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Research Article

Fibonacci Numbers Related to Some Subclasses of Bi-Univalent Functions

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This research paper introduces the novel subclass $\Upsilon_{\Sigma}^{\vartheta,\beta,\mu}(\tilde{q})$ of bi-univalent functions that are connected to Fibonacci numbers. Our main contributions in this study involve establishing constraints on the absolute values of the second coefficient $|a_2|$ and the third coefficient $|a_3|$ for functions within this specific subclass. In addition, we provide solutions to Fekete–Szegö functional problems. Furthermore, our investigation reveals intriguing outcomes resulting from the specific parameter values used in our main findings.

1. Introduction and Preliminaries

The Fibonacci numbers form a recursive sequence of integers. The sequence is defined recursively as $F_0 = 0$, $F_1 = 1$, and for $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$. This sequence is generated by adding the two preceding terms [1]. It starts with the following elements: the given sequence follows the Fibonacci sequence pattern, where each number is the sum of the two previous numbers.

The Fibonacci numbers have unique characteristics and are applied in various fields. Fractal patterns are evident in the growth of biological entities, such as tree branching and leaf arrangement. These concepts have diverse applications in mathematics and science, including number theory, physics, and geometry. The Fibonacci sequence can be generated using various methods, such as matrix multiplication, generating functions, and combinatorial techniques. This sequence exhibits intriguing features and connections with mathematical entities such as the golden ratio, Pell numbers, and Lucas numbers. Moreover, Fibonacci numbers are used in financial markets, particularly in technical analysis. Traders and analysts rely on Fibonacci retracement levels to identify potential support and resistance levels on financial charts, aiding in making well-informed trading decisions. The significance of Fibonacci numbers is apparent in multiple fields such as mathematics, computer science, biology, and finance. This is due to their natural appearance and unique mathematical characteristics. By understanding and utilizing Fibonacci numbers, we can enhance our comprehension of the inherent patterns in nature and access useful problem-solving and analytical tools [2].

One possible approach is to use mathematical analysis or computational techniques that leverage the properties of Fibonacci numbers in the study or implementation of bi-univalent functions. Fibonacci numbers often demonstrate intriguing mathematical properties related to sequences, series, and mathematical structures. These properties could be applied in the analysis of functions, particularly bi-univalent functions, especially in cases where understanding the behavior or properties of these functions involves intricate mathematical patterns or sequences. Furthermore, studying Fibonacci numbers could spark new mathematical techniques or approaches that may be useful in the analysis or manipulation of bi-univalent functions. Mathematical concepts and tools that are developed in one area of mathematics frequently have unforeseen applications in seemingly unrelated fields. In conclusion, exploring mathematical connections and utilizing the properties of Fibonacci numbers could potentially result in insights, techniques, or approaches that improve the understanding or application of bi-univalent functions in mathematical analysis and computation.

Let $\mathscr{A}$ denote the class of analytic functions in the open unit disk $\nabla = \{\varsigma\colon |\varsigma| < 1\}$ with $f(0) = 0$ and $f'(0) = 1$ and having the following form:

$$f(\varsigma) = \varsigma + \sum_{n=2}^{\infty} a_n \varsigma^n, \quad (\varsigma \in \nabla). \tag{1}$$

Let Q denote the class of functions of the form as follows:

$$q(\varsigma) = 1 + q_1\varsigma + q_2\varsigma^2 + q_3\varsigma^3 + \cdots, \quad (\varsigma \in \nabla). \tag{2}$$

Here, $q(\varsigma)$ is analytic and is called Carathéodory functions with $\mathrm{Re}\{q(\varsigma)\} > 0$ (see [3]). Also, $q \in Q$ if a Schwarz function w exists and $q(\varsigma) = 1 + w(\varsigma)/1 - w(\varsigma)$. Let $Q(\alpha)$ and $0 \le \alpha < 1$ be the class of analytic functions q in ∇ with $q(0) = 1$ and $\mathrm{Re}\{q(\varsigma)\} > 0$. Recently, Sokół [4] and Dziok et al. [5] investigated categories $\mathscr{S}\mathscr{L}(\tilde{q})$, comprising shell-like functions characterized by $\varsigma f'(\varsigma)/f(\varsigma) \prec \tilde{q}(\varsigma)$ and $\mathscr{K}\mathscr{S}\mathscr{L}(\tilde{q})$, consisting of convex shell-like functions characterized by $1 + \varsigma^2 f''(\varsigma)/f'(\varsigma) \prec \tilde{q}(\varsigma)$. Here, $\tilde{q}(\varsigma) = 1 + \tau^2\varsigma^2/1 - \tau\varsigma - \tau^2\varsigma^2$, with $\tau = 1 - \sqrt{5}/2 \approx -0.618$ as defined in [6, 7]. It is essential to note that the function $\tilde{q}$ is univalent in $|\varsigma| < 3 - \sqrt{5}/2 \approx -0.38$, but it is not univalent in ∇. Since τ satisfies the equation $\tau^2 = 1 + \tau$, this expression can be employed to calculate higher powers τ^n as a linear combination of lower powers, ultimately reducing to a combination of τ and 1. Fibonacci numbers σ_n are generated from these recursive relationships.

$$\tau^n = \sigma_n \tau + \sigma_{n-1}. \tag{3}$$

Also, Raina and Sokół [7] taking $\tau\varsigma = t$ proved that

$$\tilde{q}(\varsigma) = \frac{1 + \tau^2\varsigma^2}{1 - \tau\varsigma - \tau^2\varsigma^2} = \sum_{n=1}^{\infty} (\sigma_{n-1} + \sigma_{n+1})\tau^n\varsigma^n, \tag{4}$$

where

$$\begin{aligned} \sigma_n &= \frac{(1-\tau)^n - \tau}{\sqrt{5}}, \\ \tau &= \frac{1 - \sqrt{5}}{2}, \quad \mathrm{n} = 1, 2, \cdots. \end{aligned} \tag{5}$$

This proves that

$$\begin{aligned} \sigma_0 &= 0, \\ \sigma_1 &= 1, \\ \sigma_{\mathrm{n}+2} &= \sigma_{\mathrm{n}} + \sigma_{\mathrm{n}+1}, \quad \mathrm{n} = 0, 1, 2, \cdots. \end{aligned} \tag{6}$$

Hence,

$$\begin{aligned} \tilde{q}(\varsigma) &= 1 + \sum_{n=1}^{\infty} \tilde{q}_n \varsigma^n \\ &= 1 + (\sigma_0 + \sigma_2)\tau\varsigma + (\sigma_1 + \sigma_3)\tau^2\varsigma^2 + \sum_{n=3}^{\infty} (\sigma_{n-3} + \sigma_{n-2} + \sigma_{n-1} + \sigma_n)\tau^n\varsigma^n \\ &= 1 + \tau\varsigma + 3\tau^2\varsigma^2 + 4\tau^3\varsigma^3 + 7\tau^4\varsigma^4 + 11\tau^5\varsigma^5 + \cdots. \end{aligned} \tag{7}$$

We note that $\tilde{q} \in Q(\alpha)$ with $\alpha = \sqrt{5}/10 \approx 0.2236$ (see [7]).

For two analytic functions f and g in ∇, if there exists an analytic function w such that

$$\begin{aligned} &w(0) = 0, \quad |\mathrm{w}(\varsigma)| < 1 \quad \text{and} \\ &f(\varsigma) = g(w(\varsigma)), \quad (\varsigma \in \nabla), \end{aligned} \tag{8}$$

where f is said to be subordinate to g and is denoted by $f \prec g$ or $f(\varsigma) \prec g(\varsigma)$.

Also, when g is univalent in ∇,

$$\begin{aligned} &f(0) = g(0) \quad \text{and} \\ &f(\nabla) \subset g(\nabla) \Longleftrightarrow f \prec g, \quad (\varsigma \in \nabla). \end{aligned} \tag{9}$$

Further, for all functions $f \in \mathscr{A}$ univalent in ∇, we will denote it by $\mathscr{S}$. So, every $f \in \mathscr{S}$ has an inverse f^{-1}, defined by

$$\begin{aligned} &f^{-1}(f(\varsigma)) = \varsigma(\varsigma \in \nabla) \text{ and} \\ &f\left(f^{-1}(w)\right) = w\left(|w| < r_0(f); r_0(f) \ge \frac{1}{4}\right), \end{aligned} \tag{10}$$

where

$$\begin{aligned} f^{-1}(w) = w - a_2 w^2 &+ \left(2a_2^2 - a_3\right)w^3 \\ &- \left(5a_2^3 - 5a_2a_3 + a_4\right)w^4 + \cdots. \end{aligned} \tag{11}$$

A function $f \in \mathscr{A}$ belongs to the set Σ, which includes all bi-univalent functions within ∇ if both $f(\varsigma)$ and $f^{-1}(\varsigma)$ are univalent within ∇. Additional information about properties of the class Σ can be found in [8–28].

Motivated by the works of Ali et al. [29] and Orhan et al. [30], the following will be introduced.

Definition 1. A function $f \in \mathscr{A}$ is in the class $\Upsilon_{\Sigma}^{\vartheta,\beta,\mu}(\tilde{q})$, where β and $\mu \geq 0, \vartheta \geq 1$, if the following equation is satisfied:

$$(1-\vartheta)\left(\frac{f(\varsigma)}{\varsigma}\right)^{\mu} + \vartheta\left(f'(\varsigma)\right)^{1-\mu} + \beta\varsigma f''(\varsigma) \prec \tilde{q}(\varsigma) = \frac{1+\tau^2\varsigma^2}{1-\tau\varsigma-\tau^2\varsigma^2}, \quad (\varsigma \in \nabla), \tag{12}$$

and g is the inverse of f given by (11),

$$(1-\vartheta)\left(\frac{g(\omega)}{\omega}\right)^{\mu} + \vartheta\left(g'(\omega)\right)^{1-\mu} + \beta\varsigma g''(\omega) \prec \tilde{q}(\varsigma) = \frac{1+\tau^2 w^2}{1-\tau w-\tau^2 w^2}, \quad (w \in \nabla), \tag{13}$$

where $\tau = 1-\sqrt{5}/2 \approx -0.618$.

If we specify the value of ϑ, β, and μ where $\tau = 1-\sqrt{5}/2$, the class $\Upsilon_{\Sigma}^{\vartheta,\beta,\mu}(\tilde{q})$ is reduced to many subclasses. For example,

(i) If $\vartheta = 1$, a function $f(\varsigma) = \varsigma + a_2\varsigma^2 + a_3\varsigma^3 + \cdots \in \Upsilon_{\Sigma}^{\beta,\mu}(\tilde{q}) = \Upsilon_{\Sigma}^{1,\beta,\mu}(\tilde{q})$, where β and $\mu \geq 0$, if the following equation is satisfied:

$$\left(f'(\varsigma)\right)^{1-\mu} + \beta\varsigma f''(\varsigma) \prec \tilde{q}(\varsigma) = \frac{1+\tau^2\varsigma^2}{1-\tau\varsigma-\tau^2\varsigma^2}, \quad (\varsigma \in \nabla), \tag{14}$$

and for g given by (11),

$$\left(g'(\omega)\right)^{1-\mu} + \beta\varsigma g''(\omega) \prec \tilde{q}(\varsigma) = \frac{1+\tau^2 w^2}{1-\tau w-\tau^2 w^2}, \quad (w \in \nabla). \tag{15}$$

(ii) If $\vartheta = 1$ and $\beta = 0$, a function $f(\varsigma) = \varsigma + a_2\varsigma^2 + a_3\varsigma^3 + \cdots \in \Upsilon_{\Sigma}^{\mu}(\tilde{q}) = \Upsilon_{\Sigma}^{1,0,\mu}(\tilde{q})$, where $\mu \geq 0$, if the following equation is satisfied:

$$\left(f'(\varsigma)\right)^{1-\mu} \prec \tilde{q}(\varsigma) = \frac{1+\tau^2\varsigma^2}{1-\tau\varsigma-\tau^2\varsigma^2}, \quad (\varsigma \in \nabla), \tag{16}$$

and for g given by (11),

$$\left(g'(\omega)\right)^{1-\mu} \prec \tilde{q}(\varsigma) = \frac{1+\tau^2 w^2}{1-\tau w-\tau^2 w^2}, \quad (w \in \nabla). \tag{17}$$

(iii) If $\vartheta = 1$ and $\beta = \mu = 0$, a function $f(\varsigma) = \varsigma + a_2\varsigma^2 + a_3\varsigma^3 + \cdots \in \Upsilon_{\Sigma}(\tilde{q}) = \Upsilon_{\Sigma}^{1,0,0}(\tilde{q})$, if the following equation is satisfied:

$$f'(\varsigma) \prec \tilde{q}(\varsigma) = \frac{1+\tau^2\varsigma^2}{1-\tau\varsigma-\tau^2\varsigma^2}, \quad (\varsigma \in \nabla), \tag{18}$$

and for g given by (11),

$$g'(w) \prec \tilde{q}(\varsigma) = \frac{1+\tau^2 w^2}{1-\tau w-\tau^2 w^2}, \quad (w \in \nabla). \tag{19}$$

(iv) If $\beta = 0$, a function $f(\varsigma) = \varsigma + a_2\varsigma^2 + a_3\varsigma^3 + \cdots \in \Upsilon_{\Sigma}^{\vartheta,\mu}(\tilde{q}) = \Upsilon_{\Sigma}^{\vartheta,0,\mu}(\tilde{q})$, where $\vartheta \geq 1$ and $\mu \geq 0$, if the following equation is satisfied:

$$(1-\vartheta)\left(\frac{f(\varsigma)}{\varsigma}\right)^{\mu} + \vartheta\left(f'(\varsigma)\right)^{1-\mu} \prec \tilde{q}(\varsigma) = \frac{1+\tau^2\varsigma^2}{1-\tau\varsigma-\tau^2\varsigma^2}, \quad (\varsigma \in \nabla), \tag{20}$$

and for g given by (11),

$$(1-\vartheta)\left(\frac{g(\omega)}{\omega}\right)^{\mu} + \vartheta\left(g'(\omega)\right)^{1-\mu} \prec \tilde{q}(\varsigma) = \frac{1+\tau^2 w^2}{1-\tau w-\tau^2 w^2}, \quad (w \in \nabla). \tag{21}$$

(v) If $\beta = 0$ and $\mu = 1$, a function $f(\varsigma) = \varsigma + a_2\varsigma^2 + a_3\varsigma^3 + \cdots \in \Upsilon_{\Sigma}^{\vartheta}(\tilde{q}) = \Upsilon_{\Sigma}^{\vartheta,0,1}(\tilde{q})$, where $\vartheta \geq 1$, if the following is satisfied:

$$(1-\vartheta)\left(\frac{f(\varsigma)}{\varsigma}\right) + \vartheta \prec \tilde{q}(\varsigma) = \frac{1+\tau^2\varsigma^2}{1-\tau\varsigma-\tau^2\varsigma^2}, \quad (\varsigma \in \nabla), \tag{22}$$

and for g given by (11),

$$(1-\vartheta)\left(\frac{g(\omega)}{\omega}\right)+\vartheta \prec \tilde{q}(\varsigma)=\frac{1+\tau^2 w^2}{1-\tau w-\tau^2 w^2}, \quad (w \in \nabla), \tag{23}$$

(vi) If $\beta=0$ and $\mu=0$, a function $f(\varsigma)=\varsigma+a_2\varsigma^2+a_3\varsigma^3+\cdots \in \Upsilon_\Sigma^\vartheta(\tilde{q})=\Upsilon_\Sigma^{\vartheta,0,0}(\tilde{q})$, where $\vartheta\geq 1$, if the following equation is satisfied:

$$(1-\vartheta)+\vartheta(f'(\varsigma)) \prec \tilde{q}(\varsigma)=\frac{1+\tau^2\varsigma^2}{1-\tau\varsigma-\tau^2\varsigma^2}, \quad (\varsigma \in \nabla), \tag{24}$$

and for g given by (11),

$$(1-\vartheta)+\vartheta(g'(\omega)) \prec \tilde{q}(\varsigma)=\frac{1+\tau^2 w^2}{1-\tau w-\tau^2 w^2}, \quad (w \in \nabla). \tag{25}$$

(vii) If $\mu=0$, a function $f(\varsigma)=\varsigma+a_2\varsigma^2+a_3\varsigma^3+\cdots \in \Upsilon_\Sigma^\vartheta(\tilde{q})=\Upsilon_\Sigma^{\vartheta,\beta,0}(\tilde{q})$, where $\beta\geq 0, \vartheta\geq 1$, if the following equation is satisfied:

$$(1-\vartheta)+\vartheta(f'(\varsigma))+\beta\varsigma f''(\varsigma) \prec \tilde{q}(\varsigma)=\frac{1+\tau^2\varsigma^2}{1-\tau\varsigma-\tau^2\varsigma^2}, \quad (\varsigma \in \nabla), \tag{26}$$

and for g given by (11),

$$(1-\vartheta)+\vartheta(g'(\omega))+\beta\varsigma g''(\omega) \prec \tilde{q}(\varsigma)=\frac{1+\tau^2 w^2}{1-\tau w-\tau^2 w^2}, \quad (w \in \nabla). \tag{27}$$

To prove our results, we need the following lemma.

Lemma 2 (see [31]). *If $c \in C$, then $|c_i|\leq 2$ for each i, where C is the family of all functions c analytic in ∇ for which*

$$\mathrm{Re}\{c(\varsigma)\}>0, c(\varsigma)=1+c_1\varsigma+c_2\varsigma^2+\cdots, \quad (\varsigma \in \nabla). \tag{28}$$

2. Estimation of Coefficients and Fekete–Szegö Inequality for the Function Class $\Upsilon_\Sigma^{\vartheta,\beta,\mu}(\tilde{q})$

In this section, we estimate the Taylor–Maclaurin coefficients and Fekete–Szegö inequality for functions in the class $\Upsilon_\Sigma^{\vartheta,\beta,\mu}(\tilde{q})$.

Theorem 3. *Let $f(\varsigma)$ given by (1) be in the function class $\Upsilon_\Sigma^{\vartheta,\beta,\mu}(\tilde{q})$, where β, $\mu\geq 0$, and $\vartheta\geq 1$. Then,*

$$|a_2|\leq\sqrt{\frac{2}{F}}|\tau|, |a_3|\leq\frac{|\tau|(F+\tau[2\mu(1-4\vartheta)+6\vartheta+12\beta])}{[\mu(1-4\vartheta)+3\vartheta+6\beta]F}, \tag{29}$$

and for $\varepsilon\in\mathbb{R}$,

$$|a_3-\varepsilon a_2^2|\leq\begin{cases}\dfrac{|\tau|}{\mu(1-4\vartheta)+3\vartheta+6\beta}; & 0\leq|\epsilon(\varepsilon)|\leq\dfrac{|\tau|}{4[\mu(1-4\vartheta)+3\vartheta+6\beta]},\\[2ex] 4|\epsilon(\varepsilon)|; & |\epsilon(\varepsilon)|\geq\dfrac{|\tau|}{4[\mu(1-4\vartheta)+3\vartheta+6\beta]},\end{cases} \tag{30}$$

where $F=\tau[\mu^2(3\vartheta+1)+\mu(1-11\vartheta)+6\vartheta+12\beta]+2(1-3\tau)[\mu(1-3\vartheta)+2(\vartheta+\beta)]^2$ and $\epsilon(\varepsilon)=(1-\varepsilon)\tau^2/2F$.

Proof. Since $f\in\Upsilon_\Sigma^{\vartheta,\beta,\mu}(\tilde{q})$, then there exist two functions u and u, analytic in ∇ with $u(0)=v(0)=0$, $|u(\varsigma)|<1$, and $|v(w)|<1$ for all $\varsigma, w\in\nabla$. So, from (7) and (8), we have

$$(1-\vartheta)\left(\frac{f(\varsigma)}{\varsigma}\right)^\mu+\vartheta(f'(\varsigma))^{1-\mu}+\beta\varsigma f''(\varsigma)=\tilde{q}(u(\varsigma)), \tag{31}$$

and for $f^{-1}=g$,

$$(1-\vartheta)\left(\frac{g(\omega)}{\omega}\right)^{\mu}+\vartheta\left(g'(\omega)\right)^{1-\mu}+\beta\varsigma g''(\omega)=\widetilde{q}(\nu(\varsigma)). \tag{32}$$

Using the fact that $q\prec\widetilde{q}$ and $q(\varsigma)=1+q_1\varsigma+q_2\varsigma^2+q_3\varsigma^3+\cdots$, therefore, we define

$$t(\varsigma)=\frac{1+u(\varsigma)}{1-u(\varsigma)}=1+t_1\varsigma+t_2\varsigma^2+\cdots. \tag{33}$$

Note that $t(\varsigma)\in Q$. As a consequence,

$$u(\varsigma)=\frac{t(\varsigma)-1}{t(\varsigma)+1}=\frac{t_1}{2}\varsigma+\left(u_2-\frac{u_1^2}{2}\right)\frac{\varsigma^2}{2}+\left(u_3-u_1u_2+\frac{u_1^3}{4}\right)\frac{\varsigma^3}{2}+\cdots, \tag{34}$$

and

$$\begin{aligned}\widetilde{q}(u(\varsigma))&=1+\widetilde{q}_1\left(\frac{u_1}{2}\varsigma+\left(u_2-\frac{u_1^2}{2}\right)\frac{\varsigma^2}{2}+\left(u_3-u_1u_2+\frac{u_1^3}{4}\right)\frac{\varsigma^3}{2}+\cdots\right)\\&\quad+\widetilde{q}_2\left(\frac{u_1}{2}\varsigma+\left(u_2-\frac{u_1^2}{2}\right)\frac{\varsigma^2}{2}+\left(u_3-u_1u_2+\frac{u_1^3}{4}\right)\frac{\varsigma^3}{2}+\cdots\right)^2\\&\quad+\widetilde{q}_3\left(\frac{u_1}{2}\varsigma+\left(u_2-\frac{u_1^2}{2}\right)\frac{\varsigma^2}{2}+\left(u_3-u_1u_2+\frac{u_1^3}{4}\right)\frac{\varsigma^3}{2}+\cdots\right)^3+\cdots\\&=1+\frac{\widetilde{q}_1u_1}{2}\varsigma+\left(\frac{1}{2}\left(u_2-\frac{u_1^2}{2}\right)\widetilde{q}_1+\frac{u_1^2}{4}\widetilde{q}_2\right)\varsigma^2\\&\quad+\left(\frac{1}{2}\left(u_3-u_1u_2+\frac{u_1^3}{4}\right)\widetilde{q}_1+\frac{u_1}{2}\left(u_2-\frac{u_1^2}{2}\right)\widetilde{q}_2+\frac{u_1^3}{8}\widetilde{q}_3\right)\varsigma^3+\cdots.\end{aligned} \tag{35}$$

Similarly, there exists a function $\nu\in\mathscr{A}$ such that $|\nu(w)|<1$ in ∇ and $q(w)=\widetilde{q}(\nu(w))$. Therefore, the function

$$d(w)=\frac{1+\nu(w)}{1-\nu(w)}=1+d_1w+d_2w^2+\cdots\in Q. \tag{36}$$

It follows that

$$\nu(w)=\frac{d(w)-1}{d(w)+1}=\frac{\nu_1}{2}w+\left(\nu_2-\frac{\nu_1^2}{2}\right)\frac{w^2}{2}+\left(\nu_3-\nu_1\nu_2+\frac{\nu_1^3}{4}\right)\frac{w^3}{2}+\cdots, \tag{37}$$

and

$$\begin{aligned}\widetilde{q}(\nu(w))&=1+\widetilde{q}_1\left(\frac{\nu_1}{2}w+\left(\nu_2-\frac{\nu_1^2}{2}\right)\frac{w^2}{2}+\left(\nu_3-\nu_1\nu_2+\frac{\nu_1^3}{4}\right)\frac{w^3}{2}+\cdots\right)\\&\quad+\widetilde{q}_2\left(\frac{\nu_1}{2}w+\left(\nu_2-\frac{\nu_1^2}{2}\right)\frac{w^2}{2}+\left(\nu_3-\nu_1\nu_2+\frac{\nu_1^3}{4}\right)\frac{w^3}{2}+\cdots\right)^2\\&\quad+\widetilde{q}_3\left(\frac{\nu_1}{2}w+\left(\nu_2-\frac{\nu_1^2}{2}\right)\frac{w^2}{2}+\left(\nu_3-\nu_1\nu_2+\frac{\nu_1^3}{4}\right)\frac{w^3}{2}+\cdots\right)^3+\cdots\\&=1+\frac{\widetilde{q}_1\nu_1}{2}w+\left(\frac{1}{2}\left(\nu_2-\frac{\nu_1^2}{2}\right)\widetilde{q}_1+\frac{\nu_1^2}{4}\widetilde{q}_2\right)w^2\\&\quad+\left(\frac{1}{2}\left(\nu_3-\nu_1\nu_2+\frac{\nu_1^3}{4}\right)\widetilde{q}_1+\frac{\nu_1}{2}\left(\nu_2-\frac{\nu_1^2}{2}\right)\widetilde{q}_2+\frac{\nu_1^3}{8}\widetilde{q}_3\right)w^3+\cdots.\end{aligned} \tag{38}$$

By (31), (32), (35), and (38), we have

$$[\mu(1-3\vartheta)+2(\vartheta+\beta)]a_2=\frac{u_1\tau}{2}, \tag{39}$$

$$\left[\frac{\mu(\mu-1)(3\vartheta+1)}{2}\right]a_2^2+[\mu(1-4\vartheta)+3\vartheta+6\beta]a_3=\frac{1}{2}\left(u_2-\frac{u_1^2}{2}\right)\tau+\frac{3u_1^2}{4}\tau^2, \tag{40}$$

$$-[\mu(1-3\vartheta)+2(\vartheta+\beta)]a_2=\frac{v_1\tau}{2}, \tag{41}$$

and

$$\left[\frac{\mu^2(3\vartheta+1)+\mu(3-19\vartheta)+12\vartheta+24\beta}{2}\right]a_2^2-[\mu(1-4\vartheta)+3\vartheta+6\beta]a_3, \tag{42}$$

$$=\frac{1}{2}\left(v_2-\frac{v_1^2}{2}\right)\tau+\frac{3v_1^2}{4}\tau^2. \tag{43}$$

From (39) and (41), we get

$$u_1=-v_1 \tag{44}$$

and

$$2[\mu(1-3\vartheta)+2(\vartheta+\beta)]^2a_2^2=\frac{\left(u_1^2+v_1^2\right)\tau^2}{4}. \tag{45}$$

Adding (40)–(42), we have

$$\left[\mu^2(3\vartheta+1)+\mu(1-11\vartheta)+6\vartheta+12\beta\right]a_2^2=\frac{1}{2}(u_2+v_2)\tau-\frac{1}{4}\left(u_1^2+v_1^2\right)\tau+\frac{3}{4}\left(u_1^2+v_1^2\right)\tau^2. \tag{46}$$

By substituting (45) in (46), we get

$$a_2^2=\frac{(u_2+v_2)\tau^2}{2\left(\tau\left[\mu^2(3\vartheta+1)+\mu(1-11\vartheta)+6\vartheta+12\beta\right]+2(1-3\tau)[\mu(1-3\vartheta)+2(\vartheta+\beta)]^2\right)}. \tag{47}$$

Now, using Lemma 2, we get

$$\left|a_2\right|\leq\frac{\sqrt{2}|\tau|}{\sqrt{\tau\left[\mu^2(3\vartheta+1)+\mu(1-11\vartheta)+6\vartheta+12\beta\right]+2(1-3\tau)[\mu(1-3\vartheta)+2(\vartheta+\beta)]^2}}. \tag{48}$$

Then, by subtracting (42) from (40), we get

$$a_3=\frac{(u_2-v_2)\tau}{2[2\mu(1-4\vartheta)+6\vartheta+12\beta]}+a_2^2. \tag{49}$$

Hence, by Lemma 2, we have

$$\begin{aligned}\left|a_3\right|&\leq\frac{\left(\left|u_2\right|+\left|v_2\right|\right)|\tau|}{2[2\mu(1-4\vartheta)+6\vartheta+12\beta]}+\left|a_2\right|^2\\&\leq\frac{2|\tau|}{[2\mu(1-4\vartheta)+6\vartheta+12\beta]}+\left|a_2\right|^2.\end{aligned} \tag{50}$$

By (48), we obtain

$$\left|a_3\right| \leq \frac{|\tau|\left(F + \tau[2\mu(1-4\vartheta)+6\vartheta+12\beta]\right)}{[\mu(1-4\vartheta)+3\vartheta+6\beta]F}, \tag{51}$$

where $F = \tau[\mu^2(3\vartheta+1)+\mu(1-11\vartheta)+6\vartheta+12\beta]+2(1-3\tau)\,[\mu(1-3\vartheta)+2(\vartheta+\beta)]^2$.

From (49), we get

$$a_3 - \varepsilon a_2^2 = \frac{(u_2-v_2)\tau}{2[2\mu(1-4\vartheta)+6\vartheta+12\beta]} + (1-\varepsilon)a_2^2. \tag{52}$$

By substituting (47) in (52), we get

$$\begin{aligned} a_3 - \varepsilon a_2^2 &= \frac{(u_2-v_2)\tau}{2[2\mu(1-4\vartheta)+6\vartheta+12\beta]} + \frac{(1-\varepsilon)(u_2-v_2)\tau^2}{2F} \\ &= \left(\epsilon(\varepsilon) + \frac{|\tau|}{2[2\mu(1-4\vartheta)+6\vartheta+12\beta]}\right)u_2 + \left(\epsilon(\varepsilon) - \frac{|\tau|}{2[2\mu(1-4\vartheta)+6\vartheta+12\beta]}\right)v_2, \end{aligned} \tag{53}$$

where $\epsilon(\varepsilon) = (1-\varepsilon)\tau^2/2F$.

Taking the modulus of (53), we have

$$\left|a_3 - \varepsilon a_2^2\right| \leq \begin{cases} \dfrac{2|\tau|}{[2\mu(1-4\vartheta)+6\vartheta+12\beta]}; & 0 \leq |\epsilon(\varepsilon)| \leq \dfrac{|\tau|}{2[2\mu(1-4\vartheta)+6\vartheta+12\beta]}, \\ 4|\epsilon(\varepsilon)|; & |\epsilon(\varepsilon)| \geq \dfrac{|\tau|}{2[2\mu(1-4\vartheta)+6\vartheta+12\beta]}. \end{cases} \tag{54}$$

□

3. Some Corollary Functions in Subclass $\Upsilon_{\Sigma}^{\vartheta,\beta,\mu}(\widetilde{q})$

In this section, we provide specific examples of subclasses within $\Upsilon_{\Sigma}^{\vartheta,\beta,\mu}(\widetilde{q})$.

Corollary 4. *Let $f(\varsigma) \in \Upsilon_{\Sigma}^{\beta,\mu}(\widetilde{q})$, where β and $\mu \geq 0$. Then,*

$$\left|a_2\right| \leq \sqrt{\frac{2}{F_1}}|\tau|, \left|a_3\right| \leq \frac{|\tau|\left(F_1 + 6\tau[2\beta-\mu+1]\right)}{3[2\beta-\mu+1]F_1}, \tag{55}$$

and for $\varepsilon \in \mathbb{R}$,

$$\left|a_3 - \varepsilon a_2^2\right| \leq \begin{cases} \dfrac{|\tau|}{3[2\beta-\mu+1]}; & 0 \leq |\epsilon(\varepsilon)| \leq \dfrac{|\tau|}{12[2\beta-\mu+1]}, \\ 4|\epsilon(\varepsilon)|; & |\epsilon(\varepsilon)| \geq \dfrac{|\tau|}{12[2\beta-\mu+1]}, \end{cases} \tag{56}$$

where $F_1 = 2\tau[2\mu^2-5\mu+6\beta+3]+8(1-3\tau)[\beta-\mu+1)]^2$ and $\epsilon(\varepsilon) = ((1-\varepsilon)\tau^2)/2F_1$.

Corollary 5. *Let $f(\varsigma) \in \Upsilon_{\Sigma}^{\mu}(\widetilde{q})$, where $\mu \geq 0$. Then,*

$$\left|a_2\right| \leq \sqrt{\frac{2}{F_2}}|\tau|, \left|a_3\right| \leq \frac{|\tau|\left(F_2 + 6\tau[1-\mu]\right)}{3[1-\mu]F_2}, \tag{57}$$

and for $\varepsilon \in \mathbb{R}$,

$$\left|a_3 - \varepsilon a_2^2\right| \leq \begin{cases} \dfrac{|\tau|}{3[1-\mu]}; & 0 \leq |\epsilon(\varepsilon)| \leq \dfrac{|\tau|}{12[1-\mu]}, \\ 4|\epsilon(\varepsilon)|; & |\epsilon(\varepsilon)| \geq \dfrac{|\tau|}{12[1-\mu]}, \end{cases} \tag{58}$$

where $F_2 = 2\tau[2\mu^2-5\mu+3]+8(1-3\tau)[1-\mu]^2$ and $\epsilon(\varepsilon) = (1-\varepsilon)\tau^2/2F_2$.

Corollary 6. *Let $f(\varsigma) \in \Upsilon_{\Sigma}(\widetilde{q})$. Then,*

$$\begin{aligned} \left|a_2\right| &\leq \frac{|\tau|}{\sqrt{4-9\tau}}, \\ \left|a_3\right| &\leq \frac{2(2-3\tau)|\tau|}{3(4-9\tau)}, \end{aligned} \tag{59}$$

and for $\varepsilon \in \mathbb{R}$,

$$\left|a_3 - \varepsilon a_2^2\right| \leq \begin{cases} \dfrac{|\tau|}{3}; & 0 \leq |\epsilon(\varepsilon)| \leq \dfrac{|\tau|}{12}, \\ 4|\epsilon(\varepsilon)|; & |\epsilon(\varepsilon)| \geq \dfrac{|\tau|}{12}, \end{cases} \tag{60}$$

where $\epsilon(\varepsilon) = (1-\varepsilon)\tau/4(4-9\tau)$.

Corollary 7. *Let $f(\varsigma) \in \Upsilon_{\Sigma}^{\vartheta,\mu}(\widetilde{q})$, where $\mu \geq 0$ and $\vartheta \geq 1$. Then,*

$$\left|a_2\right| \leq \sqrt{\frac{2}{F_3}}|\tau|, \left|a_3\right| \leq \frac{|\tau|\left(F_3 + \tau[2\mu(1-4\vartheta)+6\vartheta]\right)}{[\mu(1-4\vartheta)+3\vartheta]F_3}, \tag{61}$$

and for $\varepsilon \in \mathbb{R}$,

$$\left|a_3-\varepsilon a_2^2\right| \leq \begin{cases} \dfrac{2|\tau|}{[2\mu(1-4\vartheta)+6\vartheta]}; & 0 \leq |\epsilon(\varepsilon)| \leq \dfrac{|\tau|}{2[2\mu(1-4\vartheta)+6\vartheta]}, \\ 4|\epsilon(\varepsilon)|; & |\epsilon(\varepsilon)| \geq \dfrac{|\tau|}{2[2\mu(1-4\vartheta)+6\vartheta]}, \end{cases} \quad (62)$$

where $F_3=\tau[\mu^2(3\vartheta+1)+\mu(1-11\vartheta)+6\vartheta]+2(1-3\tau)[\mu(1-3\vartheta)+2\vartheta]^2$ *and* $\epsilon(\varepsilon)=(1-\varepsilon)\tau^2/2F_3$.

Corollary 8. *Let* $f(\varsigma)\in \Upsilon_\Sigma^{\vartheta}(\tilde{q})$, *where* $\vartheta \geq 1$. *Then,*

$$\left|a_2\right| \leq \sqrt{\frac{2}{F_4}}|\tau|, \left|a_3\right| \leq \frac{|\tau|\left(F_4+2\tau[1-\vartheta]\right)}{[1-\vartheta]F_4}, \quad (63)$$

and for $\varepsilon \in \mathbb{R}$,

$$\left|a_3-\varepsilon a_2^2\right| \leq \begin{cases} \dfrac{|\tau|}{1-\vartheta}; & 0 \leq |\epsilon(\varepsilon)| \leq \dfrac{|\tau|}{4[1-\vartheta]}, \\ 4|\epsilon(\varepsilon)|; & |\epsilon(\varepsilon)| \geq \dfrac{|\tau|}{4[1-\vartheta]}, \end{cases} \quad (64)$$

where $F_4=\tau[\mu^2(3\vartheta+1)+\mu(1-11\vartheta)+6\vartheta]+2(1-3\tau)[\mu(1-3\vartheta)+2\vartheta]^2$ *and* $\epsilon(\varepsilon)=(1-\varepsilon)\tau^2/2F_4$.

4. Conclusions

In this study, we introduced a new subclass of standardized analytic functions and bi-univalent functions related to Fibonacci numbers. This subclass is denoted as $\Upsilon_\Sigma^{\vartheta,\beta,\mu}(\tilde{q})$. Bounds for the Taylor coefficients $|a_2|$ and $|a_3|$ were determined, and solutions to Fekete–Szegö functional problems were provided for functions falling under this category. This research could pave the way for further exploration into the development of new categories of analytic and bi-univalent functions associated with Fibonacci numbers.

In future research, the study can be extended to various conic regions, such as three-leaf domains, Ozaki-type bi-close-to-convex, and bi-concave functions involving a modified Caputo's fractional operator linked with a three-leaf function [32].

Data Availability

The data used to support the finding of this study are cited at relevant places within the text as references.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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