Partial Differential Equations in Applied Mathematics 12 (2024) 100950

Contents lists available at ScienceDirect

Partial Differential Equations in Applied Mathematics

journal homepage: www.elsevier.com/locate/padiff

New modifications of natural transform iterative method and q-homotopy analysis method applied to fractional order KDV-Burger and Sawada–Kotera equations

Muayyad Mahmood Khalil [a], Siddiq Ur Rehman [b,c,d], Ali Hasan Ali [e,f,*], Rashid Nawaz [g], Belal Batiha [h]

[a] Department of Mathematics, College of Education for Pure Sciences, Tikrit 34001 University, Tikrit, Iraq
[b] Department of Mathematics, Abdul Wali Khan University Mardan, Mardan 23200, Pakistan
[c] Department of Business Management, Al-imam University College, 34011 Balad, Iraq
[d] Technical Engineering College, Al-Ayen University, 64001, Dhi Qar, Iraq
[e] Institute of Mathematics, University of Debrecen, Pf.400, H-4002, Debrecen, Hungary
[f] Jadara University Research Center, Jadara University, 21110 Irbid, Jordan
[g] Department of Mathematics, University of South Australia, Adelaide, SA 5000, Australia
[h] Mathematics Department, Faculty of Science and Information Technology, Jadara University, Irbid 21110, Jordan

ARTICLE INFO

Keywords:
Fractional calculus
Fifth-order Sawada–Kotera equation
Fractional nonlinear KDV-burgers
q-homotopy analysis method
Caputo operator
Natural transform iterative method

ABSTRACT

This manuscript presents enhanced versions of two methods: the natural transform iterative method (NTIM) and the q-homotopy analysis method (q-HAM). These methods harness concepts from Fractional Calculus, particularly leveraging the Caputo fractional derivative operator, to successfully manage the complexities of fractional-order systems. To validate their accuracy and efficiency, we applied the proposed techniques to FPDEs like the fractional-order KDV-Burger and fifth-order Sawada–Kotera equations. Our outcomes, which closely resemble the exact solutions, demonstrate how useful NTIM and q-HAM are for solving difficult FPDEs and improving the study of fractional calculus.

1. Introduction

The mathematicians Leibniz and L'Hospital discussed fractional calculus (FC) in a 1695 early letter. Calculus in integer order is extended to any order by FC. Because fractional calculus may accurately describe a variety of non-linear processes, numerous writers have recently begun to research this subject. Standard differential equations can be extended to fractional differential equations (FDEs), which have effects on material properties that are both non-local and genetic. Several distinguished researchers explored and debated the concept of (FC), and they developed innovative definitions that provided the foundation for the field.[1–4] Nonlinear fractional differential equations offer a more accurate representation of physical and biological systems by accounting for their complex behaviors, such as non-locality and memory.[5] There is a growing dependence on (FDEs) for both the analysis of dynamical systems and the construction of non-linear models. (FC) theory has been used to study and analyze a variety of phenomena and has been connected to real-world applications, such as financial models,[6] a noisy surroundings,[7] optics,[8] and others.[9,10] The properties of non-linear problems found in nature are represented by a wide range of solutions to (FDEs). The general classes of FDEs, including mixed, sequential, hybrid, and many others that are still relatively unexplored in this field, were reviewed by specialists. Experiential approaches play a major role in helping to comprehend the dynamics of systems that are modeled by different algorithms in the area of nonlinear analysis. The (FDEs) characterizing non-linear systems are difficult to solve accurately, so we use several analytical and numerical methods. In comparison to conventional numerical approaches, computational methods have some advantages and produce analytical solutions. Since discretization is not used, there are no rounding off errors, and little computer power or memory is needed. Researchers have recently discovered how to solve fractional ordinary differential equations using the Adomian decomposition approach.[11] The spectral Tau method[12] is applied to solve a general family of fractional differential equations involving Liouville-Caputo fractional derivatives with a linear functional argument. Ref. 13 combined two different fractional derivatives with the natural decomposition approach, the Caputo–Fabrizio (CF) and the Atangana–Baleanu derivative in the Caputo manner (ABC), to solve the

* Corresponding author.
E-mail address: ali.hasan@science.unideb.hu (A.H. Ali).

https://doi.org/10.1016/j.padiff.2024.100950
Received 14 August 2024; Received in revised form 25 September 2024; Accepted 5 October 2024
Available online 11 October 2024

fractional-order Kaup–Kupershmidt (KK) equation. For the solution of fractional local Poisson equations, Ref. 14 used a hybrid approach. This approach combines two widely used methods, the homotopy perturbation method (HPM) and the Shehu transform. An innovative approch known as the fractional natural decomposition method (FNDM) was used by Ref. 15 to solve numerically for time-fractional coupled Burgers equations. The Elzaki transform is utilized in conjunction with novel approaches, such as the (HPM) and new iteration method, by Ref. 16 to solve the "fractional advection–dispersion equation. Ref. 17 solved the non-linear homogeneous gas dynamics equation using the (q-HAM).

The time fractional non-linear coupled ITO systems and non-linear KDV systems[18] were analyzed by Rashid et al.. Analytical solutions for the drainage equation in time-fractional foam were also achieved by Ref. 19 using (q-HAM). Nonlinear fractional equations were solved using (Oq-HAM) by Ref. 20. Ref. 21 fractional (PDEs) are the first set of problems to which the optimal auxiliary function technique (OAFM) has been applied. Two analytical techniques were employed by Ref. 22 to solve the time-fractional system of PDEs. The Mohand variational iteration transform approach (MVITA) and the new approximate analytical approach (NAAA) are the recently presented methods. Ref. 23 investigated the solution of the nonlinear fractional forced Korteweg–de Vries (FF-KdV) problem using the natural decomposition approach with nonsingular kernel derivatives. Ref. 24 found analytical solutions for the fractional order nonlinear Foam Drainage equation through a novel modification of the well-known OAFM. Ref. 25 developed and examined a comprehensive regulatory framework to control insulin and blood glucose levels in people with diabetes mellitus. They examine and illustrate this novel mathematical model of diabetes in fractional order by means of the ABC fractional derivative. Use the Laplace transform approach for Adomian decomposition method (LADM). Two standard homotopy base techniques were also used to assess results. Ref. 26 investigate for the first time the connections between Lie symmetry groups and basic solutions for a class of conformable time-fractional (PDEs) with varying coefficients. The considered equations are solved in a group-invariant manner by applying the symmetry group technique. Afterward, get the corresponding fundamental solutions for these PDEs by considering the group invariant solutions' inverse Laplace transform.

The Laguerre wavelet-oriented numerical[27] scheme were utilized to solve nonlinear first and second-order delay differential equations (DDEs). A hybrid difference scheme with appropriate quadrature rules on a Shishkin-type mesh is constructed to deal with singularly perturbed Volterra integro-differential equations[28] with delay. The coupled non-linear higher order BVPs were solved using the shooting technique.[29] A novel numerical approach, combining conformable derivatives, finite differences, and non-polynomial splines, were employed to solve the nonlinear inhomogeneous time-fractional Burgers-Huxley equation.[30] The technique of variational iteration and the (ADM), two recently developed methods for analysis, are applied by Ref. 31 to solve linear (FPDEs) that arise in fluid mechanics. In the Caputo meaning, the fractional derivatives are explained. Two alternate approaches in applied mathematics can be utilized to find approximate and analytical solutions for various kinds of (FDEs). Ref. 32 used (HAM) to solve nonlinear (FPDEs). A method is devised depending on the (HAM) to approximate the solution of the following equations: fractional KdV, K(2,2), Burgers, BBM-Burgers, cubic Boussinesq, coupled KdV, and Boussinesq-like B(m,n) with initial conditions imposed by substituting fractional times derivatives for some integer-order time derivatives.

Ref. 33 explored two forms of (FEs) in the Caputo sense using the (q-HAM): the time-fractional Newell–Whitehead equation (FNWE) and the time-fractional generalized Hirota–Satsuma coupled KdV system (HS-cKdVS). These two techniques provide highly accurate and low-processing approximation and precise solutions to difficult non-linear problems, in contrast to other approaches that could need extremely large computations. The natural transform, enhanced by the M-derivative,[34] provides a more generalized framework for fractional transforms, offering new properties and relationships for certain functions. FPDEs have been solved using the NTIM approach, which combines the NIM and natural transform and q-HAM method. We found that the suggested approach was simple to use and provided a good approximation solution for both non-linear and fractional linear differential equations. The proposed techniques have recently been applied to solve non-integer order differential equations by Ref. 35. The same approach was utilized by Ref. 36 to solve the biological population model (FBPM).

The subsequent segments are arranged as follows: We provide some essential concepts and notations for the natural transform and fractional calculus used in the current framework in Section 2. Section 3 briefly discusses the general idea behind the recommended methods. Section 4 is on implementing the recommended methods for fractional-order KDV-Burger and fifth-order Sawada–Kotera equations. In Section 5, numerical findings and observations are provided. Section 6 contains the conclusion, at last.

2. Preliminaries

In this section, we defined several fundamental concepts related to fractional calculus.[37]

2.1. Definition

Using the RL technique $\tilde{\varpi}(\tilde{\kappa},\tilde{\mathrm{T}})\ \epsilon\ C_\eta(\eta\ \geq\ -1)$, the fractional integral for a function is as follows:

$$J^{\phi}\tilde{\varpi}(\tilde{\kappa},\tilde{\mathrm{T}})=\frac{1}{\Gamma(\phi)}\int_0^{\tilde{\mathrm{T}}}(\tilde{\mathrm{T}}-\psi)^{\phi-1}\mathrm{T}\varpi(\tilde{\kappa},\psi)d\psi$$

$$J^{0}\tilde{\varpi}(\tilde{\kappa},\tilde{\mathrm{T}})=\tilde{\varpi}(\tilde{\kappa},\tilde{\mathrm{T}}).$$

2.2. Definition

According to the Caputo method, the fractional-order derivative is as follows

$$D^{\phi}_{\tilde{\mathrm{T}}}\varpi(\tilde{\kappa},\tilde{\mathrm{T}})=\frac{1}{\Gamma(\bar{n}-\phi)}\int_0^{\tilde{\mathrm{T}}}(\tilde{\mathrm{T}}-\psi)^{\bar{n}-\phi-1}\varpi^{\bar{n}}(\tilde{\kappa},\psi)d\psi,\qquad \bar{n}-1\ \leq\ \phi\ \leq\ \bar{n},\ n\ \epsilon\ N. \tag{1}$$

2.3. Definition

The natural transform of $\psi(t)$ is given by

$$N^{+}\Big[\psi(t)\Big]=R(\delta,u)=\frac{1}{u}\int_0^{\infty}e^{\frac{-\delta t}{u}}(\psi(t))dt. \tag{2}$$

Where δ and u are positive transform variables.

2.4. Definition

If $R(\delta,u)$ is the natural transform of $\psi(t)$, then the inverse natural transform is described as follows:

$$N^{-}\Big[R(\delta,u)\Big]=\psi(t)=\frac{1}{2\pi\imath}\int_{c-\imath\infty}^{c+\imath\infty}e^{\frac{\delta t}{u}}(R(\delta,u))d\delta. \tag{3}$$

The integral is represented in the complex plane with $\delta=a+b\imath$ along the line $\delta=c$, where $c\ \epsilon\ \mathbb{R}$.

2.5. Definition

The nth derivative of the natural transform of $\psi(t)$ is given

$$N^{+}\Big[\psi^{n}(t)\Big]=R_n(\delta,u)=\frac{\delta^n}{u^n}R(\delta,u)-\Sigma_{k=0}^{n-1}\frac{\delta^{n-k-1}}{u^{n-k}}\Big[\psi^{n}(0)\Big],\quad n\ \geq\ 1. \tag{4}$$

3. Methodology

Due to their distinct benefits, The KDV-Burger and Sawada–Kotera equations can be solved using NTIM and q-HAM since they can manage the complexity and nonlinearity that come with fractional partial differential equations. These techniques are ideal for capturing the complex dynamics of such nonlinear systems because they offer great precision and convergence without the need for discretization or linearization.

3.1. q-HAM

Take into account the following differential equation.[38]

$$\psi\left(\beta(\tilde{\gamma},\tilde{\sigma})\right)-\varphi(\tilde{\gamma},\tilde{\sigma})=0. \tag{5}$$

where $\varphi(\tilde{\gamma},\tilde{\sigma})$ is a known function, $\beta(\tilde{\gamma},\tilde{\sigma})$ is an unspecified function, ψ is a nonlinear operator, $(\tilde{\gamma},\tilde{\sigma})$ and signifies independent variables. Let us create the aforementioned equation for so-called zero-order deformation:

$$(1-\tilde{n}\varrho)\zeta\left[\psi(\tilde{\gamma},\tilde{\sigma};\varrho)-\beta_0(\tilde{\gamma},\tilde{\sigma})\right]=\varrho\hbar H(\tilde{\gamma},\tilde{\sigma})\left[\psi\left(\beta(\tilde{\gamma},\tilde{\sigma};\varrho)\right)-\varphi(\tilde{\gamma},\tilde{\sigma})\right], \tag{6}$$

where $n\geq 1$, $\varrho\ \epsilon\ [0,\frac{1}{\tilde{n}}]$ indicates the deformation equation, ζ is the auxiliary linear operator with the condition $\zeta[\varphi]=0$, when $\varphi=0$, $\hbar\neq 0$ is an auxiliary parameter, signifies a non-zero auxiliary function.

$$\psi(\tilde{\gamma},\tilde{\sigma};0)=\beta_0(\tilde{\gamma},\tilde{\sigma}),\qquad \psi(\tilde{\gamma},\tilde{\sigma};\frac{1}{\tilde{n}})=\beta(\tilde{\gamma},\tilde{\sigma}), \tag{7}$$

respectively. The solution so differs from the first guess $\beta_0(\tilde{\gamma},\tilde{\sigma})$ to the solution $\beta(\tilde{\gamma},\tilde{\sigma})$ as rises from 0 to $\frac{1}{\tilde{n}}$. We may suppose that all of them can be wisely selected if given the freedom to pick $\beta_0(\tilde{\gamma},\tilde{\sigma})$, ζ, $\hbar$, $H(\tilde{\gamma},\tilde{\sigma})$ such that the solution $\psi(\tilde{\gamma},\tilde{\sigma};\varrho)$ of Eq. (2) exists for $\varrho\ \epsilon\left[0,\frac{1}{\tilde{n}}\right]$. One has expanded $\psi(\tilde{\gamma},\tilde{\sigma};\varrho)$ in the Taylor series.

$$\psi(\tilde{\gamma},\tilde{\sigma};\varrho)=\beta_0(\tilde{\gamma},\tilde{\sigma})+\sum_{\eta=1}^{\infty}\beta_\eta(\upsilon)\varrho^\eta. \tag{8}$$

Where

$$\beta_\eta(\tilde{\gamma},\tilde{\sigma})=\frac{1}{\eta!}\frac{\partial^\eta\psi(\tilde{\gamma},\tilde{\sigma};\varrho)}{\partial\varrho^\eta}|_{\varrho=0}. \tag{9}$$

Assume $\beta_0(\tilde{\gamma},\tilde{\sigma})$, ζ, $\hbar$, $H(\tilde{\gamma},\tilde{\sigma})$ are selected in such a way that they cause the series (8) to converge at $\varrho=\frac{1}{\tilde{n}}$ and:

$$\beta(\tilde{\gamma},\tilde{\sigma})=\psi(\tilde{\gamma},\tilde{\sigma};\frac{1}{\tilde{n}})=\beta_0(\tilde{\gamma},\tilde{\sigma})+\sum_{\eta=1}^{\infty}\beta_\eta(\tilde{\gamma},\tilde{\sigma})(\frac{1}{\tilde{n}})^\eta. \tag{10}$$

The vector's definition $\beta_r(\tilde{\gamma},\tilde{\sigma})=\{\beta_0(\tilde{\gamma},\tilde{\sigma}),\beta_1(\tilde{\gamma},\tilde{\sigma}),\beta_2(\tilde{\gamma},\tilde{\sigma}),\dots,\beta_r(\tilde{\gamma},\tilde{\sigma})\}$, the η^{th} order deformation equation can be obtained by differentiating (6)η times with regard to ϱ, setting $\varrho=0$, and then dividing them by $\varrho!$.

$$\zeta\left[\beta_\eta(\tilde{\gamma},\tilde{\sigma})-\kappa_\eta\beta_{\eta-1}(\tilde{\gamma},\tilde{\sigma})\right]=\hbar H(\tilde{\gamma},\tilde{\sigma})R_\eta(\beta_{\eta-1}(\tilde{\gamma},\tilde{\sigma})). \tag{11}$$

Where

$$R_\eta(\beta_{\eta-1}(\tilde{\gamma},\tilde{\sigma}))=\frac{1}{(\eta-1)!}\frac{\partial^{\eta-1}(\psi\left(\beta(\tilde{\gamma},\tilde{\sigma})\right)-\varphi(\tilde{\gamma},\tilde{\sigma}))}{\partial\varrho^{\eta-1}}|_{\varrho=0} \tag{12}$$

and

$$\kappa_\eta=\begin{cases}0, & \eta\leq 1;\\ n, & \text{otherwise}\end{cases} \tag{13}$$

It should be highlighted that the linear Eq. (11) with linear boundary conditions derived from the original issue governs for $\beta_{\eta-1}(\tilde{\gamma},\tilde{\sigma})$ for $\eta\geq 1$. Because of the factor $(\frac{1}{\tilde{n}})^\eta$, there are higher chances for convergence[39] to happen or it may even happen considerably faster than with the usual HAM. Remember that the standard HAM can be attained in the case of $\tilde{n}=1$ in (6).

3.2. NTIM

Take into consideration FDE of the following form[37]:

$$D_{\tilde{\sigma}}^\eta\left(\bar{\Theta}(\tilde{\gamma},\tilde{\sigma})\right)=f(\tilde{\gamma},\tilde{\sigma})+\varpi\left(\bar{\Theta}(\tilde{\gamma},\tilde{\sigma})\right)+\tilde{\psi}\left(\bar{\Theta}(\tilde{\gamma},\tilde{\sigma})\right),\ \tilde{\gamma},\tilde{\sigma}\geq 0,\tilde{m}-1\leq\beta\leq\tilde{m}, \tag{14}$$

where the caputo fractional derivative of order $\eta,\tilde{m}\ \epsilon\ \mathbf{N}$ and $(\gamma=y_1,y_2,\dots,y_{\tilde{m}})$ are represented by $D_{\tilde{\sigma}}^\eta$. $\tilde{\varpi}$ and $\tilde{\psi}$, respectively, stand for the linear and non-linear functions. The function $f(\tilde{\gamma},\tilde{\sigma})$ is known. The corresponding initial condition are given as follows

$$\bar{\Theta}(\tilde{\gamma},0)=\tilde{\varphi}(\tilde{\gamma}). \tag{15}$$

Using the natural transform, Eq. (14) has undergone a natural transformation.

$$\aleph^+\left[D_{\tilde{\sigma}}^\eta\left[\bar{\Theta}\left(\tilde{\gamma},\tilde{\sigma}\right)\right]\right]=\aleph^+\left[f(\tilde{\gamma},\tilde{\sigma})\right]+\aleph^+\left[\varpi\left(\bar{\Theta}(\tilde{\gamma},\tilde{\sigma})\right)+\tilde{\psi}\left(\bar{\Theta}(\tilde{\gamma},\tilde{\sigma})\right)\right]. \tag{16}$$

Using the natural transform differentiation characteristic, Eq. (16) can be represented in one way as

$$\frac{s^\eta}{\bar{\Theta}^\eta}\aleph^+\left[\bar{\Theta}\left(\tilde{\gamma},\tilde{\sigma}\right)\right]-\frac{s^{\eta-1}}{\bar{\Theta}^\eta}\bar{\Theta}(\tilde{\gamma},0)=\aleph^+\left[f(\tilde{\gamma},\tilde{\sigma})\right]+\aleph^+\left[\varpi\left(\bar{\Theta}(\tilde{\gamma},\tilde{\sigma})\right)+\tilde{\psi}\left(\bar{\Theta}(\tilde{\gamma},\tilde{\sigma})\right)\right]. \tag{17}$$

When we arrange Eq. (17), we obtain

$$\aleph^+\left[\bar{\Theta}\left(\tilde{\gamma},\tilde{\sigma}\right)\right]=\frac{\varphi(\tilde{\gamma})}{s}+\frac{\bar{\Theta}^\eta}{s^\eta}\left(\aleph^+[f(\tilde{\gamma},\tilde{\sigma})]\right)+\frac{\bar{\Theta}^\eta}{s^\eta}\left(\aleph^+\left[\varpi\left(\bar{\Theta}(\tilde{\gamma},\tilde{\sigma})\right)+\tilde{\psi}\left(\bar{\Theta}(\tilde{\gamma},\tilde{\sigma})\right)\right]\right). \tag{18}$$

When computing the NTIM solution, $\bar{\Theta}(\tilde{\gamma},\tilde{\sigma})$ is extended as

$$\bar{\Theta}(\tilde{\gamma},\tilde{\sigma})=\sum_{\tilde{i}=0}^{\infty}\bar{\Theta}_i(\tilde{\gamma},\tilde{\sigma}). \tag{19}$$

and the non-linear term $\tilde{\psi}\left(\bar{\Theta}(\tilde{\gamma},\tilde{\sigma})\right)$ is defined as

$$\tilde{\psi}\left(\sum_{\tilde{m}=0}^{\infty}\bar{\Theta}_{\tilde{m}}(\tilde{\gamma},\tilde{\sigma})\right)=\tilde{\psi}\left(\bar{\Theta}_0(\tilde{\gamma},\tilde{\sigma})\right)+\sum_{\tilde{m}=1}^{\infty}\left\{\tilde{\psi}\left(\sum_{\tilde{j}=0}^{i}\bar{\Theta}_{\tilde{j}}(\tilde{\gamma},\tilde{\sigma})\right)-\tilde{\psi}\left(\sum_{\tilde{j}=0}^{\tilde{j}-1}\bar{\Theta}_{\tilde{j}}(\tilde{\gamma},\tilde{\sigma})\right)\right\}. \tag{20}$$

Using Eq. (19) and Eq. (20) in Eq. (18), we obtain

$$\aleph^+\left[\sum_{\tilde{i}=1}^{\infty}\bar{\Theta}_{\tilde{i}}\right]=\frac{\varphi(\tilde{\gamma})}{s}+\frac{\bar{\Theta}^\eta}{s^\eta}\left(\aleph^+[f(\tilde{\gamma},\tilde{\sigma})]\right)+\frac{\bar{\Theta}^\eta}{s^\eta}\left[\aleph^+\left[\sum_{\tilde{m}=0}^{\infty}\varpi\left(\bar{\Theta}_{\tilde{m}}\right)+\tilde{\psi}(\bar{\Theta}_0)+\sum_{\tilde{m}=1}^{\infty}\left\{\tilde{\psi}(\sum_{\tilde{j}=0}^{\tilde{m}}\bar{\Theta}_{\tilde{j}})-\tilde{\psi}(\sum_{\tilde{j}=0}^{\tilde{m}-1}\bar{\Theta}_{\tilde{j}})\right\}\right]\right]. \tag{21}$$

Utilizing the recursive relationship

$$\aleph^+\left[\bar{\Theta}_0(\tilde{\gamma},\tilde{\sigma})\right]=\frac{\varphi(\tilde{\gamma})}{s}+\frac{\bar{\Theta}^\eta}{s^\eta}\left(\aleph^+[f(\tilde{\gamma},\tilde{\sigma})]\right),$$

$$\aleph^+\left[\bar{\Theta}_1(\tilde{\gamma},\tilde{\sigma})\right]=\frac{\bar{\Theta}^\eta}{s^\eta}\left[\varpi(\bar{\Theta}_0(\tilde{\gamma},\tilde{\sigma}))+\tilde{\psi}(\bar{\Theta}_0(\tilde{\gamma},\tilde{\sigma}))\right],$$

$$\aleph^+\left[\bar{\Theta}_2(\tilde{\gamma},\tilde{\sigma})\right]=\frac{\bar{\Theta}^\eta}{s^\eta}\left[\varpi(\bar{\Theta}_1(\tilde{\gamma},\tilde{\sigma}))+\tilde{\psi}(\bar{\Theta}_0(\tilde{\gamma},\tilde{\sigma})+\bar{\Theta}_1(\tilde{\gamma},\tilde{\sigma}))-\tilde{\psi}(\bar{\Theta}_0(\tilde{\gamma},\tilde{\sigma}))\right],$$

$$\vdots$$

$$\aleph^+\left[\bar{\Theta}_{\tilde{i}+1}(\tilde{\gamma},\tilde{\sigma})\right]=\frac{\bar{\Theta}^\eta}{s^\eta}\Big[\varpi(\bar{\Theta}_{\tilde{i}}(\tilde{\gamma},\tilde{\sigma}))+\tilde{\psi}(\bar{\Theta}_0(\tilde{\gamma},\tilde{\sigma})+\bar{\Theta}_1(\tilde{\gamma},\tilde{\sigma})+\cdots+\bar{\Theta}_{\tilde{i}}(\tilde{\gamma},\tilde{\sigma}))-\tilde{\psi}(\bar{\Theta}_0(\tilde{\gamma},\tilde{\sigma})+\bar{\Theta}_1(\tilde{\gamma},\tilde{\sigma})+\cdots+\bar{\Theta}_{\tilde{i}}(\tilde{\gamma},\tilde{\sigma}))\Big],\tilde{i}\geq 0. \tag{22}$$

Using Eq. (22) as an inverse natural transform, we get

$$\bar{\Theta}_0(\tilde{\gamma},\tilde{\sigma})=\aleph^-\left[\frac{\varphi(\tilde{\gamma})}{s}+\frac{\bar{\Theta}^\eta}{s^\eta}\left(\aleph^+[f(\tilde{\gamma},\tilde{\sigma})]\right)\right]$$

$$\bar{\Theta}_1(\tilde{\gamma},\tilde{\sigma})=\aleph^-\left[\frac{\bar{\Theta}^\eta}{s^\eta}\left[\varpi(\bar{\Theta}_0(\tilde{\gamma},\tilde{\sigma}))+\tilde{\psi}(\bar{\Theta}_0(\tilde{\gamma},\tilde{\sigma}))\right]\right],$$

$$\bar{\Theta}_1(\tilde{\gamma},\tilde{\sigma}) = \aleph^-\left[\frac{\bar{\Theta}^\eta}{s^\eta}\left[\varpi(\bar{\Theta}_1(\tilde{\gamma},\tilde{\sigma})) + \tilde{\psi}(\bar{\Theta}_0(\tilde{\gamma},\tilde{\sigma}) + \bar{\Theta}_1(\tilde{\gamma},\tilde{\sigma})) - \tilde{\psi}(\bar{\Theta}_0(\tilde{\gamma},\tilde{\sigma}))\right]\right],$$

$$\vdots$$

$$\begin{aligned}\bar{\Theta}_{\tilde{i}+1}(\tilde{\gamma},\tilde{\sigma}) =& \aleph^-\left[\frac{\bar{\Theta}^\eta}{s^\eta}\left[\varpi(\bar{\Theta}_{\tilde{i}}(\tilde{\gamma},\tilde{\sigma})) + \tilde{\psi}(\bar{\Theta}_0(\tilde{\gamma},\tilde{\sigma}) + \bar{\Theta}_1(\tilde{\gamma},\tilde{\sigma}) + \cdots + \bar{\Theta}_{\tilde{i}}(\tilde{\gamma},\tilde{\sigma}))\right.\right.\\ &\left.\left.- \tilde{\psi}(\bar{\Theta}_0(\tilde{\gamma},\tilde{\sigma}) + \bar{\Theta}_1(\tilde{\gamma},\tilde{\sigma}) + \cdots + \bar{\Theta}_{\tilde{i}}(\tilde{\gamma},\tilde{\sigma}))\right]\right], \tilde{i} \geq 0.\end{aligned} \tag{23}$$

After adding up all the components, the approximate solutions to Eqs. (14) and (16) using NTIM are given as follows

$$\bar{\Theta}(\tilde{\gamma},\tilde{\sigma}) = \bar{\Theta}_0(\tilde{\gamma},\tilde{\sigma}) + \bar{\Theta}_1(\tilde{\gamma},\tilde{\sigma}) + \cdots + \bar{\Theta}_{\tilde{m}-1}(\tilde{\gamma},\tilde{\sigma}),\ \ \tilde{m}\ \epsilon\ N. \tag{24}$$

Ref. 40 assert that NTIM convergence is equal to NIM convergence.

4. Analysis of the methods

This part involves testing the suggested approach for fifth-order Sawada–Kotera equations and time-fractional order nonlinear KDV-Burger equations. Mathematica 13.2 has been utilized for an extensive computation.

4.1. Time fractional nonlinear KDV-Burger equation

Examine the time-fractional nonlinear KDV-Burgers equation.

$$D_\sigma^\beta \Phi(\gamma,\sigma) - 6\Phi(\gamma,\sigma)\Phi_\gamma(\gamma,\sigma) + \Phi_{\gamma\gamma\gamma}(\gamma,\sigma) = 0, \quad 0 \leq \beta \leq 1, \tag{25}$$

with corresponding condition

$$\Phi(\gamma,0) = -\frac{2e^{k\gamma}k^2}{(1+e^{k\gamma})^2}. \tag{26}$$

The exact solution of Eq. (25) for $\beta = 1$ is given as follows

$$\Phi(\gamma,\sigma) = -\frac{(2e^{k(\gamma-k^2\sigma)}k^2)}{(1+e^{k(\gamma-k^2\sigma)})^2}. \tag{27}$$

q-HAM Solution

Applying q-HAM, we can solve Eq. (25) and obtain the iterative terms of $\Phi(\gamma,\sigma)$ as follows:

$$\Phi_0(\gamma,\sigma) = -\frac{2k^2e^{k\gamma}}{\left(e^{k\gamma}+1\right)^2},$$

$$\Phi_1(\gamma,\sigma) = \frac{4hk^5\sigma^\beta\sinh^4\left(\frac{k\tilde{\gamma}}{2}\right)csch^3(k\gamma)}{\beta\Gamma(\beta)},$$

$$\Phi_2(\gamma,\sigma) = \frac{1}{4}hk^5\sigma^\beta sech^4\left(\frac{k\gamma}{2}\right)\left(\frac{(h+n)\sinh(k\gamma)}{\Gamma(\beta+1)} - \frac{hk^3\sigma^\beta(\cosh(k\gamma)-2)}{\Gamma(2\beta+1)}\right),$$

taking the sum of all components we get $\Phi_{q-HAM}(\gamma,\sigma)$

$$\Phi_{q-HAM}(\gamma,\sigma) = \Phi_0(\gamma,\sigma) + \Phi_1(\gamma,\sigma) + \Phi_2(\gamma,\sigma), \tag{28}$$

$$\begin{aligned}&\Phi_{q-HAM}(\gamma,\sigma)\\ &= \frac{1}{4}k^2sech^2\left(\frac{k\gamma}{2}\right)\left(\frac{hk^3\sigma^\beta\left(\frac{2(h+2n)\tanh\left(\frac{k\gamma}{2}\right)}{\Gamma(\beta+1)} - \frac{hk^3\sigma^\beta(\cosh(k\gamma)-2)sech^2\left(\frac{k\gamma}{2}\right)}{\Gamma(2\beta+1)}\right)}{n^2} - 2\right).\end{aligned} \tag{29}$$

NTIM Solution

On solving Eq. (25) by utilizing NTIM, we get the iterative terms of $\Phi(\gamma,\sigma)$ as follows,

$$\Phi_0(\gamma,\sigma) = -\frac{2k^2e^{k\gamma}}{\left(e^{k\tilde{\gamma}}+1\right)^2},$$

$$\Phi_1(\gamma,\sigma) = -\frac{4k^5\sigma^\beta\sinh^4\left(\frac{k\gamma}{2}\right)csch^3(k\gamma)}{\Gamma(\beta+1)},$$

$$\begin{aligned}\Phi_2(\gamma,\sigma) =& \frac{2k^5e^{k\gamma}\sigma^\beta}{\left(e^{k\gamma}+1\right)^7}\left(-\frac{12k^6\Gamma(2\beta+1)e^{k\gamma}(e^{k\gamma}-1)(-4e^{k\gamma}+e^{2k\gamma}+1)\sigma^{2\beta}}{\Gamma(\beta+1)^2\Gamma(3\beta+1)}\right.\\ &- \frac{k^3(e^{k\gamma}+1)^3(-4e^{k\gamma}+e^{2k\gamma}+1)\sigma^\beta}{\Gamma(2\beta+1)}\\ &\left.+ \frac{(e^{k\gamma}-1)(e^{k\gamma}+1)^2(-10e^{k\gamma}+e^{2k\gamma}+1)}{\Gamma(\beta+1)}\right),\end{aligned}$$

by adding all the components $\Phi_{NTIM}(\gamma,\sigma)$ are produced

$$\Phi_{NTIM}(\gamma,\sigma) = \Phi_0(\gamma,\sigma) + \Phi_1(\gamma,\sigma) + \Phi_2(\gamma,\sigma), \tag{30}$$

$$\begin{aligned}\Phi_{NTIM}(\gamma,\sigma) =& \frac{1}{4}k^2sech^2\left(\frac{k\gamma}{2}\right)\\ &\times\left(-\frac{3k^9\Gamma(2\beta+1)\sigma^{3\beta}(\cosh(k\gamma)-2)\tanh\left(\frac{k\gamma}{2}\right)sech^4\left(\frac{k\gamma}{2}\right)}{\Gamma(\beta+1)^2\Gamma(3\beta+1)}\right.\\ &\left.+ \frac{k^6\sigma^{2\beta}\left(3sech^2\left(\frac{k\gamma}{2}\right)-2\right)}{\Gamma(2\beta+1)} - \frac{6k^3\sigma^\beta\tanh\left(\frac{k\gamma}{2}\right)sech^2\left(\frac{k\gamma}{2}\right)}{\Gamma(\beta+1)} - 2\right).\end{aligned} \tag{31}$$

4.2. Time-fractional fifth-order Sawada–Kotera equation

Examine the Fifth-Order Sawada–Kotera Equation for Time-Fractionality.

$$\begin{aligned}&D_\sigma^\beta\Phi(\gamma,\sigma) + 45\Phi^2(\gamma,\sigma)\Phi_\gamma(\gamma,\sigma) + 15\Phi_\gamma(\gamma,\sigma)\Phi_{\gamma\gamma}(\gamma,\sigma) + 15\Phi(\gamma,\mathfrak{J})\Phi_{\gamma\gamma\gamma}(\gamma,\sigma)\\ &\quad + \Phi_{\gamma\gamma\gamma\gamma\gamma}(\gamma,\sigma) = 0, \quad 0 \leq \beta \leq 1,\end{aligned} \tag{32}$$

corresponding initial conditions are given as follows

$$\Phi(\gamma,0) = 2k^2sech^2(k(\gamma-\lambda)), \tag{33}$$

The exact solution of Eq. (32) when $\beta = 1$ is

$$\Phi(\gamma,\sigma) = 2k^2sech^2\left(k\left(x-16k^4\lambda\sigma\right)\right), \tag{34}$$

q-HAM Solution

$$\Phi_0(\gamma,\sigma) = 2k^2sech^2(k(\gamma-\lambda)), \tag{35}$$

$$\Phi_1(\gamma,\sigma) = -\frac{64hk^7\sigma^\beta\tanh(k(\gamma-\lambda))sech^2(k(\gamma-\lambda))}{\beta\Gamma(\beta)}, \tag{36}$$

$$\begin{aligned}\Phi_2(\gamma,\sigma) =& -\frac{2^{5-2\beta}hk^7\sigma^\beta sech^4(k(\gamma-\lambda))}{\beta\Gamma(\beta)\Gamma\left(\beta+\frac{1}{2}\right)}\\ &\times\left(4^\beta\Gamma\left(\beta+\frac{1}{2}\right)(h+n)\sinh(2k(\gamma-\lambda))\right.\\ &\left.- 32\sqrt{\pi}hk^5\sigma^\beta(\cosh(2k(\gamma-\lambda))-2)\right),\end{aligned} \tag{37}$$

by adding all the components $\Phi_{q-HAM}(\gamma,\sigma)$ is generated

$$\Phi_{q-HAM}(\gamma,\sigma) = \Phi_0(\gamma,\sigma) + \Phi_1(\gamma,\sigma) + \Phi_2(\gamma,\sigma), \tag{38}$$

Table 1
The exact solution of the fractional order KDV Burger problem $\Phi(\gamma,\sigma)$ where $\beta=1, k=0.1, h=-1$ and $n=1$ is compared with NTIM and q-HAM.

σ	γ	$\Phi_{NTIM}(\gamma,\sigma)$	$\Phi_{q-HAM}(\gamma,\sigma)$	$\Phi_{Exact}(\gamma,\sigma)$	Abs Diff NTIM	Abs Diff q-HAM
0.1	0.01	−0.005	−0.005	−0.005	5×10^{-10}	1.1459×10^{-18}
0.1	0.035	−0.00499999	−0.00499999	−0.00499999	1.74998×10^{-9}	3.45384×10^{-18}
0.1	0.06	−0.00499996	−0.00499996	−0.00499996	2.99992×10^{-9}	6.48997×10^{-18}
0.1	0.085	−0.00499992	−0.00499991	−0.00499991	4.24978×10^{-9}	6.44754×10^{-18}
0.3	0.01	−0.005	−0.005	−0.005	1.5×10^{-9}	1.87872×10^{-17}
0.3	0.035	−0.00499999	−0.00499999	−0.00499999	5.24995×10^{-9}	7.74923×10^{-17}
0.3	0.06	−0.00499997	−0.00499996	−0.00499996	8.99977×10^{-9}	1.33097×10^{-16}
0.3	0.085	−0.00499993	−0.00499992	−0.00499992	1.27493×10^{-8}	1.8708×10^{-16}
0.5	0.01	−0.005	−0.005	−0.005	2.5×10^{-9}	9.06158×10^{-17}
0.5	0.035	−0.005	−0.00499999	−0.00499999	8.74992×10^{-9}	3.50904×10^{-16}
0.5	0.06	−0.00499998	−0.00499996	−0.00499996	1.49996×10^{-8}	6.1359×10^{-16}
0.5	0.085	−0.00499994	−0.00499992	−0.00499992	2.12489×10^{-8}	8.70942×10^{-16}

$$\begin{aligned}\Phi_{q-HAM}(\gamma,\sigma)=&k^2 sech^4(k(\gamma-\lambda))\Bigg(-\frac{2^{5-2\beta}hk^5\sigma^\beta}{\beta n^2\Gamma(\beta)\Gamma\left(\beta+\frac{1}{2}\right)}\\&\times\left(4^\beta\Gamma\left(\beta+\frac{1}{2}\right)(h+2n)\sinh(2k(\gamma-\lambda))\right.\\&\left.-32\sqrt{\pi}hk^5\sigma^\beta(\cosh(2k(\gamma-\lambda))-2)\right)\\&+\cosh(2k(\gamma-\lambda))+1\Bigg).\end{aligned}\tag{39}$$

NTIM Solution

$$\Phi_0(\gamma,\sigma)=2k^2 sech^2(k(\gamma-\lambda)),\tag{40}$$

$$\Phi_1(\gamma,\sigma)=\frac{64k^7\sigma^\beta\tanh(k(\gamma-\lambda))sech^2(k(\gamma-\lambda))}{\Gamma(\beta+1)},\tag{41}$$

$$\begin{aligned}\Phi_2(\gamma,\sigma)=&4k^7\sigma^\beta sech^4(k(\gamma-\lambda))\Bigg(\frac{256k^5\sigma^\beta(\cosh(2k(\gamma-\lambda))-2)}{\Gamma(2\beta+1)}\\&-\frac{(302\sinh(k(\gamma-\lambda))-57\sinh(3k(\gamma-\lambda))+\sinh(5k(\gamma-\lambda)))sech^3(k(\tilde{\gamma}-\lambda))}{\Gamma(\beta+1)}\\&+\frac{1}{\Gamma(\beta+1)^2\Gamma(3\beta+1)}\Big(15360k^{10}\Gamma(2\beta+1)\sigma^{2\beta}(35\sinh(k(\gamma-\lambda))\\&-12\sinh(3k(\gamma-\lambda))\\&+\sinh(5k(\gamma-\lambda)))sech^5(k(\gamma-\lambda))\Big)+\frac{1}{\Gamma(\beta+1)^3\Gamma(4\beta+1)}\Big(2949120k^{15}\Gamma(3\beta\\&+1)\sigma^{3\beta}(\cosh(2k(\gamma-\lambda))-2)\tanh^2(k(\gamma-\lambda))sech^4(k(\gamma-\lambda))\Big)\Bigg),\end{aligned}\tag{42}$$

by adding all the components $\Phi_{NTIM}(\gamma,\sigma)$ is generated

$$\Phi_{NTIM}(\gamma,\sigma)=\Phi_0(\gamma,\sigma)+\Phi_1(\gamma,\sigma)+\Phi_2(\gamma,\sigma),\tag{43}$$

$$\begin{aligned}\Phi_{NTIM}(\gamma,\sigma)=&2k^2 sech^2(k(\gamma-\lambda))\\&\times\Bigg(1+\frac{512k^{10}\sigma^{2\beta}(\cosh(2k(\gamma-\lambda))-2)sech^2(k(\gamma-\lambda))}{\Gamma(2\beta+1)}\\&+\frac{120k^5\sigma^\beta(\sinh(3k(\gamma-\lambda))-5\sinh(k(\gamma-\lambda)))sech^5(k(\gamma-\lambda))}{\Gamma(\beta+1)}\\&+\frac{1}{\Gamma(\beta+1)^2\Gamma(3\beta+1)}\Big(30720k^{15}\Gamma(2\beta+1)\sigma^{3\beta}(35\sinh(k(\gamma-\lambda))\\&-12\sinh(3k(\gamma\\&-\lambda))+\sinh(5k(\gamma-\lambda)))sech^7(k(\gamma-\lambda))\Big)\\&+\frac{1}{\Gamma(\beta+1)^3\Gamma(4\beta+1)}\Big(5898240k^{20}\Gamma(3\beta\\&+1)\sigma^{4\beta}(\cosh(2k(\gamma-\lambda))-2)\tanh^2(k(\gamma-\lambda))sech^6(k(\gamma-\lambda))\Big)\Bigg).\end{aligned}\tag{44}$$

5. Results & discussion

Tables 1 and 2 show the comparative numerical solutions obtained by q-HAM and NTIM, along with the exact solution for the KdV equation $\Phi(\gamma,\mathfrak{I})$, while keeping $\beta=1$, $k=0.1$, $h=-1$, and $n=1$. The absolute differences obtained by NTIM, q-HAM, and OAFM are given in Table 3 for the KdV equation $\Phi(\gamma,\mathfrak{I})$, with the same parameter values. This indicates that the solutions acquired by NTIM and q-HAM are more appropriate than those obtained by OAFM.

Tables 4 and 5 represent solutions obtained by NTIM and q-HAM for the fifth-order Sawada–Kotera equation $\Phi(\gamma,\mathfrak{I})$, again with $\beta=1$, $k=0.1$, $h=-1$, and $n=1$. The absolute differences acquired by NTIM, q-HAM, and OAFM are given in Table 6. The 3D representation of the solutions obtained by NTIM, q-HAM, and the exact solution for the KdV-Burger equation, when $\beta=1$, $k=0.1$, $h=-1$, and $n=1$, are shown in Figs. 1(a)–1(c). Graphs 2 and 3 show how changing the parameter β affects the solutions of the KdV-Burgers equation. As β increases, the solution curves become more steep and concentrated near the peak. This indicates that a higher β value results in a stronger and more concentrated wave-like phenomenon. Figs. 6(a)–6(c) illustrates the 3D graph of solutions for the fifth-order Sawada–Kotera equation $\Phi(\gamma,\mathfrak{I})$, also with $\beta=1$, $k=0.1$, $h=-1$, and $n=1$. Variations in the parameter β are shown in Figs. 7 and 8. The KdV-Burger and fifth-order Sawada–Kotera equations are mathematical models that are non-linear and explain wave behaviors in different physical systems. The KdV-Burger equation represents water waves and traffic flow, whereas the Sawada–Kotera equation is employed in plasma physics and optical fibers. The KdV-Burger and Sawada–Kotera equations both have connections to delay differential equations. Time delays present in these equations have the potential to impact wave patterns. By investigating delay differential equations, we can acquire a more profound understanding. comprehension of the KdV-Burger and Sawada–Kotera equations as well as their practical uses. Understanding how the parameter β affects the solutions can aid in analyzing and managing the behavior of these physical systems.

Figs. 4 and 5 display the absolute discrepancies in the estimated solutions from NTIM, q-HAM and exact solutions for the KdV-Burgers equation. Likewise, Figs. 9 and 10 show the absolute discrepancies in the Sawada–Kotera equation, contrasting the exact results with those estimated using NTIM and q-HAM. These error plots are necessary to evaluate the convergence and accuracy of both techniques. A lower absolute error implies a better approximation, indicating that these methods can effectively model the behavior of the corresponding non-linear partial differential equations. Through analyzing these error graphs, we can acquire valuable understanding of the strengths and limitations of each method and assess their effectiveness in addressing different nonlinear issues.

Table 2
The exact solution of the fractional order KDV Burger problem $\Phi(\gamma,\sigma)$ where $\beta=1, k=0.1, h=-1$ and $n=1$ is compared with NTIM and q-HAM.

γ	σ	$\Phi_{NTIM}(\gamma,\sigma)$	$\Phi_{q-HAM}(\gamma,\sigma)$	$\Phi_{Exact}(\gamma,\sigma)$	Abs Diff NTIM	Abs Diff q-HAM
0.1	0.1	−0.00499988	−0.00499988	−0.00499988	4.99965×10^{-9}	8.72559×10^{-18}
0.1	0.2	−0.00499989	−0.00499988	−0.00499988	9.99929×10^{-9}	6.65322×10^{-17}
0.1	0.3	−0.0049999	−0.00499988	−0.00499988	1.49989×10^{-8}	2.23727×10^{-16}
0.1	0.4	−0.0049999	−0.00499988	−0.00499988	1.99986×10^{-8}	5.28014×10^{-16}
0.3	0.1	−0.0049989	−0.00499888	−0.00499888	1.49904×10^{-8}	2.65681×10^{-17}
0.3	0.2	−0.00499892	−0.00499889	−0.00499889	2.99809×10^{-8}	2.01327×10^{-16}
0.3	0.3	−0.00499894	−0.0049989	−0.0049989	4.49713×10^{-8}	6.74329×10^{-16}
0.3	0.4	−0.00499896	−0.0049989	−0.0049989	5.99618×10^{-8}	1.59303×10^{-15}
0.5	0.1	−0.00499691	−0.00499689	−0.00499689	2.49558×10^{-8}	3.99143×10^{-17}
0.5	0.2	−0.00499695	−0.0049969	−0.0049969	4.99115×10^{-8}	3.31551×10^{-16}
0.5	0.3	−0.00499699	−0.00499691	−0.00499691	7.48673×10^{-8}	1.11864×10^{-15}
0.5	0.4	−0.00499703	−0.00499693	−0.00499693	9.98231×10^{-8}	2.65445×10^{-15}

Table 3
The comparison of Absoulte Differences obtained by NTIM, q-HAM of fractional order KDV Burger equation $\Phi(\gamma,\sigma)$ with OAFM, when $\beta=1,\ t=0.1,\ k=0.1,\ h=-1$ and $n=1$.

γ	Φ_{NTIM}	Φ_{q-HAM}	Φ_{Exact}	Abs Diff NTIM	Abs Diff q-HAM	Abs Diff OAFM[21]
0.25	−0.00499924	−0.00499923	−0.00499923	1.24945×10^{-8}	2.22666×10^{-17}	1.25899×10^{-9}
0.5	−0.00499691	−0.00499689	−0.00499689	2.49558×10^{-8}	3.99143×10^{-17}	2.97366×10^{-9}
0.75	−0.00499303	−0.00499299	−0.00499299	3.73509×10^{-8}	6.24799×10^{-17}	1.15346×10^{-9}
1.	−0.0049876	−0.00498755	−0.00498755	4.96471×10^{-8}	8.3695×10^{-17}	1.26266×10^{-8}

Table 4
The comparison between NTIM, q-HAM and the exact solution of fifth-order Sawada–Kotera equation $\Phi(\gamma,\sigma)$ when $\beta=1, \lambda=0.1, k=0.1, h=-1$ and $n=1$.

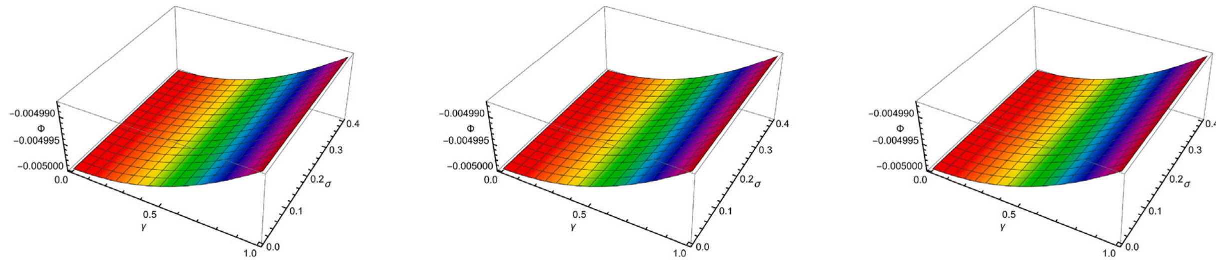
σ	γ	$u_{NTIM}(\gamma,\sigma)$	$\Phi_{q-HAM}(\gamma,\sigma)$	$\Phi_{Exact}(\gamma,\sigma)$	Abs Diff NTIM	Abs Diff q-HAM
0.1	0.	0.019998	0.019998	0.02	1.9519×10^{-6}	2.00627×10^{-6}
0.1	0.025	0.0199989	0.0199989	0.0199999	9.64134×10^{-7}	1.00492×10^{-6}
0.1	0.05	0.0199995	0.0199995	0.0199995	2.36717×10^{-8}	3.52495×10^{-9}
0.1	0.075	0.0199999	0.0199999	0.0199989	1.01147×10^{-6}	9.97873×10^{-7}
0.1	0.1	0.02	0.02	0.019998	1.99922×10^{-6}	1.99922×10^{-6}
0.3	0.	0.0199981	0.019998	0.02	1.85599×10^{-6}	2.01911×10^{-6}
0.3	0.025	0.019999	0.0199989	0.0199999	8.92516×10^{-7}	1.01488×10^{-6}
0.3	0.05	0.0199996	0.0199995	0.0199995	7.09848×10^{-8}	1.06053×10^{-8}
0.3	0.075	0.0199999	0.0199999	0.0199989	1.03447×10^{-6}	9.93673×10^{-7}
0.3	0.1	0.02	0.02	0.019998	1.9979×10^{-6}	1.9979×10^{-6}
0.5	0.	0.0199982	0.019998	0.02	1.76012×10^{-6}	2.03199×10^{-6}
0.5	0.025	0.0199991	0.0199989	0.0199999	8.20939×10^{-7}	1.02488×10^{-6}
0.5	0.05	0.0199996	0.0199995	0.0199995	1.18257×10^{-7}	1.77261×10^{-8}
0.5	0.075	0.0199999	0.0199999	0.0199989	1.05743×10^{-6}	9.89432×10^{-7}
0.5	0.1	0.02	0.02	0.019998	1.99654×10^{-6}	1.99654×10^{-6}

Table 5
The comparison between NTIM, q-HAM and the exact solution of fifth-order Sawada–Kotera equation $\Phi(\gamma,\sigma)$ when $\beta=1, \lambda=0.1, k=0.1, h=-1$ and $n=1$.

γ	σ	$\Phi_{NTIM}(\gamma,\sigma)$	$\Phi_{q-HAM}(\gamma,\sigma)$	$\Phi_{Exact}(\gamma,\sigma)$	Abs Diff NTIM	Abs Diff q-HAM
0.1	0.1	0.02	0.02	0.019998	1.99922×10^{-6}	1.99922×10^{-6}
0.1	0.2	0.02	0.02	0.019998	1.99857×10^{-6}	1.99857×10^{-6}
0.1	0.3	0.02	0.02	0.019998	1.9979×10^{-6}	1.9979×10^{-6}
0.1	0.4	0.02	0.02	0.019998	1.99723×10^{-6}	1.99723×10^{-6}
0.1	0.5	0.02	0.02	0.019998	1.99654×10^{-6}	1.99654×10^{-6}
0.3	0.1	0.0199919	0.019992	0.019982	9.89362×10^{-6}	1.00022×10^{-5}
0.3	0.2	0.0199918	0.019992	0.019982	9.79589×10^{-6}	1.00131×10^{-5}
0.3	0.3	0.0199917	0.019992	0.019982	9.69815×10^{-6}	1.00239×10^{-5}
0.3	0.4	0.0199916	0.0199921	0.019982	9.60041×10^{-6}	1.00348×10^{-5}
0.3	0.5	0.0199915	0.0199921	0.019982	9.50265×10^{-6}	1.00456×10^{-5}
0.5	0.1	0.0199678	0.0199681	0.0199501	1.77573×10^{-5}	1.79732×10^{-5}
0.5	0.2	0.0199677	0.0199681	0.0199501	1.75638×10^{-5}	1.79956×10^{-5}
0.5	0.3	0.0199675	0.0199681	0.0199501	1.73702×10^{-5}	1.80179×10^{-5}
0.5	0.4	0.0199673	0.0199681	0.0199501	1.71766×10^{-5}	1.80402×10^{-5}
0.5	0.5	0.0199671	0.0199682	0.0199501	1.6983×10^{-5}	1.80625×10^{-5}

Table 6
Comparison of Absolute Differences obtained by NTIM, q-HAM of fifth-order Sawada–Kotera equation $\Phi(\gamma,\sigma)$ with OAFM, when $\beta=1, x=0.1, k=0.1, h=-1$ and $n=1$.

σ	Φ_{NTIM}	Φ_{q-HAM}	Φ_{Exact}	Abs Diff NTIM	Abs Diff q-HAM	Abs Diff OAFM[21]
0.1	0.02	0.02	0.019998	1.99922×10^{-6}	1.99922×10^{-6}	1.4687×10^{-5}
0.2	0.02	0.02	0.019998	1.99857×10^{-6}	1.99857×10^{-6}	2.93741×10^{-5}
0.3	0.02	0.02	0.019998	1.9979×10^{-6}	1.9979×10^{-6}	4.40611×10^{-5}
0.4	0.02	0.02	0.019998	1.99723×10^{-6}	1.99723×10^{-6}	5.87481×10^{-5}
0.5	0.02	0.02	0.019998	1.99654×10^{-6}	1.99654×10^{-6}	7.34351×10^{-5}



(a) NTIM Solution 3D Graph (b) q-HAM Solution 3D Graph (c) Exact Solution 3D Graph

Fig. 1. The 3D comparison of NTIM, q-HAM and exact solution of KDV-Burgers equation $\Phi(\gamma,\sigma)$ in Fig. 1(a)–1(c) when $\beta = 1$, $k = 0.1$, $h = -1$, $n = 1$ for case 1.

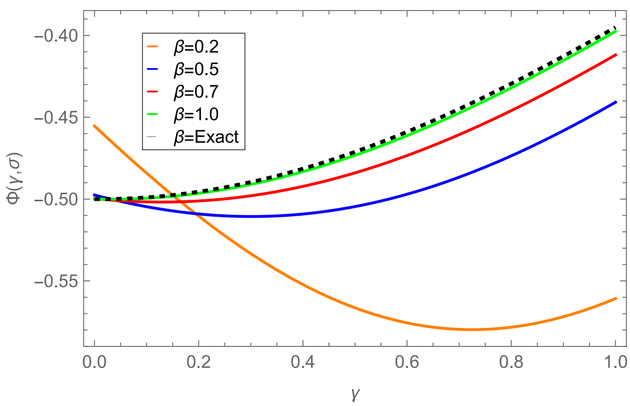


Fig. 2. Effect of different values of β on KDV-Burgers equation, while keeping $k = 1$, $t = 0.01$ fixed using NTIM.

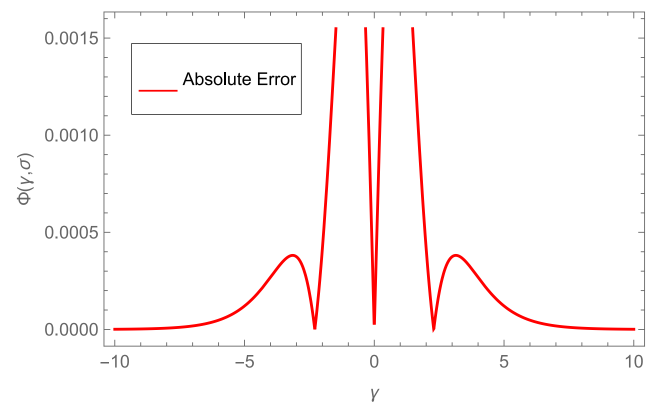


Fig. 4. For the fractional-order KDV-Burger equation, the 2D graph illustrates the absolute errors obtained by NTIM at $\beta = 1$, $t = 0.01$ and $k = 1$.

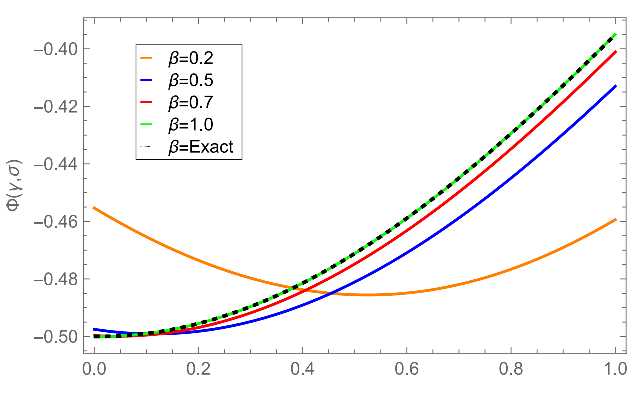


Fig. 3. Effect of different values of β on KDV-Burgers equation, while keeping $k = 1$, $t = 0.01$, $n = 1$ and $h = -1$ fixed using q-HAM.

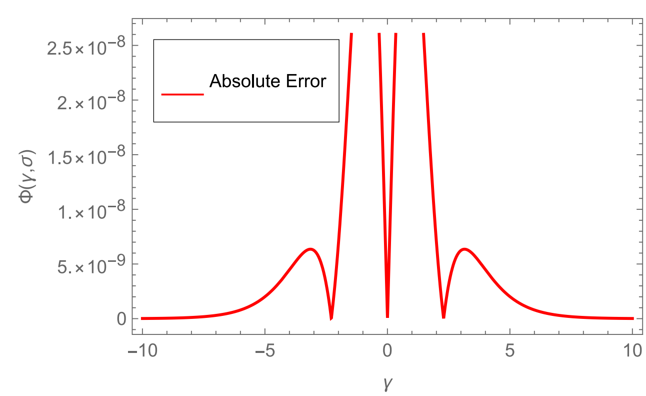
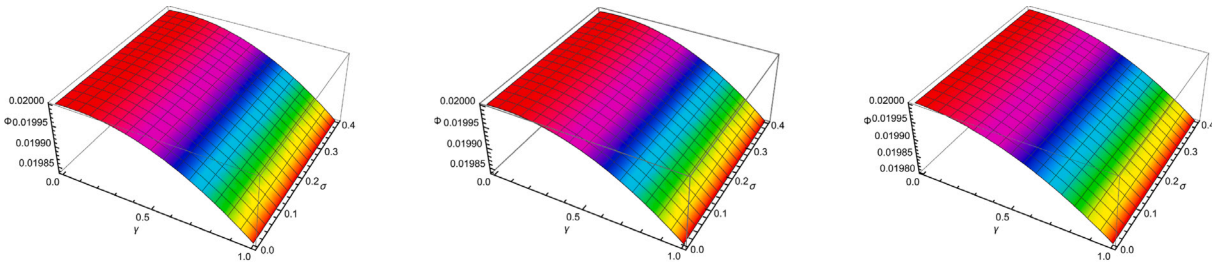


Fig. 5. For the fractional-order KDV-Burger equation, the 2D graph illustrates the absolute errors obtained by the q-HAM at $\beta = 1$, $t = 0.01$, $h = -1$, $n = 1$ and $k = 1$.

6. Conclusion

Finding exact solutions for fractional partial differential equations can be challenging in many cases. Specifically, when utilized for intricate models like the KDV-Burger and Sawada–Kotera equations, the NTIM and q-HAM techniques provide a new approach for resolving FPDEs with high precision and effectiveness. The numbers given back the clear connection between the precise and analytical outcomes. The created charts demonstrated the accuracy of the techniques. Different fractional orders are chosen as solutions and plotted to aid in comprehending the behavior of the issues being studied. The reliability of the technique has been proven by the convergence phenomenon. These advancements expand the scope of FPDEs suitable for NTIM and q-HAM, while also providing new opportunities for enhancing analytical and numerical methods for fractional-order systems in various scientific fields.



(a) NTIM Solution 3D Graph

(b) q-HAM Solution 3D Graph

(c) Exact Solution 3D Graph

Fig. 6. The 3D comparison of NTIM, q-HAM and exact solution of fractional-order Sawada–Kotera equation $\Phi(\gamma, \sigma)$ in Fig. 6(a)–6(c) when $\beta = 1$, $k = 0.1$, $h = -1$, $n = 1$.

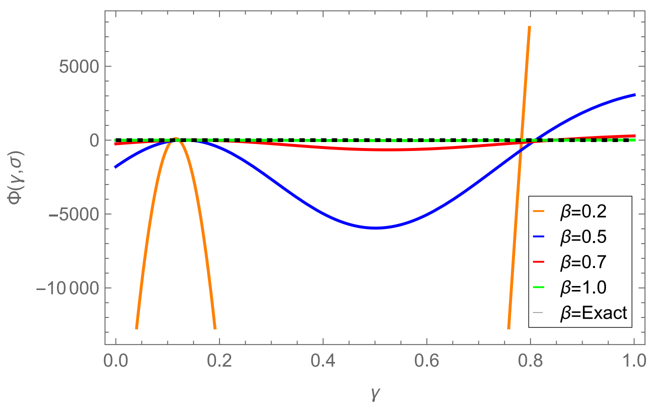


Fig. 7. Variations in β values and their impact on the fractional-order Sawada–Kotera equation with $k = 1$ and $t = 0.1$ set using NTIM.

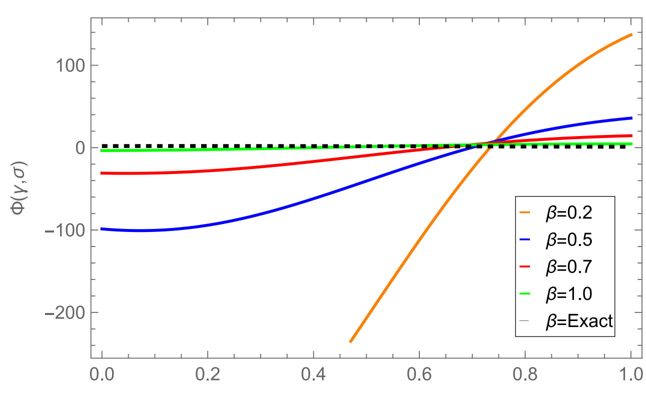


Fig. 8. Variations in β values and their impact on the fractional-order Sawada–Kotera equation, with $k = 1$, $t = 0.1$, $n = 1$, and $h = -1$ fixed through the use of q-HAM.

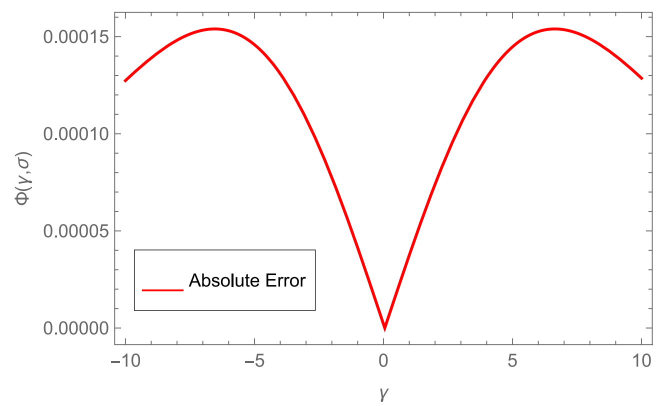


Fig. 9. 2D surface shows the absolute errors, attained by the NTIM for the fractional-order Sawada–Kotera equation at $\beta = 1$, $t = 0.01$, $\lambda = 0.1$ and $k = 0.1$.

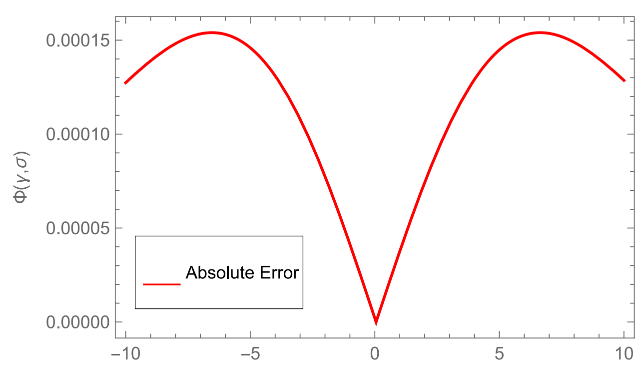


Fig. 10. 2D surface shows the absolute errors, attained by the q-HAM for the fractional-order Sawada–Kotera equation at $\beta = 1$, $t = 0.01$, $h = -1$, $n = 1$, $\lambda = 0.1$ and $k = 0.1$.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

References

1. Liouville J. Mémoire sur quelques questions de géométrie et de mécanique, et sur un nouveau genre de calcul pour résoudre ces questions. *J Ecole Polytéch*. 1832;13:1–69.
2. Podlubny I. *Fractional Differential Equations*. New York, NY, USA: Academic Press; 1999.

3. Povstenko Y. *Linear Fractional Diffusion-Wave Equation for Scientists and Engineers*. New York, NY, USA: Birkhäuser; 2015.
4. Riemann GFB. Versuch einer allgemeinen Auffassung der integration und differentiation. In: *Gesammelte Mathematische Werke*. Leipzig, Germany: Teubner; 1896.
5. Mohammed Pshtiwan Othman, Machado José António Tenreiro, Guirao Juan LG, Agarwal Ravi P. Adomian decomposition and fractional power series solution of a class of nonlinear fractional differential equations. *Mathematics*. 2021;9(9):1070.
6. Sweilam NH, Hasan MMA, Baleanu D. New studies for general fractional financial models of awareness and trial advertising decisions. *Chaos Solitons Fractals*. 2017;104:772–784.
7. Liu DY, Gibaru O, Perruquetti W, Laleg-Kirati TM. Fractional order differentiation by integration and error analysis in noisy environment. *IEEE Trans Autom Control*. 2015;60:2945–2960.
8. Esen A, Sulaiman TA, Bulut H, Baskonus HM. Optical solitons to the space–time fractional (1+1)-dimensional coupled nonlinear Schrödinger equation. *Optik*. 2018;167:150–156.
9. Caponetto R, Dongola G, Fortuna L, Gallo A. New results on the synthesis of FO-PID controllers. *Commun Nonlinear Sci Numer Simul*. 2010;15:997–1007.
10. Veeresha P, Prakasha DG, Baskonus HM. Novel simulations to the time-fractional Fisher's equation. *Math Sci*. 2019:1–10. http://dx.doi.org/10.1007/s40096-019-0276-6.
11. Momani S, Odibat Z. Analytical solution of a time-fractional Navier–Stokes equation by Adomian decomposition method. *Appl Math Comput*. 2006;177:488–494.
12. Srivastava Hari Mohan, Gusu Daba Meshesha, Mohammed Pshtiwan Othman, Wedajo Gidisa, Nonlaopon Kamsing, Hamed Yasser S. Solutions of general fractional-order differential equations by using the spectral Tau method. *Fractal Fract*. 2021;6(1):7.
13. Shah NA, Hamed YS, Abualnaja KM, Chung JD, Khan A. A comparative analysis of fractional-order Kaup-Kupershmidt equation within different operators. *Symmetry*. 2022;14:986.
14. Alqhtani M, Saad KM, Weera W, Hamanah WM. Analysis of the fractional-order local Poisson equation in fractal porous media. *Symmetry*. 2022;14:1323.
15. Prakasha DG, Veeresha P, Rawashdeh MS. Numerical solution for (2+1)-dimensional time-fractional coupled Burger equations using fractional natural decomposition method. *Math Methods Appl Sci*. 2019;42:3409–3427.
16. Aljahdaly NH, Naeem M, Arefin MA. A comparative analysis of fractional space-time advection-dispersion equation via semi-analytical methods. *J Funct Spaces*. 2022;2022:4856002.
17. El-Tawil MA, Huseen SN. The q-homotopy analysis method (q-HAM). *Int J Appl Math Mech*. 2012;8:51–75.
18. Nawaz Rashid, Iqbal Aaqib, Bakhtiar Hina, Alhilfi Wissal Audah, Fewster-Young Nicholas, Ali Ali Hasan, Poţclean Ana Danca. A new extension of optimal auxiliary function method to fractional non-linear coupled ITO system and time fractional non-linear KDV system. *Axioms*. 2023;12(9):881.
19. Iyiola OS, Soh ME, Enyi CD. Generalized homotopy analysis method (q-HAM) for solving foam drainage equation of time fractional type. *Math Eng Sci Aerosp (MESA)*. 2013;4(4):229–440.
20. Gepreel KA. Optimal Q. Homotopy analysis method for nonlinear fractional dynamics equations. *Jokull J*. 2014;68:317–326.
21. Zada Laiq, Nawaz Rashid, Alqudah Mohammad A, Nisar Kottakkaran Sooppy. A new technique for approximate solution of fractional-order partial differential equations. *Fractals*. 2022;30(01):2240015.
22. Alderremy Aisha Abdullah, Shah Rasool, Shah Nehad Ali, Aly Shaban, Nonlaopon Kamsing. Comparison of two modified analytical approaches for the systems of time fractional partial differential equations. *AIMS Math*. 2023;8(3):7142–7162.
23. AlBaidani Mashael M, Aljuaydi Fahad, Alharthi NS, Khan Adnan, Ganie Abdul Hamid. Study of fractional forced KdV equation with Caputo–Fabrizio and Atangana–Baleanu–Caputo differential operators. *AIP Adv*. 2024;14(1).
24. Ismael Mimoon, Hat Saba, Alabdali Osama, Lone Showkat Ahmad, Ali Ali Hasan. Modified optimal auxiliary functions method for approximate-analytical solutions in fractional order nonlinear foam drainage equations. *Partial Differ Equ Appl Math*. 2024;10:100700.
25. Khirsariya Sagar R, Rao Snehal B, Hathiwala Gautam S. Investigation of fractional diabetes model involving glucose–insulin alliance scheme. *Int J Dyn Control*. 2024;12(1):1–14.
26. Cheng Xiaoyu, Wang Lizhen. Fundamental solutions and conservation laws for conformable time fractional partial differential equation. *J Comput Appl Math*. 2024;437:115434.
27. Srinivasa Kumbinarasaiah, Mundewadi Ravikiran Ashok. Wavelets approach for the solution of nonlinear variable delay differential equations. *Int J Math Comput Eng*. 2023.
28. Erdogan Fevzi. A second order numerical method for singularly perturbed Volterra integro-differential equations with delay. *Int J Math Comput Eng*. 2024.
29. Javeed Shumaila, Hincal Evren. Solving coupled non-linear higher order BVPs using improved shooting method. *Int J Math Comput Eng*. 2024.
30. Yousif Majeed A, Agarwal Ravi P, Mohammed Pshtiwan Othman, Lupas Alina Alb, Jan Rashid, Chorfi Nejmeddine. Advanced methods for conformable time-fractional differential equations: Logarithmic non-polynomial splines. *Axioms*. 2024;13(8):551.
31. Momani Shaher, Odibat Zaid. Analytical approach to linear fractional partial differential equations arising in fluid mechanics. *Phys Lett A*. 2006;355(4):271–279.
32. Dehghan Mehdi, Manafian Jalil, Saadatmandi Abbas. Solving nonlinear fractional partial differential equations using the homotopy analysis method. *Numer Methods Partial Differential Equations*. 2010;26(2):448–479.
33. Liu Di, Gu Qiongya, Wang Lizhen. Q-homotopy analysis method for time fractional Newell-Whitehead equation and time fractional generalized Hirota-Satsuma coupled KdV system. *Commun Theor Phys (Beijing)*. 2024.
34. Bas Erdal, Karaoglan Merve. Representation of solution the M-Sturm–Liouville problem with natural transform. *Int J Math Comput Eng*. 2023.
35. Nawaz Rashid, Ali Nasir, Zada Laiq, Nisar Kottakkkaran Sooppy, Alharthi MR, Jamshed Wasim. Extension of natural transform method with Daftardar-Jafari polynomials for fractional order differential equations. *Alex Eng J*. 2021;60(3):3205–3217.
36. Ali Nasir, Nawaz Rashid, Zada Laiq, Mouldi Abir, Bouzgarrou Souhail Mohamed, Sene Ndolane. Analytical approximate solution of the fractional order biological population model by using natural transform. *J Nanomater*. 2022;2022:1–12.
37. Rehman Siddiq Ur, Nawaz Rashid, Zia Faisal, Fewster-Young Nicholas, Ali Ali Hasan. A comparative analysis of noyes-field model for the non-linear Belousov–Zhabotinsky reaction using two reliable techniques. *Alex Eng J*. 2024;93:259–279.
38. El-Tawil MA, Huseen SN. The q-homotopy analysis method (q-HAM). *Int J Appl Math Mech*. 2012;8(15):51–75.
39. El-Tawil Magdy A, Huseen Shaheed N. On convergence of the q-homotopy analysis method. *Int J Contemp Math Sci*. 2013;8(10):481–497.
40. Bhalekar Sachin, Daftardar-Gejji Varsha. Existence and uniqueness theorems for fractional differential equations: A new approach. *Anal Appl*. 2006;316:753–763.