu,p}\varphi(s)(\eta). \tag{3.3}$$

By condition  $\mathbb{V}(a) = \mathbb{V}_a$ , we have

$$\mathbb{V}(\eta) = \frac{w(a)}{w(\eta)}\mathbb{V}_a + {}^{\mathbb{P}-\mathbb{C}}I_{a,w}^{\sigma,\mu,p}\varphi(s)(\eta).$$

This means that

$$\mathbb{V}(\eta_1^-) = \frac{w(a)}{w(\eta_1)}\mathbb{V}_a + {}^{\mathbb{P}-\mathbb{C}}I_{\eta,w}^{\sigma,\mu,p}\varphi(s)(\eta_1).$$

By  $(\mathbb{V}(\eta_1^-) = \mathbb{V}(\eta_1^+) - \mathbb{A}_1\mathbb{V}(\eta_1^-))$ , we get

$$\mathbb{V}(\eta_1^+) = \frac{w(a)}{w(\eta_1)}\mathbb{V}_a + {}^{\mathbb{P}-\mathbb{C}}I_{\eta,w}^{\sigma,\mu,p}\varphi(s)(\eta_1) + \mathbb{A}_1\mathbb{V}(\eta_1^-). \tag{3.4}$$

If  $\eta \in (\eta_1, \eta_2]$ , then  ${}^{\mathbb{P}-\mathbb{C}}D_{[\eta],w}^{\sigma,\mu,p}\mathbb{V}(\eta) = \varphi(\eta)$ ,  $[\eta] = \eta_1$ . Thus, by taking the operator  ${}^{\mathbb{P}-\mathbb{C}}I_{\eta_1,w}^{\sigma,\mu,p}$  on both sides, we get

$$\mathbb{V}(\eta) = \frac{w(\eta_1)}{w(\eta)}\mathbb{V}(\eta_1) + {}^{\mathbb{P}-\mathbb{C}}I_{\eta_1,w}^{\sigma,\mu,p}\varphi(s)(\eta). \tag{3.5}$$

Thus, by (3.4) and (3.5), we have

$$\begin{aligned} \mathbb{V}(\eta) &= \frac{w(\eta_1)}{w(\eta)}\left(\frac{w(a)}{w(\eta_1)}\mathbb{V}_a + {}^{\mathbb{P}-\mathbb{C}}I_{\eta,w}^{\sigma,\mu,p}\varphi(s)(\eta_1) + \mathbb{A}_1\mathbb{V}(\eta_1^-)\right) + {}^{\mathbb{P}-\mathbb{C}}I_{\eta,w}^{\sigma,\mu,p}\varphi(s)(\eta) \\ &= \frac{w(a)}{w(\eta)}\mathbb{V}_a + \frac{w(\eta_1)}{w(\eta)}{}^{\mathbb{P}-\mathbb{C}}I_{\eta,w}^{\sigma,\mu,p}\varphi(s)(\eta_1) + \frac{w(\eta_1)}{w(\eta)}\mathbb{A}_1\mathbb{V}(\eta_1^-) + {}^{\mathbb{P}-\mathbb{C}}I_{\eta,w}^{\sigma,\mu,p}\varphi(s)(\eta). \end{aligned}$$

This means that

$$\mathbb{V}(\eta_2^-) = \frac{w(a)}{w(\eta_2)}\mathbb{V}_a + \frac{w(\eta_1)}{w(\eta_2)}{}^{\mathbb{P}-\mathbb{C}}I_{\eta,w}^{\sigma,\mu,p}\varphi(s)(\eta_1) + \frac{w(\eta_1)}{w(\eta_2)}\mathbb{A}_1\mathbb{V}(\eta_1^-) + {}^{\mathbb{P}-\mathbb{C}}I_{\eta_1,w}^{\sigma,\mu,p}\varphi(s)(\eta_2).$$

By condition  $(\mathbb{V}(\eta_2^-) = \mathbb{V}(\eta_2^+) - \mathbb{A}_2\mathbb{V}(\eta_2^-))$ , we get

$$\begin{aligned} \mathbb{V}(\eta_2^+) &= \frac{w(a)}{w(\eta_2)}\mathbb{V}_a + \frac{w(\eta_1)}{w(\eta_2)}{}^{\mathbb{P}-\mathbb{C}}I_{\eta,w}^{\sigma,\mu,p}\varphi(s)(\eta_1) + {}^{\mathbb{P}-\mathbb{C}}I_{\eta_1,w}^{\sigma,\mu,p}\varphi(s)(\eta_2) \\ &\quad + \frac{w(\eta_1)}{w(\eta_2)}\mathbb{A}_1\mathbb{V}(\eta_1^-) + \mathbb{A}_2\mathbb{V}(\eta_2^-). \end{aligned} \tag{3.6}$$

If  $\eta \in (\eta_2, \eta_3]$ , then  ${}^{\mathbb{P}-\mathbb{C}}D_{[\eta],w}^{\sigma,\mu,p}\mathbb{V}(\eta) = \varphi(\eta)$ ,  $[\eta] = \eta_2$ . Thus, by taking the operator  ${}^{\mathbb{P}-\mathbb{C}}I_{\eta_2,w}^{\sigma,\mu,p}$  on both sides, we get

$$\mathbb{V}(\eta) = \frac{w(a)}{w(\eta)}\mathbb{V}(\eta_2^+) + {}^{\mathbb{P}-\mathbb{C}}I_{\eta_2,w}^{\sigma,\mu,p}\varphi(s)(\eta). \tag{3.7}$$

Thus, by (3.6) and (3.7), we have

$$\begin{aligned} \mathbb{V}(\eta) &= \frac{w(\eta_2)}{w(\eta)}\left(\frac{w(a)}{w(\eta_2)}\mathbb{V}_a + \frac{w(\eta_1)}{w(\eta_2)}{}^{\mathbb{P}-\mathbb{C}}I_{\eta,w}^{\sigma,\mu,p}\varphi(s)(\eta_1) \right. \\ &\quad \left. + {}^{\mathbb{P}-\mathbb{C}}I_{\eta_1,w}^{\sigma,\mu,p}\varphi(s)(\eta_2) + \frac{w(\eta_1)}{w(\eta_2)}\mathbb{A}_1\mathbb{V}(\eta_1^-) + \mathbb{A}_2\mathbb{V}(\eta_2^-)\right) + {}^{\mathbb{P}-\mathbb{C}}I_{\eta_2,w}^{\sigma,\mu,p}\varphi(s)(\eta) \\ &= \frac{w(a)}{w(\eta)}\mathbb{V}_a + \frac{w(\eta_1)}{w(\eta)}{}^{\mathbb{P}-\mathbb{C}}I_{\eta,w}^{\sigma,\mu,p}\varphi(s)(\eta_1) + \frac{w(\eta_2)}{w(\eta)}{}^{\mathbb{P}-\mathbb{C}}I_{\eta_1,w}^{\sigma,\mu,p}\varphi(s)(\eta_2) \\ &\quad + \frac{w(\eta_1)}{w(\eta)}\mathbb{A}_1\mathbb{V}(\eta_1^-) + \frac{w(\eta_2)}{w(\eta)}\mathbb{A}_2\mathbb{V}(\eta_2^-) + {}^{\mathbb{P}-\mathbb{C}}I_{\eta_2,w}^{\sigma,\mu,p}\varphi(s)(\eta). \end{aligned}$$

This means that

$$\begin{aligned} \mathbb{V}(\eta_3^-) &= \frac{w(a)}{w(\eta_3)} \mathbb{V}_a + \frac{w(\eta_1)}{w(\eta_3)} \mathbb{I}_{\eta,w}^{\sigma,\mu,p} \varphi(s)(\eta_1) + \frac{w(\eta_2)}{w(\eta_3)} \mathbb{I}_{\eta_1,w}^{\sigma,\mu,p} \varphi(s)(\eta_2) \\ &\quad + \frac{w(\eta_1)}{w(\eta_3)} \mathbb{A}_1 \mathbb{V}(\eta_1^-) + \frac{w(\eta_2)}{w(\eta_3)} \mathbb{A}_2 \mathbb{V}(\eta_2^-) + \mathbb{I}_{\eta_2,w}^{\sigma,\mu,p} \varphi(s)(\eta_3). \end{aligned}$$

After impulse  $(\mathbb{V}(\eta_3^-) = \mathbb{V}(\eta_3^+) - \mathbb{A}_3 \mathbb{V}(\eta_3^-))$ , we get

$$\begin{aligned} \mathbb{V}(\eta_3) &= \frac{w(a)}{w(\eta_3)} \mathbb{V}_a + \frac{w(\eta_1)}{w(\eta_3)} \mathbb{I}_{\eta,w}^{\sigma,\mu,p} \varphi(s)(\eta_1) + \frac{w(\eta_2)}{w(\eta_3)} \mathbb{I}_{\eta_1,w}^{\sigma,\mu,p} \varphi(s)(\eta_2) \\ &\quad + \mathbb{I}_{\eta_2,w}^{\sigma,\mu,p} \varphi(s)(\eta_3) + \frac{w(\eta_1)}{w(\eta_3)} \mathbb{A}_1 \mathbb{V}(\eta_1^-) + \frac{w(\eta_2)}{w(\eta_3)} \mathbb{A}_2 \mathbb{V}(\eta_2^-) + \mathbb{A}_3 \mathbb{V}(\eta_3^-). \end{aligned} \tag{3.8}$$

Assume that

$$\begin{aligned} \mathbb{V}(\eta_k) &= \frac{w(a)}{w(\eta_k)} \mathbb{V}_a + \frac{w(\eta_1)}{w(\eta_k)} \mathbb{I}_{\eta,w}^{\sigma,\mu,p} \varphi(s)(\eta_1) + \frac{w(\eta_2)}{w(\eta_k)} \mathbb{I}_{\eta_1,w}^{\sigma,\mu,p} \varphi(s)(\eta_2) \\ &\quad + \dots + \frac{w(\eta_k)}{w(\eta_k)} \mathbb{I}_{\eta_{k-1},w}^{\sigma,\mu,p} \varphi(s)(\eta_k) + \frac{w(\eta_1)}{w(\eta_k)} \mathbb{A}_1 \mathbb{V}(\eta_1^-) \\ &\quad + \frac{w(\eta_2)}{w(\eta_k)} \mathbb{A}_2 \mathbb{V}(\eta_2^-) + \dots + \frac{w(\eta_k)}{w(\eta_k)} \mathbb{A}_k \mathbb{V}(\eta_k^-). \end{aligned} \tag{3.9}$$

Then, inductively, for  $\eta \in (\eta_k, \eta_{k+1}]$ , then  $\mathbb{I}_{[\eta,w]}^{\sigma,\mu,p} \mathbb{V}(\eta) = \varphi(\eta)$ ,  $[\eta] = \eta_k$ . Thus, by taking the operator  $\mathbb{I}_{\eta_k,w}^{\sigma,\mu,p}$  on both sides, we get

$$\mathbb{V}(\eta) = \frac{w(\eta_k)}{w(\eta)} \mathbb{V}(\eta_k) + \mathbb{I}_{\eta_k,w}^{\sigma,\mu,p} \varphi(s)(\eta). \tag{3.10}$$

Thus, by (3.9) and (3.10), we get

$$\begin{aligned} \mathbb{V}(\eta) &= \frac{w(a)}{w(\eta)} \mathbb{V}_a + \sum_{i=1}^k \frac{w(\eta_i)}{w(\eta)} \mathbb{I}_{\eta_{i-1},w}^{\sigma,\mu,p} \varphi(s)(\eta_i) + \mathbb{I}_{\eta_k,w}^{\sigma,\mu,p} \varphi(s)(\eta) \\ &\quad + \sum_{i=1}^k \frac{w(\eta_i)}{w(\eta)} \mathbb{A}_i \mathbb{V}(\eta_i^-). \end{aligned}$$

Thus (3.2) is satisfied. Conversely, assume that  $\mathbb{V}$  satisfies the Eq. (3.2). If  $\eta \in [a, \eta_1]$ . Replace  $\eta$  by  $a$  in (3.2), then, we get  $\mathbb{V}(a) = \mathbb{V}_a$ . On the other hand, applying  $\mathbb{I}_{a,w}^{\sigma,\mu,p}$  on both sides of (3.2), and using Theorem 1, we get

$$\mathbb{I}_{a,w}^{\sigma,\mu,p} \mathbb{V}(\eta) = \varphi(\eta).$$

For  $\eta \in (\eta_k, \eta_{k+1}]$ . By the same technique of case  $\eta \in [a, \eta_1]$ , we can easily prove case  $\eta \in (\eta_k, \eta_{k+1}]$ .  $\square$

**Theorem 3** Let  $\sigma \in (0, 1)$ ,  $\min(\mu, p) > 0$ , and let  $\mathbb{Y} : J \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  be a continuous function. If  $\mathbb{V} \in \mathcal{PC}(J)$  satisfies the model (1.1), then, in the light of Theorem 2,  $\mathbb{V}$  satisfies the following integral equations

$$\mathbb{V}(\eta) = \begin{cases} \frac{w(a)}{w(\eta)} \mathbb{V}_a + \mathbb{I}_{a,w}^{\sigma,\mu,p} \mathbb{Y}(s, \mathbb{V}(s), \mathbb{I}_{a,w}^{\sigma,\mu,p} \mathbb{V}(s))(\eta), \eta \in [a, \eta_1], \\ \frac{w(a)}{w(\eta)} \mathbb{V}_a + \sum_{i=1}^k \frac{w(\eta_i)}{w(\eta)} \mathbb{I}_{\eta_{i-1},w}^{\sigma,\mu,p} \mathbb{Y}(s, \mathbb{V}(s), \mathbb{I}_{\eta_{i-1},w}^{\sigma,\mu,p} \mathbb{V}(s))(\eta_i) \\ + \mathbb{I}_{\eta_k,w}^{\sigma,\mu,p} \mathbb{Y}(s, \mathbb{V}(s), \mathbb{I}_{\eta_k,w}^{\sigma,\mu,p} \mathbb{V}(s))(\eta) + \sum_{i=1}^k \frac{w(\eta_i)}{w(\eta)} \mathbb{A}_i \mathbb{V}(\eta_i^-), \eta \in (\eta_k, \eta_{k+1}]. \end{cases} \tag{3.11}$$

Consider the continuous operator  $\Xi : \mathcal{PC}(J) \rightarrow \mathcal{PC}(J)$  defined by

$$\Xi(\mathbb{V})(\eta) = \begin{cases} \frac{w(a)}{w(\eta)} \mathbb{V}_a + \mathbb{I}_{a,w}^{\sigma,\mu,p} \mathbb{Y}(s, \mathbb{V}(s), \mathbb{I}_{a,w}^{\sigma,\mu,p} \mathbb{V}(s))(\eta), \eta \in [a, \eta_1], \\ \frac{w(a)}{w(\eta)} \mathbb{V}_a + \sum_{i=1}^k \frac{w(\eta_i)}{w(\eta)} \mathbb{I}_{\eta_{i-1},w}^{\sigma,\mu,p} \mathbb{Y}(s, \mathbb{V}(s), \mathbb{I}_{\eta_{i-1},w}^{\sigma,\mu,p} \mathbb{V}(s))(\eta_i) \\ + \mathbb{I}_{\eta_k,w}^{\sigma,\mu,p} \mathbb{Y}(s, \mathbb{V}(s), \mathbb{I}_{\eta_k,w}^{\sigma,\mu,p} \mathbb{V}(s))(\eta) + \sum_{i=1}^k \frac{w(\eta_i)}{w(\eta)} \mathbb{A}_i \mathbb{V}(\eta_i^-), \eta \in (\eta_k, \eta_{k+1}]. \end{cases} \tag{3.12}$$

We noted that the fixed points of the operator  $\Xi$  is a solutions of model (1.1).

**Criteria required for existence of solution**

In this subsection, we derive sufficient conditions that ensure the existence of a solution to model (1.1). Having established the equivalent integral formulation. We will use arguments based on Schaefer’s fixed-point theorem (demonstrating operator continuity and compactness via the Arzelà–Ascoli theorem, along with boundedness) to show that such a fixed point exists under certain conditions on the system parameters and nonlinearities. See Refs. 34,35.

**Theorem 4** Assume that (H<sub>1</sub>) and (H<sub>2</sub>) hold. If  $(Q_1 \varrho_{\mathbb{Y}} + \frac{m|w(T)|}{w^*} L_{\mathbb{A}}) < 1$ , where

$$Q_1 : = \frac{(m + 1) |w(T)|}{w^* (1 - \varrho'_{\mathbb{Y}})} \left( \frac{1 - \sigma}{\mathbb{P}(\sigma)} + \ln p \frac{\sigma}{\mathbb{P}(\sigma)} \frac{(T - a)^\mu}{\Gamma(\mu + 1)} \right),$$

$$w^* = \min_{\eta \in \mathcal{I}} |w(\eta)|,$$

then the model (1.1) possess at least one solution.

**Proof** To apply fixed point approach, we define a closed ball set  $\mathbb{M}_R$  as follows

$$\mathbb{M}_R = \{ \mathbb{V} \in \mathbb{R} : \|\mathbb{V}\|_{\mathcal{PC}(J)} \leq R \},$$

with radius  $R$  such that

$$R \geq \frac{\frac{|w(a)|}{w^*} \mathbb{V}_a + Q_1 \mathcal{O}_{\mathbb{Y}}}{1 - (Q_1 \varrho_{\mathbb{Y}} + \frac{m|w(T)|}{w^*} L_{\mathbb{A}})},$$

where  $\mathcal{O}_{\mathbb{Y}} = \max_{\eta \in \mathcal{I}} |\mathbb{Y}(\eta, 0, 0)|$ . The goal is to establish that the operator  $\Xi$  as specified in (3.12), contains a fixed point. To accomplish this, the proof is organized into the following steps.

*Step 1*  $\Xi(\mathbb{M}_R) \subset \mathbb{M}_R$ . For any  $\mathbb{V} \in \mathbb{M}_R$ , we have

$$|\Xi(\mathbb{V}(\eta))| \leq \frac{|w(a)|}{|w(\eta)|} \mathbb{V}_a + \sum_{i=1}^k \frac{|w(\eta_i)|}{|w(\eta)|} \mathbb{I}_{\eta_{i-1}, w}^{\sigma, \mu, p} \left| \mathbb{Y}(s, \mathbb{V}(s), {}^{\mathbb{P}-\mathbb{C}}\mathbb{D}_{\eta_{i-1}, w}^{\sigma, \mu, p} \mathbb{V}(s)) \right| (\eta_i) + {}^{\mathbb{P}-\mathbb{C}}\mathbb{I}_{\eta_k, w}^{\sigma, \mu, p} \left| \mathbb{Y}(s, \mathbb{V}(s), {}^{\mathbb{P}-\mathbb{C}}\mathbb{D}_{\eta_k, w}^{\sigma, \mu, p} \mathbb{V}(s)) \right| (\eta) + \sum_{i=1}^k \frac{|w(\eta_i)|}{|w(\eta)|} |\mathbb{A}_i \mathbb{V}(\eta_i^-)|. \tag{3.13}$$

For  $i = 1, 2, \dots, k, k + 1$ , with the fact  ${}^{\mathbb{P}-\mathbb{C}}\mathbb{D}_{\eta_{i-1}, w}^{\sigma, \mu, p} \mathbb{V}(\eta) = \mathbb{Y}(\eta, \mathbb{V}(\eta), {}^{\mathbb{P}-\mathbb{C}}\mathbb{D}_{\eta_{i-1}, w}^{\sigma, \mu, p} \mathbb{V}(\eta))$  and by (H<sub>1</sub>), we have

$$\begin{aligned} |{}^{\mathbb{P}-\mathbb{C}}\mathbb{D}_{\eta_{i-1}, w}^{\sigma, \mu, p} \mathbb{V}(\eta)| &= |\mathbb{Y}(\eta, \mathbb{V}(\eta), {}^{\mathbb{P}-\mathbb{C}}\mathbb{D}_{\eta_{i-1}, w}^{\sigma, \mu, p} \mathbb{V}(\eta))| \\ &= |\mathbb{Y}(\eta, \mathbb{V}(\eta), {}^{\mathbb{P}-\mathbb{C}}\mathbb{D}_{\eta_{i-1}, w}^{\sigma, \mu, p} \mathbb{V}(\eta)) - \mathbb{Y}(\eta, 0, 0) + \mathbb{Y}(\eta, 0, 0)| \\ &\leq |\mathbb{Y}(\eta, \mathbb{V}(\eta), {}^{\mathbb{P}-\mathbb{C}}\mathbb{D}_{\eta_{i-1}, w}^{\sigma, \mu, p} \mathbb{V}(\eta)) - \mathbb{Y}(\eta, 0, 0)| + |\mathbb{Y}(\eta, 0, 0)| \\ &\leq \varrho_{\mathbb{Y}} \|\mathbb{V}\|_{\mathcal{PC}(J)} + \varrho'_{\mathbb{Y}} |{}^{\mathbb{P}-\mathbb{C}}\mathbb{D}_{\eta_{i-1}, w}^{\sigma, \mu, p} \mathbb{V}(\eta)| + |\mathbb{Y}(\eta, 0, 0)|. \end{aligned}$$

Therefore we get

$$|{}^{\mathbb{P}-\mathbb{C}}\mathbb{D}_{\eta_{i-1}, w}^{\sigma, \mu, p} \mathbb{V}(\eta)| \leq \frac{\varrho_{\mathbb{Y}} \|\mathbb{V}\|_{\mathcal{PC}(J)} + \mathcal{O}_{\mathbb{Y}}}{1 - \varrho'_{\mathbb{Y}}}.$$

Thus, we have

$$\begin{aligned} &{}^{\mathbb{P}-\mathbb{C}}\mathbb{I}_{\eta_{i-1}, w}^{\sigma, \mu, p} \left| \mathbb{Y}(s, \mathbb{V}(s), {}^{\mathbb{P}-\mathbb{C}}\mathbb{D}_{\eta_{i-1}, w}^{\sigma, \mu, p} \mathbb{V}(s)) \right| (\eta_i) \\ &\leq \frac{1 - \sigma}{\mathbb{P}(\sigma)} \left| \mathbb{Y}(s, \mathbb{V}(s), {}^{\mathbb{P}-\mathbb{C}}\mathbb{D}_{\eta_{i-1}, w}^{\sigma, \mu, p} \mathbb{V}(s)) \right| (\eta_i) \\ &\quad + \ln p \frac{\sigma}{\mathbb{P}(\sigma)} {}^{\mathbb{RL}}\mathbb{I}_{\eta_{i-1}, w}^{\mu} \left| \mathbb{Y}(s, \mathbb{V}(s), {}^{\mathbb{P}-\mathbb{C}}\mathbb{D}_{\eta_{i-1}, w}^{\sigma, \mu, p} \mathbb{V}(s)) \right| (\eta_i) \\ &\leq \frac{\varrho_{\mathbb{Y}} \|\mathbb{V}\|_{\mathcal{PC}(J)} + \mathcal{O}_{\mathbb{Y}}}{1 - \varrho'_{\mathbb{Y}}} \left( \frac{1 - \sigma}{\mathbb{P}(\sigma)} + \ln p \frac{\sigma}{\mathbb{P}(\sigma)} {}^{\mathbb{RL}}\mathbb{I}_{\eta_{i-1}, w}^{\mu}(1) (\eta_i) \right) \\ &\leq \frac{\varrho_{\mathbb{Y}} \|\mathbb{V}\|_{\mathcal{PC}(J)} + \mathcal{O}_{\mathbb{Y}}}{1 - \varrho'_{\mathbb{Y}}} \left( \frac{1 - \sigma}{\mathbb{P}(\sigma)} + \ln p \frac{\sigma}{\mathbb{P}(\sigma)} \frac{(\eta_i - \eta_{i-1})^\mu}{\Gamma(\mu + 1)} \right). \end{aligned} \tag{3.14}$$

Hence, by (3.13) and (3.14), we have

$$\begin{aligned}
 |\Xi(\mathbb{V}(\eta))| &\leq \frac{|w(a)|}{w^*} \mathbb{V}_a + \sum_{i=1}^k \frac{|w(\eta_i)| \varrho_{\mathbb{Y}}}{w^*(1-\varrho'_{\mathbb{Y}})} \left( \frac{1-\sigma}{\mathbb{P}(\sigma)} + \ln p \frac{\sigma}{\mathbb{P}(\sigma)} \frac{(\eta_i - \eta_{i-1})^\mu}{\Gamma(\mu+1)} \right) \|\mathbb{V}\|_{\mathcal{PC}(J)} \\
 &\quad + \sum_{i=1}^k \frac{|w(\eta_i)| \mathcal{O}_{\mathbb{Y}}}{w^*(1-\varrho'_{\mathbb{Y}})} \left( \frac{1-\sigma}{\mathbb{P}(\sigma)} + \ln p \frac{\sigma}{\mathbb{P}(\sigma)} \frac{(\eta_i - \eta_{i-1})^\mu}{\Gamma(\mu+1)} \right) \\
 &\quad + \frac{\varrho_{\mathbb{Y}}}{1-\varrho'_{\mathbb{Y}}} \left( \frac{1-\sigma}{\mathbb{P}(\sigma)} + \ln p \frac{\sigma}{\mathbb{P}(\sigma)} \frac{(\eta - \eta_k)^\mu}{\Gamma(\mu+1)} \right) \|\mathbb{V}\|_{\mathcal{PC}(J)} \\
 &\quad + \frac{\mathcal{O}_{\mathbb{Y}}}{1-\varrho'_{\mathbb{Y}}} \left( \frac{1-\sigma}{\mathbb{P}(\sigma)} + \ln p \frac{\sigma}{\mathbb{P}(\sigma)} \frac{(\eta - \eta_k)^\mu}{\Gamma(\mu+1)} \right) + \sum_{i=1}^k \frac{|w(\eta_i)|}{w^*} L_{\mathbb{A}} |\mathbb{V}(\eta)| \\
 &\leq \frac{|w(a)|}{w^*} \mathbb{V}_a + \frac{(m+1)|w(T)| \varrho_{\mathbb{Y}}}{w^*(1-\varrho'_{\mathbb{Y}})} \left( \frac{1-\sigma}{\mathbb{P}(\sigma)} + \ln p \frac{\sigma}{\mathbb{P}(\sigma)} \frac{(T-a)^\mu}{\Gamma(\mu+1)} \right) \|\mathbb{V}\|_{\mathcal{PC}(J)} \\
 &\quad + \frac{(m+1)|w(T)| \mathcal{O}_{\mathbb{Y}}}{w^*(1-\varrho'_{\mathbb{Y}})} \left( \frac{1-\sigma}{\mathbb{P}(\sigma)} + \ln p \frac{\sigma}{\mathbb{P}(\sigma)} \frac{(T-a)^\mu}{\Gamma(\mu+1)} \right) + \frac{m|w(T)|}{w^*} L_{\mathbb{A}} \|\mathbb{V}\|_{\mathcal{PC}(J)} \\
 &\leq \frac{|w(a)|}{w^*} \mathbb{V}_a + Q_1 \mathcal{O}_{\mathbb{Y}} + \left( Q_1 \varrho_{\mathbb{Y}} + \frac{m|w(T)|}{w^*} L_{\mathbb{A}} \right) \|\mathbb{V}\|_{\mathcal{PC}(J)}.
 \end{aligned}$$

This implies that

$$\|\Xi(\mathbb{V})\|_{\mathcal{PC}(J)} \leq \frac{|w(a)|}{w^*} \mathbb{V}_a + Q_1 \mathcal{O}_{\mathbb{Y}} + \left( Q_1 \varrho_{\mathbb{Y}} + \frac{m|w(T)|}{w^*} L_{\mathbb{A}} \right) R \leq R.$$

This means that  $\Xi(\mathbb{M}_R) \subset \mathbb{M}_R$ .

Step 2 In this step, we prove that  $\Xi$  is a continuous and compact. Let  $\mathbb{V}_n$  be a sequence in  $\mathbb{M}_R$ , such that  $\mathbb{V}_n \rightarrow \mathbb{V}$  in  $\mathbb{M}_R$ . Let

$${}^{\mathbb{P}-\mathbb{C}}D_{\eta_{i-1},w}^{\sigma,\mu,p} \mathbb{V}_n(s) = \mathbb{Y}(s, \mathbb{V}_n(s), {}^{\mathbb{P}-\mathbb{C}}D_{\eta_{i-1},w}^{\sigma,\mu,p} \mathbb{V}_n(s)).$$

This means that

$$\begin{aligned}
 & \left| {}^{\mathbb{P}-\mathbb{C}}D_{\eta_{i-1},w}^{\sigma,\mu,p} \mathbb{V}_n(s) - {}^{\mathbb{P}-\mathbb{C}}D_{\eta_{i-1},w}^{\sigma,\mu,p} \mathbb{V}(s) \right| \\
 &= \left| \mathbb{Y}(s, \mathbb{V}_n(s), {}^{\mathbb{P}-\mathbb{C}}D_{\eta_{i-1},w}^{\sigma,\mu,p} \mathbb{V}_n(s)) - \mathbb{Y}(s, \mathbb{V}(s), {}^{\mathbb{P}-\mathbb{C}}D_{\eta_{i-1},w}^{\sigma,\mu,p} \mathbb{V}(s)) \right|.
 \end{aligned}$$

By (H<sub>1</sub>), we have

$$\begin{aligned}
 & \left| {}^{\mathbb{P}-\mathbb{C}}D_{\eta_{i-1},w}^{\sigma,\mu,p} \mathbb{V}_n(s) - {}^{\mathbb{P}-\mathbb{C}}D_{\eta_{i-1},w}^{\sigma,\mu,p} \mathbb{V}(s) \right| \\
 & \leq \varrho_{\mathbb{Y}} |\mathbb{V}_n(s) - \mathbb{V}(s)| + \varrho'_{\mathbb{Y}} \left| {}^{\mathbb{P}-\mathbb{C}}D_{\eta_{i-1},w}^{\sigma,\mu,p} \mathbb{V}_n(s) - {}^{\mathbb{P}-\mathbb{C}}D_{\eta_{i-1},w}^{\sigma,\mu,p} \mathbb{V}(s) \right|.
 \end{aligned}$$

This implies that

$$\left| {}^{\mathbb{P}-\mathbb{C}}D_{\eta_{i-1},w}^{\sigma,\mu,p} \mathbb{V}_n(s) - {}^{\mathbb{P}-\mathbb{C}}D_{\eta_{i-1},w}^{\sigma,\mu,p} \mathbb{V}(s) \right| \leq \frac{\varrho_{\mathbb{Y}}}{1-\varrho'_{\mathbb{Y}}} |\mathbb{V}_n(s) - \mathbb{V}(s)|.$$

Thus, we have

$$\begin{aligned}
 & \left| \mathbb{Y}(s, \mathbb{V}_n(s), {}^{\mathbb{P}-\mathbb{C}}D_{\eta_{i-1},w}^{\sigma,\mu,p} \mathbb{V}_n(s)) - \mathbb{Y}(s, \mathbb{V}(s), {}^{\mathbb{P}-\mathbb{C}}D_{\eta_{i-1},w}^{\sigma,\mu,p} \mathbb{V}(s)) \right| \\
 & \leq \frac{\varrho_{\mathbb{Y}}}{1-\varrho'_{\mathbb{Y}}} |\mathbb{V}_n(s) - \mathbb{V}(s)|.
 \end{aligned} \tag{3.15}$$

Hence, by (3.14) and (3.15), we obtain

$$\begin{aligned}
 & {}^{\mathbb{P}-\mathbb{C}}I_{\eta_{i-1},w}^{\sigma,\mu,p} \left| \mathbb{Y}(s, \mathbb{V}_n(s), {}^{\mathbb{P}-\mathbb{C}}D_{\eta_{i-1},w}^{\sigma,\mu,p} \mathbb{V}_n(s)) - \mathbb{Y}(s, \mathbb{V}(s), {}^{\mathbb{P}-\mathbb{C}}D_{\eta_{i-1},w}^{\sigma,\mu,p} \mathbb{V}(s)) \right| (\eta_i) \\
 &= \frac{\varrho_{\mathbb{Y}}}{(1-\varrho'_{\mathbb{Y}})} \|\mathbb{V}_n - \mathbb{V}\|_{\mathcal{PC}(J)} {}^{\mathbb{P}-\mathbb{C}}I_{\eta_{i-1},w}^{\sigma,\mu,p}(1) (\eta_i) \\
 &\leq \frac{\varrho_{\mathbb{Y}}}{(1-\varrho'_{\mathbb{Y}})} \left( \frac{1-\sigma}{\mathbb{P}(\sigma)} + \ln p \frac{\sigma}{\mathbb{P}(\sigma)} \frac{(T-a)^\mu}{\Gamma(\mu+1)} \right) \|\mathbb{V}_n - \mathbb{V}\|_{\mathcal{PC}(J)}.
 \end{aligned}$$

Then, we have

$$\begin{aligned}
 & |\Xi(\mathbb{V}_n)(\eta) - \Xi(\mathbb{V})(\eta)| \\
 & \leq \sum_{i=1}^k \frac{|w(\eta_i)|}{|w(\eta)|} \left| \mathbb{P}^{-\mathbb{C}} \mathbb{I}_{\eta_{i-1}, w}^{\sigma, \mu, p} \left| \mathbb{Y}(s, \mathbb{V}_n(s), \mathbb{P}^{-\mathbb{C}} \mathbb{D}_{\eta_{i-1}, w}^{\sigma, \mu, p} \mathbb{V}_n(s)) - \mathbb{Y}(s, \mathbb{V}(s), \mathbb{P}^{-\mathbb{C}} \mathbb{D}_{\eta_{i-1}, w}^{\sigma, \mu, p} \mathbb{V}(s)) \right| (\eta_i) \right. \\
 & \quad \left. + \mathbb{P}^{-\mathbb{C}} \mathbb{I}_{\eta_k, w}^{\sigma, \mu, p} \left| \mathbb{Y}(s, \mathbb{V}_n(s), \mathbb{P}^{-\mathbb{C}} \mathbb{D}_{\eta_k, w}^{\sigma, \mu, p} \mathbb{V}_n(s)) - \mathbb{Y}(s, \mathbb{V}(s), \mathbb{P}^{-\mathbb{C}} \mathbb{D}_{\eta_k, w}^{\sigma, \mu, p} \mathbb{V}(s)) \right| (\eta) \right. \\
 & \quad \left. + \sum_{i=1}^k \frac{|w(\eta_i)|}{|w(\eta)|} \left| \mathbb{A}_i \mathbb{V}_n(\eta_i^-) - \mathbb{A}_i \mathbb{V}(\eta_i^-) \right| \right) \\
 & \leq \left( Q_1 \varrho_{\mathbb{Y}} + \frac{m |w(T)|}{w^*} L_{\mathbb{A}} \right) \|\mathbb{V}_n - \mathbb{V}\|_{\mathcal{PC}(J)}.
 \end{aligned}$$

This implies that

$$\begin{aligned}
 \|\Xi(\mathbb{V}_n) - \Xi(\mathbb{V})\|_{\mathcal{PC}(J)} & \leq \left( Q_1 \varrho_{\mathbb{Y}} + \frac{m |w(T)|}{w^*} L_{\mathbb{A}} \right) \|\mathbb{V}_n - \mathbb{V}\|_{\mathcal{PC}(J)} \\
 & \rightarrow 0 \text{ as } \mathbb{V}_n \rightarrow \mathbb{V}.
 \end{aligned}$$

Hence,  $\Xi$  is continuous. Also,  $\Xi$  is bounded by  $R$  on  $\mathbb{M}_R$ . This implies that  $\Xi$  is uniformly bounded on  $\mathbb{M}_R$ . Next, we prove that  $\Xi$  is equicontinuous. Let  $\eta_1, \eta_2 \in J$  such that  $\eta_1 < \eta_2$ . In view of (H<sub>2</sub>) we fix

$$\widehat{\mathbb{Y}} = \sup_{(\eta, \mathbb{V}) \in J \times \mathbb{M}_R} \left| \mathbb{Y}(\eta, \mathbb{V}(\eta), \mathbb{P}^{-\mathbb{C}} \mathbb{D}_{\eta_{i-1}, w}^{\sigma, \mu, p} \mathbb{V}(\eta)) \right|.$$

Then, we have

$$\begin{aligned}
 & |\Xi(\mathbb{V}(\eta_2)) - \Xi(\mathbb{V}(\eta_1))| \\
 & = \left| \mathbb{P}^{-\mathbb{C}} \mathbb{I}_{\eta_k, w}^{\sigma, \mu, p} \mathbb{Y}(s, \mathbb{V}(s), \mathbb{P}^{-\mathbb{C}} \mathbb{D}_{\eta_k, w}^{\sigma, \mu, p} \mathbb{V}(s)) (\eta_2) \right. \\
 & \quad \left. - \mathbb{P}^{-\mathbb{C}} \mathbb{I}_{\eta_k, w}^{\sigma, \mu, p} \mathbb{Y}(s, \mathbb{V}(s), \mathbb{P}^{-\mathbb{C}} \mathbb{D}_{\eta_k, w}^{\sigma, \mu, p} \mathbb{V}(s)) (\eta_1) \right| \tag{3.16} \\
 & \leq \widehat{\mathbb{Y}} \left[ \frac{1 - \sigma}{\mathbb{P}(\sigma)} + \ln p \frac{\sigma}{\mathbb{P}(\sigma)} \frac{(\eta_1 - \eta_k)^\mu + (\eta_1 - \eta_2)^\mu - (\eta_2 - \eta_k)^\mu}{\Gamma(\mu + 1)} \right].
 \end{aligned}$$

Take  $\eta_2 \rightarrow \eta_1$ , then from (3.16), we have

$$\|\Xi(\mathbb{V}(\eta_2)) - \Xi_1(\mathbb{V}(\eta_1))\|_{\mathcal{PC}(J)} \rightarrow 0 \text{ as } \eta_2 \rightarrow \eta_1. \tag{3.17}$$

Hence,  $\Xi$  exhibits equicontinuity, we can invoke the Arzelá–Ascoli theorem to deduce that  $\Xi$  maps  $\mathbb{M}_R$  into a compact subset of itself. Consequently,  $\Xi$  is a completely continuous operator.

Step 3 In this step, we prove that the set  $F = \{\mathbb{V} \in \mathbb{M}_R : \mathbb{V} = \xi \Xi(\mathbb{V}), \xi \in (0, 1)\}$  is bounded. Let  $\mathbb{V} \in F$ . Then, we have

$$|\mathbb{V}(\eta)| = |\xi \Xi(\mathbb{V})| \leq \|\Xi_1(\mathbb{V}, \vartheta)\|_{\mathcal{PC}(J)}.$$

In view of first step, we have

$$\begin{aligned}
 \|\mathbb{V}\|_{\mathcal{PC}(J)} & = \|\xi \Xi(\mathbb{V})\|_{\mathcal{PC}(J)} \\
 & \leq \frac{|w(a)|}{w^*} \mathbb{V}_a + Q_1 \mathcal{O}_{\mathbb{Y}} + \left( Q_1 \varrho_{\mathbb{Y}} + \frac{m |w(T)|}{w^*} L_{\mathbb{A}} \right) R. \tag{3.18}
 \end{aligned}$$

This implies that

$$\|\mathbb{V}\|_{\mathcal{PC}(J)} \leq R.$$

Consequently,  $F$  is bounded, and by the prior steps, we deduce that the operator  $\Xi$  has at least one fixed point. Therefore, the model (1.1) has at least one solution.  $\square$

### Criteria required for uniqueness of solution

While Theorem 4 guarantees that our model (1.1) possesses at least one solution under the given conditions, for many practical applications, particularly in prediction and control, it is crucial to know if this solution is the only one. Uniqueness ensures that the system’s behavior is deterministic and predictable, given the initial state and parameters. Without uniqueness, different simulations starting from the same point could yield different outcomes, undermining the model’s reliability. Therefore, we now investigate stricter conditions under which the solution is unique. The primary tool for establishing uniqueness in this context is the Banach Contraction Principle.

**Theorem 5** Assume that  $(H_1)$  and  $(H_2)$  hold. If

$$Q = \left( Q_1 \varrho_{\mathbb{Y}} + \frac{m |w(T)|}{w^*} L_{\mathbb{A}} \right) < 1,$$

where

$$Q_1 := \frac{(m+1) |w(T)|}{w^* (1 - \varrho'_{\mathbb{Y}})} \left( \frac{1 - \sigma}{\mathbb{P}(\sigma)} + \ln p \frac{\sigma}{\mathbb{P}(\sigma)} \frac{(T-a)^\mu}{\Gamma(\mu+1)} \right),$$

then, the problem (1.1) has a unique solution.

**Proof** In this theorem, we consider  $\mathbb{M}_R$  which defined in Theorem 4. By the first step in Theorem 4, we have  $\Xi(\mathbb{B}_R) \subset \mathbb{M}_R$ . Now, we need only prove that the operator  $\Xi$  is contraction map by using Banach contraction principle. Let  $\mathbb{V}, \widehat{\mathbb{V}} \in \mathbb{B}_R$  and  $\eta \in \mathcal{I}$ . Then, we obtain

$$\begin{aligned} & \left| \Xi(\mathbb{V})(\eta) - \Xi(\widehat{\mathbb{V}})(\eta) \right| \\ & \leq \sum_{i=1}^k \frac{|w(\eta_i)|}{|w(\eta)|} \left| \mathbb{Y}(s, \mathbb{V}(s), {}^{\mathbb{P}-\mathbb{C}}D_{\eta_{i-1}, w}^{\sigma, \mu, p} \mathbb{V}(s)) - \mathbb{Y}(s, \widehat{\mathbb{V}}(s), {}^{\mathbb{P}-\mathbb{C}}D_{\eta_{i-1}, w}^{\sigma, \mu, p} \widehat{\mathbb{V}}(s)) \right| (\eta_i) \\ & \quad + {}^{\mathbb{P}-\mathbb{C}}I_{\eta_k, w}^{\sigma, \mu, p} \left| \mathbb{Y}(s, \mathbb{V}(s), {}^{\mathbb{P}-\mathbb{C}}D_{\eta_k, w}^{\sigma, \mu, p} \mathbb{V}(s)) - \mathbb{Y}(s, \widehat{\mathbb{V}}(s), {}^{\mathbb{P}-\mathbb{C}}D_{\eta_k, w}^{\sigma, \mu, p} \widehat{\mathbb{V}}(s)) \right| (\eta) \\ & \quad + \sum_{i=1}^k \frac{|w(\eta_i)|}{|w(\eta)|} \left| \mathbb{A}_i \mathbb{V}(\eta_i^-) - \mathbb{A}_i \widehat{\mathbb{V}}(\eta_i^-) \right| \\ & \leq \left( Q_1 \varrho_{\mathbb{Y}} + \frac{m |w(T)|}{w^*} L_{\mathbb{A}} \right) \left\| \mathbb{V} - \widehat{\mathbb{V}} \right\|_{\mathcal{PC}(J)}. \end{aligned}$$

Since  $(Q_1 \varrho_{\mathbb{Y}} + \frac{m |w(T)|}{w^*} L_{\mathbb{A}}) < 1$ , this implies that  $\Xi$  is a contraction operator. Thus, the system (1.1) has a unique solution.  $\square$

**Stability behavior**

Beyond existence and uniqueness, understanding how solutions behave under small perturbations or inaccuracies is critical for practical robustness. This leads to the concept of stability, specifically Ulam–Hyers (UH) stability. The core idea is to demonstrate that if a function approximately satisfies the differential equation (within a certain error  $\epsilon$ ), then it must be close to an actual exact solution of the equation. Establishing UH stability provides confidence that numerical approximations or small modeling errors will not lead to drastically different outcomes. Our approach will involve relating the approximate solution (satisfying inequality (3.19)) to the integral equation framework and utilizing the bounds derived previously. Before presenting the stability analysis theorem, we introduce the necessary definitions<sup>29,32,33</sup>. Let  $\epsilon > 0$  and  $\lambda_\phi : J \rightarrow [0, \infty)$  be a continuous function. We then consider the following inequalities.

$$\left| {}^{\mathbb{P}-\mathbb{C}}D_{[\eta], w}^{\sigma, \mu, p} \widehat{\mathbb{V}}(\eta) - \mathbb{Y} \left( \eta, \widehat{\mathbb{V}}(\eta), {}^{\mathbb{P}-\mathbb{C}}D_{[\eta], w}^{\sigma, \mu, p} \widehat{\mathbb{V}}(\eta) \right) \right| \leq \epsilon, \tag{3.19}$$

$$\left| {}^{\mathbb{P}-\mathbb{C}}D_{[\eta], w}^{\sigma, \mu, p} \widehat{\mathbb{V}}(\eta) - \mathbb{Y} \left( \eta, \widehat{\mathbb{V}}(\eta), {}^{\mathbb{P}-\mathbb{C}}D_{[\eta], w}^{\sigma, \mu, p} \widehat{\mathbb{V}}(\eta) \right) \right| \leq \epsilon \lambda_\phi(\eta). \tag{3.20}$$

**Definition 4** The model (1.1) is Ulam–Hyers stable if there exists a positive constant  $\mathcal{M}$ , such that for any  $\epsilon > 0$ , there is a piecewise continuous function  $\widehat{\mathbb{V}}$  that satisfies the inequality (3.19) related to problem (1.1). Furthermore, for the exact solution  $\mathbb{V}$  of the problem (1.1), the following inequality holds:

$$\left\| \widehat{\mathbb{V}} - \mathbb{V} \right\|_{\mathcal{PC}(J)} \leq \mathcal{M} \epsilon.$$

**Remark 2** A function  $\widehat{\mathbb{V}} \in \mathcal{PC}(J)$  satisfies the inequalities (3.19) if and only if there exist functions  $\mathbb{Q} \in \mathcal{PC}(J)$  such that

- (i)  $|\mathbb{Q}(\eta)| \leq \epsilon;$
- (ii)  $\begin{cases} {}^{\mathbb{P}-\mathbb{C}}D_{[\eta], w}^{\sigma, \mu, p} \widehat{\mathbb{V}}(\eta) = \mathbb{Y} \left( \eta, \widehat{\mathbb{V}}(\eta), {}^{\mathbb{P}-\mathbb{C}}D_{[\eta], w}^{\sigma, \mu, p} \widehat{\mathbb{V}}(\eta) \right) + \mathbb{Q}(\eta), \\ \Delta \widehat{\mathbb{V}}(\eta_k) = \widehat{\mathbb{V}}(\eta_k^+) - \widehat{\mathbb{V}}(\eta_k^-) = I_k \widehat{\mathbb{V}}(\eta_k), k = 1, \dots, m, \\ \widehat{\mathbb{V}}(a) = \mathbb{V}_a, \end{cases}, \eta \in J$

**Lemma 3.1** Let  $\sigma \in (0, 1)$ ,  $\min(\mu, p) > 0$ . If a function  $\widehat{V} \in \mathcal{PC}(J)$  satisfies the inequalities (3.19), then  $\widehat{V}$  satisfies the following integral inequalities

$$\left| \widehat{V}(\eta) - \mathcal{A}_{\widehat{V}}^{-\mathbb{P}-\mathbb{C}} I_{\eta_k, w}^{\sigma, \mu, p} \mathbb{Y} \left( s, \widehat{V}(s), {}^{\mathbb{P}-\mathbb{C}} D_{[\eta], w}^{\sigma, \mu, p} \widehat{V}(s) \right) (\eta) \right| \leq \epsilon K,$$

where

$$\begin{aligned} \mathcal{A}_{\widehat{V}} = & \frac{w(a)}{w(\eta)} \mathbb{V}_a - \sum_{i=1}^k \frac{w(\eta_i)}{w(\eta)} {}^{\mathbb{P}-\mathbb{C}} I_{\eta_{i-1}, w}^{\sigma, \mu, p} \mathbb{Y} \left( s, \widehat{V}(s), {}^{\mathbb{P}-\mathbb{C}} D_{[\eta], w}^{\sigma, \mu, p} \widehat{V}(s) \right) (\eta_i) \\ & - \sum_{i=1}^k \frac{w(\eta_i)}{w(\eta)} \mathbb{A}_i \widehat{V}(\eta_i^-), \end{aligned}$$

and

$$K := (m + 1) \frac{w(T)}{w^*} \left( \frac{1 - \sigma}{\mathbb{P}(\sigma)} + \ln p \frac{\sigma}{\mathbb{P}(\sigma)} \frac{(T - a)^\mu}{\Gamma(\mu + 1)} \right).$$

**Proof** By second part in Remark 2, we have

$$\begin{cases} {}^{\mathbb{P}-\mathbb{C}} D_{[\eta], w}^{\sigma, \mu, p} \widehat{V}(\eta) = \mathbb{Y} \left( \eta, \widehat{V}(\eta), {}^{\mathbb{P}-\mathbb{C}} D_{[\eta], w}^{\sigma, \mu, p} \widehat{V}(\eta) \right) + \mathbb{Q}(\eta), \\ \Delta \widehat{V}(\eta_k) = \widehat{V}(\eta_k^+) - \widehat{V}(\eta_k^-) = I_k \widehat{V}(\eta_k), k = 1, \dots, m, \\ \widehat{V}(a) = \mathbb{V}_a. \end{cases} \tag{3.21}$$

Then, by Theorem 3, the solution of (3.21) is given by

$$\mathbb{V}(\eta) = \begin{cases} \frac{w(a)}{w(\eta)} \mathbb{V}_a + {}^{\mathbb{P}-\mathbb{C}} I_{a, w}^{\sigma, \mu, p} \left[ \mathbb{Y} \left( s, \widehat{V}(s), {}^{\mathbb{P}-\mathbb{C}} D_{[\eta], w}^{\sigma, \mu, p} \widehat{V}(s) \right) + \mathbb{Q}(s) \right], \eta \in [a, \eta_1], \\ \frac{w(a)}{w(\eta)} \mathbb{V}_a + \sum_{i=1}^k \frac{w(\eta_i)}{w(\eta)} {}^{\mathbb{P}-\mathbb{C}} I_{\eta_{i-1}, w}^{\sigma, \mu, p} \left[ \mathbb{Y} \left( s, \widehat{V}(s), {}^{\mathbb{P}-\mathbb{C}} D_{[\eta], w}^{\sigma, \mu, p} \widehat{V}(s) \right) + \mathbb{Q}(s) \right] (\eta_i) \\ + {}^{\mathbb{P}-\mathbb{C}} I_{\eta_k, w}^{\sigma, \mu, p} \left[ \mathbb{Y} \left( s, \widehat{V}(s), {}^{\mathbb{P}-\mathbb{C}} D_{[\eta], w}^{\sigma, \mu, p} \widehat{V}(s) \right) + \mathbb{Q}(s) \right] (\eta) + \sum_{i=1}^k \frac{w(\eta_i)}{w(\eta)} \mathbb{A}_i \mathbb{V}(\eta_i^-), \eta \in (\eta_k, \eta_{k+1}] \end{cases}$$

For  $\eta \in [a, \eta_1]$ , we have

$$\left| \widehat{V}(\eta) - \mathcal{A}_{\widehat{V}}^{-\mathbb{P}-\mathbb{C}} I_{\eta_k, w}^{\sigma, \mu, p} \mathbb{Y} \left( s, \widehat{V}(s), {}^{\mathbb{P}-\mathbb{C}} D_{[\eta], w}^{\sigma, \mu, p} \widehat{V}(s) \right) (\eta) \right| \leq \left( \frac{1 - \sigma}{\mathbb{P}(\sigma)} + \ln p \frac{\sigma}{\mathbb{P}(\sigma)} \frac{(T - a)^\mu}{\Gamma(\mu + 1)} \right) \epsilon.$$

For  $\eta \in (\eta_k, \eta_{k+1}]$ , we have

$$\begin{aligned} & \left| \widehat{V}(\eta) - \mathcal{A}_{\widehat{V}}^{-\mathbb{P}-\mathbb{C}} I_{\eta_k, w}^{\sigma, \mu, p} \mathbb{Y} \left( s, \widehat{V}(s), {}^{\mathbb{P}-\mathbb{C}} D_{[\eta], w}^{\sigma, \mu, p} \widehat{V}(s) \right) (\eta) \right| \\ & \leq \sum_{i=1}^k \frac{w(\eta_i)}{w(\eta)} {}^{\mathbb{P}-\mathbb{C}} I_{\eta_{i-1}, w}^{\sigma, \mu, p} |\mathbb{Q}(s)| (\eta_i) + {}^{\mathbb{P}-\mathbb{C}} I_{\eta_k, w}^{\sigma, \mu, p} |\mathbb{Q}(s)| (\eta) \\ & \leq \epsilon \sum_{i=1}^k \frac{w(T)}{w^*} {}^{\mathbb{P}-\mathbb{C}} I_{\eta_{i-1}, w}^{\sigma, \mu, p} (1) (\eta_i) + \epsilon ({}^{\mathbb{P}-\mathbb{C}} I_{\eta_k, w}^{\sigma, \mu, p} (1) (\eta)) \\ & \leq \frac{w(T)}{w^*} (m + 1) \left( \frac{1 - \sigma}{\mathbb{P}(\sigma)} + \ln p \frac{\sigma}{\mathbb{P}(\sigma)} \frac{(T - a)^\mu}{\Gamma(\mu + 1)} \right) \epsilon. \end{aligned}$$

This implies that

$$\left| \widehat{V}(\eta) - \mathcal{A}_{\widehat{V}}^{-\mathbb{P}-\mathbb{C}} I_{\eta_k, w}^{\sigma, \mu, p} \mathbb{Y} \left( s, \widehat{V}(s), {}^{\mathbb{P}-\mathbb{C}} D_{[\eta], w}^{\sigma, \mu, p} \widehat{V}(s) \right) (\eta) \right| \leq \epsilon K.$$

□

In the following theorem, we establish that the problem (1.1) demonstrates stable behavior.

**Theorem 6** Assume that  $(H_1)$  and  $(H_2)$  hold. Then

$${}^{\mathbb{P}-\mathbb{C}}D_{[\eta],w}^{\sigma,\mu,p}\mathbb{V}(\eta) = \mathbb{Y} \left( \eta, \mathbb{V}(\eta), {}^{\mathbb{P}-\mathbb{C}}D_{[\eta],w}^{\sigma,\mu,p}\mathbb{V}(\eta) \right), \tag{3.22}$$

is Ulam Hyers stable, provided that

$$0 < \left( Q_1 \varrho_{\mathbb{Y}} + \frac{m |w(T)|}{w^*} L_{\mathbb{A}} \right) < 1.$$

**Proof** Let  $\epsilon > 0$  and  $\widehat{\mathbb{V}} \in \mathcal{PC}(J)$  be satisfies the inequalities (3.19) and let  $\mathbb{V} \in \mathcal{PC}(J)$  be the unique solution of the following system

$$\begin{cases} {}^{\mathbb{P}-\mathbb{C}}D_{[\eta],w}^{\sigma,\mu,p}\mathbb{V}(\eta) = \mathbb{Y} \left( \eta, \mathbb{V}(\eta), {}^{\mathbb{P}-\mathbb{C}}D_{[\eta],w}^{\sigma,\mu,p}\mathbb{V}(\eta) \right), \\ \Delta \mathbb{V}|_{\eta=\eta_k} = \Delta \widehat{\mathbb{V}}|_{\eta=\eta_k} = \mathbb{A}_k \widehat{\mathbb{V}}(\eta_k^-), k = 1, \dots, m, \\ \widehat{\mathbb{V}}(a) = \mathbb{V}(a) = \mathbb{V}_a. \end{cases}$$

Then, by by Theorem 3, we have

$$\mathbb{V}(\eta) = \mathcal{A}_{\mathbb{V}} + {}^{\mathbb{P}-\mathbb{C}}I_{\eta_k,w}^{\sigma,\mu,p}\mathbb{Y} \left( s, \mathbb{V}(s), {}^{\mathbb{P}-\mathbb{C}}D_{[\eta],w}^{\sigma,\mu,p}\mathbb{V}(s) \right) (\eta).$$

Since

$$\begin{cases} \Delta \mathbb{V}|_{\eta=\eta_k} = \Delta \widehat{\mathbb{V}}|_{\eta=\eta_k} = \mathbb{A}_k \widehat{\mathbb{V}}(\eta_k^-), k = 1, \dots, m, \\ \widehat{\mathbb{V}}(a) = \mathbb{V}(a) = \mathbb{V}_a. \end{cases}$$

We can easily prove that  $\mathcal{A}_{\mathbb{V}} = \mathcal{A}_{\widehat{\mathbb{V}}}$ . Hence, from (H<sub>2</sub>) and Lemma 3.1, for each  $\eta \in J$ , we have

$$\begin{aligned} & \left| \widehat{\mathbb{V}}(\eta) - \mathbb{V}(\eta) \right| \\ &= \left| \widehat{\mathbb{V}}(\eta) - \mathcal{A}_{\widehat{\mathbb{V}}} - {}^{\mathbb{P}-\mathbb{C}}I_{\eta_k,w}^{\sigma,\mu,p}\mathbb{Y} \left( s, \mathbb{V}(s), {}^{\mathbb{P}-\mathbb{C}}D_{[\eta],w}^{\sigma,\mu,p}\mathbb{V}(s) \right) (\eta) \right. \\ &\quad \left. - {}^{\mathbb{P}-\mathbb{C}}I_{\eta_k,w}^{\sigma,\mu,p}\mathbb{Y} \left( s, \widehat{\mathbb{V}}(s), {}^{\mathbb{P}-\mathbb{C}}D_{[\eta],w}^{\sigma,\mu,p}\widehat{\mathbb{V}}(s) \right) (\eta) + {}^{\mathbb{P}-\mathbb{C}}I_{\eta_k,w}^{\sigma,\mu,p}\mathbb{Y} \left( s, \widehat{\mathbb{V}}(s), {}^{\mathbb{P}-\mathbb{C}}D_{[\eta],w}^{\sigma,\mu,p}\widehat{\mathbb{V}}(s) \right) (\eta) \right| \\ &\leq \left| \widehat{\mathbb{V}}(\eta) - \mathcal{A}_{\widehat{\mathbb{V}}} - {}^{\mathbb{P}-\mathbb{C}}I_{\eta_k,w}^{\sigma,\mu,p}\mathbb{Y} \left( s, \widehat{\mathbb{V}}(s), {}^{\mathbb{P}-\mathbb{C}}D_{[\eta],w}^{\sigma,\mu,p}\widehat{\mathbb{V}}(s) \right) (\eta) \right| \\ &\quad + {}^{\mathbb{P}-\mathbb{C}}I_{\eta_k,w}^{\sigma,\mu,p} \left| \mathbb{Y} \left( s, \widehat{\mathbb{V}}(s), {}^{\mathbb{P}-\mathbb{C}}D_{[\eta],w}^{\sigma,\mu,p}\widehat{\mathbb{V}}(s) \right) (\eta) - \mathbb{Y} \left( s, \mathbb{V}(s), {}^{\mathbb{P}-\mathbb{C}}D_{[\eta],w}^{\sigma,\mu,p}\mathbb{V}(s) \right) (\eta) \right| \\ &\leq K\epsilon + {}^{\mathbb{P}-\mathbb{C}}I_{\eta_k,w}^{\sigma,\mu,p} \left| \mathbb{Y} \left( s, \widehat{\mathbb{V}}(s), {}^{\mathbb{P}-\mathbb{C}}D_{[\eta],w}^{\sigma,\mu,p}\widehat{\mathbb{V}}(s) \right) (\eta) - \mathbb{Y} \left( s, \mathbb{V}(s), {}^{\mathbb{P}-\mathbb{C}}D_{[\eta],w}^{\sigma,\mu,p}\mathbb{V}(s) \right) (\eta) \right|. \end{aligned} \tag{3.23}$$

Thus, by (H<sub>1</sub>), we have

$$\left\| \widehat{\mathbb{V}} - \mathbb{V} \right\|_{\mathcal{PC}(J)} \leq K\epsilon + \left( Q_1 \varrho_{\mathbb{Y}} + \frac{m |w(T)|}{w^*} L_{\mathbb{A}} \right) \left\| \widehat{\mathbb{V}} - \mathbb{V} \right\|_{\mathcal{PC}(J)}.$$

It follows that

$$\left\| \widehat{\mathbb{V}} - \mathbb{V} \right\|_{\mathcal{PC}(J)} \leq \mathcal{M}\epsilon, \tag{3.24}$$

where  $\mathcal{M} = \frac{K}{1 - \left( Q_1 \varrho_{\mathbb{Y}} + \frac{m |w(T)|}{w^*} L_{\mathbb{A}} \right)}$ . By utilizing inequality (3.24) and the definition of Ulam–Hyers stability 4, we conclude that the solution to problem (1.1) exhibits Ulam–Hyers stability. Moreover, by choosing  $\lambda_{\phi} = \epsilon \mathcal{M}$  and ensuring that  $\lambda_{\phi}(0) = 0$ , we establish that problem (1.1) also demonstrates generalized Ulam–Hyers stability.  $\square$

### Numerical scheme

This section outlines a method for finding approximate numerical solutions to the impulsive power fractional-order evolution control model:

$$\begin{cases} {}^{\mathbb{P}-\mathbb{C}}D_{[\eta],w}^{\sigma,\mu,p}\mathbb{V}(\eta) = \mathbb{Y}(\eta, \mathbb{V}(\eta)), \eta \in \mathcal{I}_c = \mathcal{I} - \{\eta_1, \eta_2, \dots, \eta_m\}, \mathcal{I} := [a, T] \\ \Delta \mathbb{V}(\eta_k) = \mathbb{V}(\eta_k^+) - \mathbb{V}(\eta_k^-) = \mathbb{A}_k \mathbb{V}(\eta_k), k = 1, \dots, m + 1, \\ \mathbb{V}(a) = \mathbb{V}_a. \end{cases} \tag{4.1}$$

From Theorem 3, we know that the solution  $\mathbb{V}$  to this problem can be expressed piecewise via an equivalent integral Eq. (3.2). This integral equation forms the foundation of our numerical approach. Since the solution is continuous between impulse points  $\eta_k$  but jumps at these points, we must construct the numerical solution interval by interval  $[a, \eta_1], [\eta_1, \eta_2], \dots, [\eta_{k-1}, \eta_k]$ . The core idea is to discretize each interval  $(\eta_{n-1}, \eta_n]$  into small time steps and approximate  $\mathbb{V}(\eta)$  at each step. Let's consider a generic interval  $(\eta_{n-1}, \eta_n]$ . We divide it into  $J_n$  small subintervals using points  $\eta_j^n = \eta_{n-1} + jh^n$ , where  $j = 0, 1, \dots, J_n$ ,  $\eta_0^n = \eta_{n-1}$ ,  $\eta_{J_n}^n = \eta_n$ , and  $h^n = (\eta_n - \eta_{n-1})/J_n$  is the step size for this interval.

Within this interval, the integral equation derived from (4.1) (following the logic of Eq. 3.11/3.12) involves calculating a power Caputo fractional integral of the function  $\mathbb{V}(s, \mathbb{V}(s))$ . The challenge lies in evaluating this integral numerically, especially since  $\mathbb{V}(s)$  inside  $\mathbb{Y}$  is the unknown function we are trying to find.

Our strategy is to approximate the integrand term  $v(s, \mathbb{V}(s)) = w(s)\mathbb{Y}(s, \mathbb{V}(s))$  over each small time step  $[\eta_{l-1}^n, \eta_l^n]$ . We use a simple approximation: the linear function (Lagrange polynomial of degree 1) that connects the known or estimated values of  $v$  at the beginning and end of the small step, namely  $v(\eta_{l-1}^n, \mathbb{V}(\eta_{l-1}^n))$  and  $v(\eta_l^n, \mathbb{V}(\eta_l^n))$ . This approximation is explicitly given by Eq. (4.2) below. Substituting this linear approximation into the fractional integral term allows us to compute the integral analytically or using simpler quadrature over each small step  $[\eta_{l-1}^n, \eta_l^n]$ . Summing these contributions up gives an approximation for the fractional integral over the interval  $[\eta_{n-1}, \eta_j^n]$ .

This leads to a step-by-step formula where we can calculate the approximate value  $\mathbb{V}(\eta_{j+1}^n)$  based on the previously calculated values  $\mathbb{V}(\eta_l^n)$  for  $l \leq j$ . The explicit scheme presented below follows the structure derived from applying the power Caputo integral definition and the piecewise linear approximation.

The solution of (4.1) is constructed piecewise over intervals  $[a, \eta_1], [\eta_1, \eta_2], \dots, (\eta_m, \eta_{m+1}]$  (where  $\eta_{m+1} = T$ ) is given by the integral equation form derived from Theorem 3. This solution accounts for the impulsive effects at  $\eta_k$ . The numerical scheme is derived by discretizing the solution over each interval  $[\eta_{n-1}, \eta_n], n = 1, 2, \dots, m + 1$ , such that  $\eta_0 = a$ . We approximate the integral terms using the approach described below.

Let  $v(s, \mathbb{V}(s)) = w(s)\mathbb{Y}(s, \mathbb{V}(s))$ . Now, we approximate the function  $v(s, \mathbb{V}(s))$  on the  $l$ -th subinterval  $[\eta_{l-1}^n, \eta_l^n]$  within the  $n$ -th main interval ( $n = 1, \dots, m + 1, l = 1, \dots, J_n$ ) by using the Lagrange interpolation polynomial through the points  $(\eta_{l-1}^n, v(\eta_{l-1}^n, \mathbb{V}(\eta_{l-1}^n)))$  and  $(\eta_l^n, v(\eta_l^n, \mathbb{V}(\eta_l^n)))$ , where  $h^n = \eta_l^n - \eta_{l-1}^n$  is the step size in the  $n$ -th interval:

$$v(s) \approx \mathcal{A}_l^n(s) = \frac{s - \eta_{l-1}^n}{\eta_l^n - \eta_{l-1}^n} v(\eta_{l-1}^n, \mathbb{V}(\eta_{l-1}^n)) + \frac{s - \eta_l^n}{\eta_l^n - \eta_{l-1}^n} v(\eta_l^n, \mathbb{V}(\eta_l^n))$$

$$= \frac{v(\eta_{l-1}^n, \mathbb{V}(\eta_{l-1}^n))}{h^n} (\eta_l^n - s) + \frac{v(\eta_l^n, \mathbb{V}(\eta_l^n))}{h^n} (s - \eta_{l-1}^n), \quad s \in [\eta_{l-1}^n, \eta_l^n]. \tag{4.2}$$

Applying this approximation (4.2) to the Riemann–Liouville integral part within the definition of the power Caputo integral

$${}^{P-C}I_{a,w}^{\sigma,\mu,p}\mathbb{Y} = \frac{1 - \sigma}{\mathcal{P}(\sigma)} \mathbb{Y} + \frac{\sigma \ln p}{\mathcal{P}(\sigma)\Gamma(\mu)w(\eta)} \int_a^\eta (\eta - s)^{\mu-1} w(s)\mathbb{Y}(s)ds, \tag{4.3}$$

over each subinterval  $[\eta_{l-1}^n, \eta_l^n]$  and summing up, leads to the general numerical scheme. Replacing the approximation (4.2) in the expanded integral forms derived from the piecewise solution, we obtain the general form of approximate solutions for the value at step  $j + 1$  in interval  $n$  (denoted  $\eta_{j+1}^n$ ):

$$\mathbb{V}(\eta_{j+1}^n) \approx \frac{w(a)}{w(\eta_{j+1}^n)} \mathbb{V}_a + \frac{1 - \sigma}{\mathcal{P}(\sigma)} \mathbb{Y}(\eta_{j+1}^n, \mathbb{V}(\eta_{j+1}^n)) \quad (\text{Implicit term evaluated at endpoint})$$

$$+ \sum_{i=1}^n \frac{\sigma \ln p}{\mathcal{P}(\sigma)\Gamma(\mu)w(\eta_{j+1}^n)} \sum_{l=1}^{J_{i,j+1}} \left[ \frac{v(\eta_{l-1}^i, \mathbb{V}(\eta_{l-1}^i))}{h^i} \int_{\eta_{l-1}^i}^{\eta_l^i} (\eta_{j+1}^n - s)^{\mu-1} (\eta_l^i - s) ds \right.$$

$$\left. + \frac{v(\eta_l^i, \mathbb{V}(\eta_l^i))}{h^i} \int_{\eta_{l-1}^i}^{\eta_l^i} (\eta_{j+1}^n - s)^{\mu-1} (s - \eta_{l-1}^i) ds \right] \tag{4.4}$$

$$+ \sum_{i=1}^{n-1} \frac{w(\eta_i)}{w(\eta_{j+1}^n)} \mathbb{A}_i \mathbb{V}(\eta_i^-),$$

where  $J_{i,j+1}$  indicates the number of steps up to the point  $\eta_{j+1}^n$  within the relevant intervals  $i = 1, \dots, n$ . Specifically,  $J_{i,j+1} = J_i$  for  $i < n$  and  $J_{n,j+1} = j + 1$ . The integrals in (4.4) can be computed analytically as follows:

$$\int_{\eta_j^n}^{\eta_{j+1}^n} (\eta_{j+1}^n - s)^{\mu-1} (\eta_l^n - s) ds = \frac{(h^n)^{\mu+1}}{\mu(\mu + 1)} \mathcal{A}_{j,l}^\mu, \tag{4.5}$$

and

$$\int_{\eta_j^n}^{\eta_{j+1}^n} (\eta_{j+1}^n - s)^{\mu-1} (s - \eta_{j-1}^n) ds = \frac{(h^n)^{\mu+1}}{\mu(\mu+1)} \mathcal{B}_{j,l}^\mu, \tag{4.6}$$

where the coefficients  $\mathcal{A}_{j,l}^\mu$  and  $\mathcal{B}_{j,l}^\mu$  are given by:

$$\begin{aligned} \mathcal{A}_{j,l}^\mu &= (j-l)^\mu (j-l+1+\mu) - (j-l+1)^{\mu+1}, \\ \mathcal{B}_{j,l}^\mu &= (j-l+1)^\mu (j-l+2+\mu) - (j-l)^\mu (j-l+2+2\mu). \end{aligned}$$

Substituting these evaluated integrals into (4.4) gives the final computable scheme. The structure presented in the paper for specific intervals ( $n = 1, 2, 3$ ) illustrates this accumulation:

$$\mathbb{V}(\eta_{j+1}) \approx \left\{ \begin{aligned} & \frac{w(a)}{w(\eta_j)} \mathbb{V}_a + \frac{1-\sigma}{\mathbb{P}(\sigma)} w(\eta_j) \mathbb{Y}(\eta_j, \mathbb{V}(\eta_j)) \\ & + \frac{\sigma(\ln p)(h^1)^\mu}{\mathbb{P}(\sigma)\Gamma(\mu+2)w(\eta_j)} \sum_{l=1}^{j+1} [w(\eta_{l-1}^1) \mathbb{Y}(\eta_{l-1}^1, \mathbb{V}(\eta_{l-1}^1)) \mathcal{A}_{j,l}^\mu \\ & \quad + w(\eta_l^1) \mathbb{Y}(\eta_l^1, \mathbb{V}(\eta_l^1)) \mathcal{B}_{j,l}^\mu], \quad \eta_j, \eta_{j+1} \in [a, \eta_1], \\ & \frac{w(a)}{w(\eta_j)} \mathbb{V}_a + \frac{1-\sigma}{\mathbb{P}(\sigma)} w(\eta_j) \mathbb{Y}(\eta_j, \mathbb{V}(\eta_j)) \\ & + \frac{\sigma(\ln p)(h^1)^\mu}{\mathbb{P}(\sigma)\Gamma(\mu+2)w(\eta_j)} \sum_{l=1}^{J_1} [w(\eta_{l-1}^1) \mathbb{Y}(\eta_{l-1}^1, \mathbb{V}(\eta_{l-1}^1)) \mathcal{A}_{J_1-1,l}^\mu \\ & \quad + w(\eta_l^1) \mathbb{Y}(\eta_l^1, \mathbb{V}(\eta_l^1)) \mathcal{B}_{J_1-1,l}^\mu] \times (\text{Factor involving } (\eta_j - \eta_1)^\mu \text{ approx?}) \\ & + \frac{\sigma(\ln p)(h^2)^\mu}{\mathbb{P}(\sigma)\Gamma(\mu+2)w(\eta_j)} \sum_{l=1}^{j+1} [w(\eta_{l-1}^2) \mathbb{Y}(\eta_{l-1}^2, \mathbb{V}(\eta_{l-1}^2)) \mathcal{A}_{j,l}^\mu \\ & \quad + w(\eta_l^2) \mathbb{Y}(\eta_l^2, \mathbb{V}(\eta_l^2)) \mathcal{B}_{j,l}^\mu] + \frac{w(\eta_1)}{w(\eta_j)} \mathbb{A}_1 \mathbb{V}(\eta_1^-), \quad \eta_j, \eta_{j+1} \in (\eta_1, \eta_2], \\ & \dots (\text{General case for } \eta \in (\eta_k, \eta_{k+1}] \text{ accumulates sums similarly}) \end{aligned} \right.$$

At each impulse time  $\eta_n$ , the value  $\mathbb{V}(\eta_n^+)$  needed to start the next interval is obtained from the

### An application with simulation results

Consider the following impulsive power caputo model

$$\left\{ \begin{aligned} {}_a^{\mathbb{P}-\mathbb{C}} D_{[n],w}^{\sigma,\mu,p} \mathbb{V}(\eta) &= \frac{\eta}{50e^\eta(1+|\mathbb{V}(\eta)|)} + \frac{\sin \eta}{(1+|\mathbb{V}(\eta)|)}, \quad \eta \in (10, 100] - \{20, 50\}, \\ \mathbb{A}_1(20^-) &= \frac{|\mathbb{V}(20^-)|}{7e^{20}(1+|\mathbb{V}(20^-)|)}, \\ \mathbb{A}_2(50^-) &= \frac{|\mathbb{V}(50^-)|}{7e^{50}(1+|\mathbb{V}(50^-)|)} \\ \mathbb{V}(10) &= 5 \end{aligned} \right. \tag{5.1}$$

This model describes a mechanical oscillator subject to fractional damping and impulsive effects. The displacement  $\mathbb{V}(\eta)$  is governed by a fractional-order differential equation involving the Power-Caputo derivative, which accounts for the system's memory effects. The external forces acting on the system are represented by the nonlinear function  $\mathbb{Y}$ , which includes both exponentially decaying and sinusoidal components, modulated by the system's displacement. Impulsive effects at specific times  $\eta_1$  and  $\eta_2$  introduce sudden changes in displacement, proportional to the current displacement and scaled by exponential functions of time. Here  $\sigma = \frac{1}{3}$ ,  $a = 10$ ,  $T = 100$ ,  $w(\eta) = \eta$  with jumps points  $\eta_1 = 20$  and  $\eta_2 = 50$ , and

$$\begin{aligned} \mathbb{Y}(\eta, \mathbb{V}(\eta)) &= \frac{\eta}{50e^\eta(1+|\mathbb{V}(\eta)|)} + \frac{\sin \eta}{(1+|\mathbb{V}(\eta)|)}, \\ \mathbb{A}_k(\eta_k) &= \frac{|\mathbb{V}(\eta_k^-)|}{7e^\eta(1+|\mathbb{V}(\eta_k^-)|)}, \quad k = 1, 2. \end{aligned}$$

Then, for  $\mathbb{V}, \widehat{\mathbb{V}} \in \mathbb{R}$ , we have

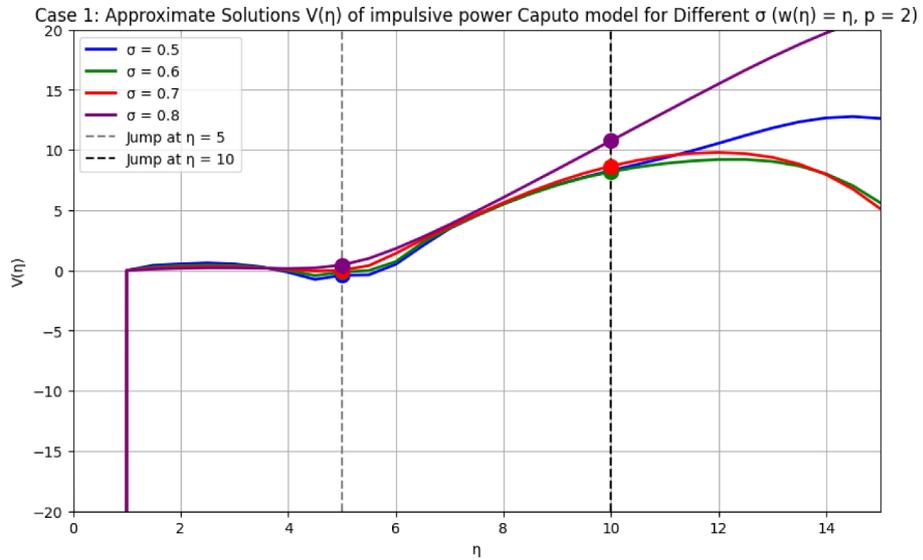
$$\left| \mathbb{Y}(\eta, \mathbb{V}(\eta)) - \mathbb{Y}(\eta, \widehat{\mathbb{V}}(\eta)) \right| \leq \frac{1}{50e^\eta} \left| \mathbb{V}(\eta) - \widehat{\mathbb{V}}(\eta) \right|,$$

and

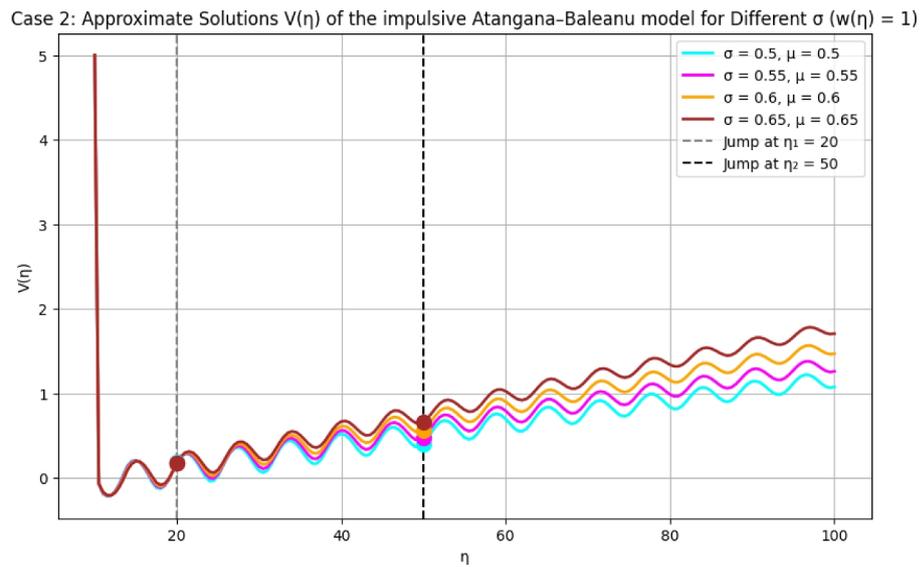
$$\left| \mathbb{A}(\mathbb{V}) - \mathbb{A}(\widehat{\mathbb{V}}) \right| \leq \frac{1}{7e^\eta} \left| \mathbb{V}(\eta) - \widehat{\mathbb{V}}(\eta) \right|.$$

Clearly,  $\varrho_{\mathbb{Y}} = \varrho'_{\mathbb{Y}} = \frac{1}{50}$  and  $L_{\mathbb{A}} = \frac{1}{7}$ ,  $m = 2$ . By some calculations, we get

$$Q_1 := \frac{(m+1)|w(T)|}{w^*(1-\varrho'_{\mathbb{Y}})} \left( \frac{1-\sigma}{\mathbb{P}(\sigma)} + \ln p \frac{\sigma}{\mathbb{P}(\sigma)\Gamma(\mu+1)} \right) \simeq 1.99,$$



**Fig. 1.** Graphical presentations of impulsive power caputo fractional model, when  $\sigma = 0.5, 0.6, 0.7, 0.8, \mu = 5, 10, p = 2, w(\eta) = \eta.$



**Fig. 2.** Graphical presentations of impulsive Atangana–Baleanu fractional model, when  $\sigma = 0.5, 0.55, 0.6, 0.65, \mu = \sigma, p = 20$  and  $w(\eta) = 1.$

and

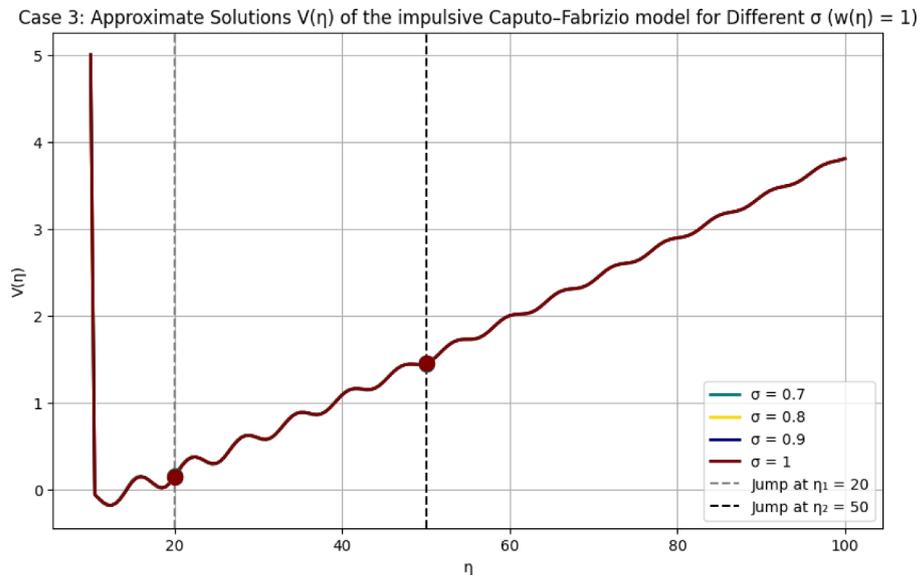
$$\left( Q_1 \varrho_{\mathbb{V}} + \frac{m |w(T)|}{w^*} L_{\mathbb{A}} \right) \simeq 0.449 < 1.$$

By examining the supplied data, we conclude that hypotheses  $(H_1)$  and  $(H_2)$  are met. Consequently, all the stipulated conditions of Theorem 4 are satisfied, which ensures that problem (1.1) exhibits at least one solution.

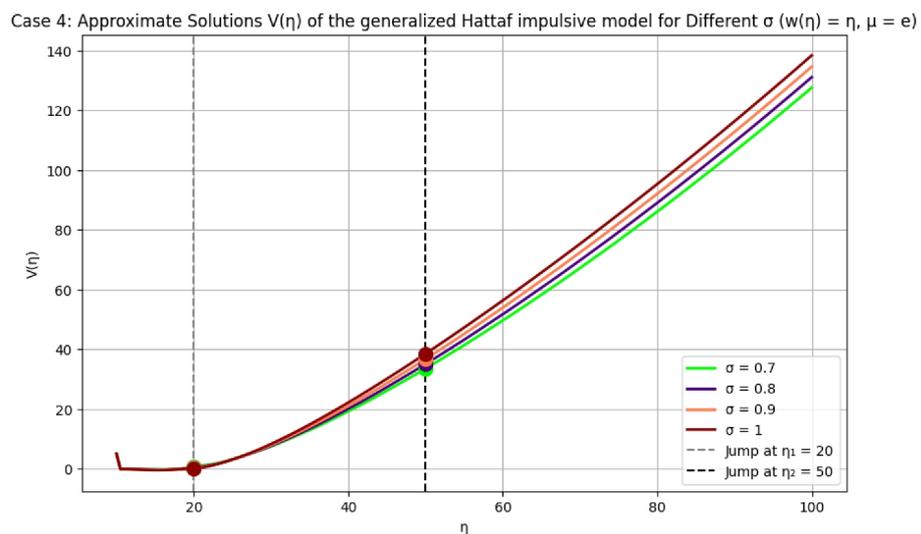
The approximate solutions of impulsive power caputo fractional model (1.1) and its symmetric cases are display in Figs. 1, 2, 3, and 4 as follows:

Case (1): The approximate solutions of impulsive power caputo fractional model, when  $\sigma = 0.5, 0.6, 0.7, 0.8, \mu = 5, p = 10, w(\eta) = \eta + 2,$  is display in Fig. 1.

Figure 1 demonstrates the behavior under the general Power Caputo operator. We observe smooth evolution of the oscillator’s displacement between the impulse points ( $\eta = 5, 10$ ), punctuated by abrupt jumps at these instances, reflecting the sudden external influences described by  $\mathbb{A}_k$ . Variations in the fractional order  $\sigma$  clearly impact the trajectory. Higher values of  $\sigma$  (closer to 1) lead to smoother, potentially less damped-looking curves



**Fig. 3.** Graphical presentations of impulsive Caputo–Fabrizio fractional model, when  $\sigma = 0.7, 0.8, 0.9, 1$ ,  $\mu = 1, p = 30$  and  $w(\eta) = 1$ .



**Fig. 4.** Graphical presentations of impulsive generalized Hattaf fractional model, when  $\sigma = 0.7, 0.8, 0.9, 1$ ,  $\mu = e, p = 30$  and  $w(\eta) = \eta + 2$ .

between impulses, suggesting stronger memory effects—the system’s future state is more influenced by its longer past history. Conversely, lower  $\sigma$  values result in paths that seem to react more quickly or perhaps exhibit slightly different oscillatory characteristics, indicating weaker memory. The parameter  $p = 2$  shapes the specific memory kernel via the PML function, influencing the precise nature of the memory decay.

Case (2): The approximate solutions of impulsive Atangana–Baleanu fractional model, when  $\sigma = 0.5, 0.55, 0.6, 0.65, \mu = \sigma, p = e$  and  $w(\eta) = 1$  is displayed in Fig. 2.

Figure 2 illustrates the dynamics using the Atangana–Baleanu (AB) operator, a special case relevant for systems with non-local, fading memory modeled by the Mittag–Leffler function. The solutions again show smooth evolution between impulses and sharp jumps. Compared to Fig. 1, the overall behavior might appear smoother, characteristic of the non-singular AB kernel, which is often considered suitable for modeling complex biological or viscoelastic systems where memory decays gradually following a power-law or stretched exponential pattern. Increasing  $\sigma$  (which also equals  $\mu$  here) again appears to smooth the trajectories and potentially increases the amplitude or persistence of oscillations between impulses, consistent with strengthening the memory effect captured by the Mittag–Leffler kernel.

Case (3): The approximate solutions of impulsive Caputo–Fabrizio fractional model, when  $\sigma = 0.7, 0.8, 0.9, 1$ ,  $\mu = 1, p = 30$  and  $w(\eta) = 1$  is displayed in Fig. 3.

Figure 3 uses the Caputo–Fabrizio (CF) operator, characterized by its non-singular exponential kernel. This operator is often employed for systems exhibiting exponential decay memory, like certain heat transfer or diffusion processes. The resulting trajectories show a distinctly different character, perhaps appearing more heavily damped or exhibiting less pronounced oscillations compared to AB. The exponential kernel implies a faster fading of memory compared to the Mittag–Leffler kernel. As  $\sigma$  increases towards 1, the system approaches standard integer-order behavior, which in this case appears as a smoother, potentially non-oscillatory drift between the impulses. The impulses still cause significant discontinuities.

Case (4): The approximate solutions of impulsive generalized Hattaf fractional model, when  $\sigma = 0.7, 0.8, 0.9, 1$ ,  $\mu = 2$ ,  $p = e$  and  $w(\eta) = \eta + 2$  is displayed in Fig. 4.

Figure 4 shows the behavior under the generalized weighted Hattaf model, another special case. Here, the parameter  $p = e$  define the kernel (based on the standard Mittag–Leffler function), and the weighting function  $w(\eta) = \eta + 2$  introduces heterogeneity (time-varying weighting). The solutions combine smooth evolution influenced by the Mittag–Leffler type kernel with the impulsive jumps. The weighting function  $w(\eta) = \eta + 2$  directly scales the memory term, meaning the influence of past states is weighted differently as time progresses. The increasing  $\sigma$  values again lead to trajectories approaching integer-order behavior, while the specific combination of  $\mu$  and the weighting function gives these curves their distinct shape compared to the other cases.

Overall, these simulations vividly illustrate several key points:

- **Impact of Fractional Order ( $\sigma$ ):** Across all models, varying  $\sigma$  significantly alters the system dynamics, controlling the degree of memory and influencing smoothness, damping, and oscillatory behavior between impulses. Higher  $\sigma$  generally implies stronger memory effects.
- **Role of Operator Kernel:** The choice of fractional operator (Power Caputo, AB, CF, Hattaf) fundamentally changes the system's response due to the different memory kernels (PML, Mittag–Leffler, Exponential). This highlights the importance of selecting an operator whose kernel appropriately reflects the physical memory process being modeled.
- **Impulsive Effects:** The impulsive terms ( $\mathbb{A}_k$ ) consistently introduce discontinuities, demonstrating the model's ability to capture sudden external events acting on the oscillator.
- **Weighting Function:** The inclusion of  $w(\eta)$  (in Cases 1 and 4) allows modeling systems where memory effects are not uniform, adding another layer of realism. These figures demonstrate the practical applicability of the theoretical results, showing how the power Caputo framework, along with its symmetric cases, can simulate complex dynamics involving both fractional-order memory and impulsive perturbations.

The simulations confirm the existence of solutions predicted by the theorems and showcase the rich variety of behaviors that can be captured by tuning the operator parameters, providing valuable insights into the behavior of physical systems like mechanical oscillators under complex conditions.

### Concluding remarks

This study successfully establishes the existence, uniqueness, and Ulam–Hyers stability of solutions for impulsive fractional differential equations (FDEs) using the generalized power Caputo fractional derivative. A key contribution is the rigorous analysis of this versatile operator, whose flexibility (owing to the parameter ' $p$ ') allows for enhanced, fine-tuned modeling of systems with diverse memory characteristics. This work represents a significant advancement by providing mathematically guaranteed predictability and reliability (well-posedness) for a broader class of impulsive systems than previously addressed with such generality. The unification of several known fractional operators (such as AB, CF, Hattaf) within this framework provides a cohesive analytical platform, impacting fields like control systems engineering and mathematical modeling in pharmacology where understanding systems with both memory and sudden changes is crucial. The methodology employed, involving integral formulations and fixed point theorems (Banach, Schaefer), demonstrates a robust approach applicable to this generalized derivative. Although the study relies on standard continuity and Lipschitz conditions for the nonlinear terms and impulses, acknowledging these limitations points towards future avenues. Extending the theoretical framework to encompass non-Lipschitz nonlinearities or singular impulses could significantly broaden the applicability of these results to more challenging real-world scenarios. The figures demonstrate the practical applicability of the theoretical results, showing how the power Caputo framework, along with its symmetric cases, can simulate complex dynamics involving both fractional-order memory and impulsive perturbations. The simulations confirm the existence of solutions predicted by the theorems and showcase the rich variety of behaviors that can be captured by tuning the operator parameters, providing valuable insights into the behavior of physical systems like mechanical oscillators under complex conditions. In future research, we will focus on developing accurate and efficient numerical methods for impulsive power Caputo systems, particularly those preserving stability, and apply the framework to real-world problems, such as PK/PD modeling with impulsive dosing or control of impacted mechanical systems, which is crucial for validation. Furthermore, we will extend the analysis to include stochastic effects and investigate the long-term asymptotic behavior of solutions representing important avenues for advancing the understanding of these complex systems.

### Data availability

All data generated or analysed during this study are included in this article.

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## Author contributions

M. Almalahi: writing—original draft preparation, conceptualization; H. Saber: writing—review and editing; A. Moutmen: writing—review and editing; Blgys Muflh: software, investigation; M. Bouye: funding acquisition, writing—review and editing K. Aldwoah: formal analysis, resources, project administration. All authors have read and agreed to the published version of the manuscript.

## Declarations

### Competing interests

The authors declare no competing interests.

### Additional information

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