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# Bounds for the Second Hankel Determinant of a General Subclass of Bi-Univalent Functions

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**Abstract**
The Hankel determinant, which plays a significant role in the theory of univalent functions, is investigated here in the context of bi-univalent analytic functions. Specifically, this paper is dedicated to deriving an upper-bound estimate for the second-order Hankel determinant for a general subclass of bi-univalent analytic functions that incorporate Gegenbauer polynomials within the unit disk. Through the careful specialization of parameters in our main result, we unveil several novel findings.

**Keywords-** Hankel determinant, Gegenhauer polynomials, Bi-univalent function, Fekete-Szegö functional.

## 1. Introduction and Definitions

Let $\mathbb{U} = \{z \in \mathbb{C}: |z| < 1\}$ and $\mathcal{A} = \{f: f$ is analytic in $\mathbb{U}$ with $f(0) = 0 = f'(0) - 1\}$. Thus, every $f \in \mathcal{A}$ can be expressed as a power series.

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

Further, let $\mathcal{S} = \{f \in \mathcal{A}: f$ is univalent in $\mathbb{U}\}$, where univalent means an injective function on $\mathbb{U}$. According to the Koebe theorem, any $f \in \mathcal{S}$ has an inverse of the form.

$$f^{-1}(w) = w - a_2w^2 + (-a_3 + 2a_2^2)w^3 - (a_4 + 5a_2^3 - 5a_2a_3)w^4 + \cdots \tag{2}$$

Moreover, a function $f \in \mathcal{A}$ is called bi-univalent in $\mathbb{U}$ if and only if $f$ and $f^{-1}$ are univalent functions in the open unit disk $\mathbb{U}$. Let $\Sigma = \{f: f$ is a bi-univalent function in $\mathbb{U}$ and given by Equation (1)$\}$. For a comprehensive source on functions in the class $\Sigma$, we refer the reader to Srivastava et al. (2010), Frasin and Aouf (2011). In fact, for more work on estimating the first two coefficients of subclasses of bi-univalent analytic functions, see Yousef et al. (2019), Amourah et al. (2022a), Illafe et al. (2022), Yousef et al. (2022), Hussen & Illafe (2023) and the references therein. In addition, if $f(z) = g(h(z))$, for some functions $f, g, h \in \mathcal{A}$ where for all $z \in \mathbb{U}, h(0) = 0$ and $|h(z)| < 1$, in that case, we state that $f$ is subordinate to $g$ in $\mathbb{U}$ and denote this as $f \prec g$. For various subclasses of bi-univalent analytic functions determined by subordination, refer to Yousef et al. (2018), Orhan et al. (2022), Al-Hawary et al. (2023), Illafe et al. (2023).

In recent years, orthogonal polynomials have been extensively researched and analyzed due to their significance in many research fields. Mathematically speaking, orthogonal polynomials often arise from solutions of ODEs when some conditions are given to specific models. To learn more about the classical orthogonal polynomials' basic definitions and essential properties, readers can refer to the sources listed in Ismail (2005), Chihara (2011), Doman (2015), Agarwal et al. (2020). For the latest advancements concerning the correlation between orthogonal polynomials and geometric function theory, see Yousef et al. (2020), Frasin et al. (2021), Srivastava et al. (2021), Frasin et al. (2022), Amourah et al. (2022b), Al-Hawary et al. (2022), Hussen and Zeyani (2023), Hussen (2024), Hussen et al. (2024).

Gegenbauer polynomials $C_n^\alpha(t)$ for $n = 3, 4, \ldots,$ and $\alpha > -\frac{1}{2}$ are obtained by the following recurrence formula:

$$\begin{aligned} C_0^\alpha(t) &= 1 \\ C_1^\alpha(t) &= 2\alpha t \\ C_2^\alpha(t) &= 2\alpha(\alpha+1)t^2 - \alpha \\ C_n^\alpha(t) &= \frac{1}{n}[2t(\alpha+n-1)C_{n-1}^\alpha(t) - (2\alpha+n-2)C_{n-2}^\alpha(t)] \end{aligned} \tag{3}$$

Note, by taking $\alpha = \frac{1}{2}$ and $\alpha = 1$ in Equation (3), one generates the Legendre polynomials $P_n(t) = C_n^{\frac{1}{2}}(t)$ and Chebyshev polynomials $U_n(t) = C_n^1(t)$, respectively. The function $H_\alpha(t,z)$, which generates any Gegenbauer polynomial, is given by:

$$H_\alpha(t,z) = \frac{1}{(1-2tz+z^2)^\alpha} \tag{4}$$

Given that $H_\alpha \in \mathcal{A}$, then it can be written in a power series expansion as follows:

$$H_\alpha(z,t) = \sum_{n=0}^{\infty} C_n^\alpha(t)z^n, \qquad z \in \mathbb{U} \text{ and } t \in [-1,1] \tag{5}$$

For non-zero natural numbers $n$ and $r$, the $r^{th}$ Hankel determinant, termed as:

$$H_r(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+r-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+r-2} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+r-1} & a_{n+r-2} & \cdots & a_{n+2r-2} \end{vmatrix}, \ (a_1 = 1).$$

The Hankel determinants' characteristics are described in (Vein and Dale, 2006). It's intriguing to observe that,

$$H_2(1)=\begin{vmatrix}a_1 & a_2\\ a_2 & a_3\end{vmatrix}=a_3-a_2^2 \text{ and } H_2(2)=\begin{vmatrix}a_2 & a_3\\ a_3 & a_4\end{vmatrix}=a_2a_4-a_3^2.$$

The determinants $H_2(1)=a_3-a_2^2$ and $H_2(2)=a_2a_4-a_3^2$ are commonly referred to as the Fekete-Szegö functional and the 2nd Hankel determinant, respectively. It is worth mentioning that generalized functional $a_3-\eta a_2^2$, for some real number $\eta$, was first introduced by Fekete and Szegö (1933). The Fekete-Szegö problem was explored to the set $\mathcal{C}_1:=\left\{f\in\mathcal{A}:\Re\left(e^{i\eta}f'(z)\right)>0,-\frac{\pi}{2}<\eta<\frac{\pi}{2},z\in\mathbb{U}\right\}$ by Srivastava et al. (2001), and an improved bound of $\left|a_3-a_2^2\right|$ for $\mathcal{C}_1$ was found. Additionally, Deniz et al. (2015) studied the estimates for the 2nd Hankel determinant of the subclasses $\mathcal{S}_\Sigma^*(\beta)$ and $\mathcal{K}_\Sigma(\beta)$. Moreover, the Fekete-Szegö functional and Hankel determinant for normalized-univalent functions were investigated by Kowalczyk et al. (2017). Several authors (Deekonda and Thoutreddy, 2015; Alarifi et al., 2017; Amourah et al., 2017; Orhan et al., 2018; Al-Shbeil et al., 2022; Srivastava et al., 2022; Orhan et al., 2023) have investigated the Hankel determinant for subclasses of analytic univalent functions. In the next section, we will introduce the general class of bi-univalent analytic functions that incorporate Gegenbauer polynomials and their subclasses.

## 2. The Class $\mathcal{B}_\Sigma^\alpha(t,\eta,\mu,\delta)$

Yousef et al. (2021) have provided the following class $\mathcal{M}_\Sigma^\alpha(\eta,\mu,\delta)$ of bi-univalent analytic functions defined as below.

***Definition* 2.1:** For $\eta\geq 1,\mu,\delta\geq 0, 0\leq\alpha<1$, and $\zeta=\frac{2\eta+\mu}{2\eta+1}$. A function $f\in\Sigma$ given by Equation (1) is said to be in the class $\mathcal{M}_\Sigma^\alpha(\eta,\mu,\delta)$ if and only if for all $z,w\in\mathbb{U}$:

$$\Re\left((1-\eta)\left(\frac{f(z)}{z}\right)^\mu+\eta f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1}+\zeta\delta zf''(z)\right)>\alpha \tag{6}$$

$$\Re\left((1-\eta)\left(\frac{g(w)}{w}\right)^\mu+\eta g'(w)\left(\frac{g(w)}{w}\right)^{\mu-1}+\zeta\delta zg''(w)\right)>\alpha \tag{7}$$

In the sequel, consider the functions $f\in\Sigma$ and $g=f^{-1}$ given by Equations (1) and (2), respectively. Let $\alpha$ be a nonzero real constant, $H_\alpha$ be given by Equation (5), $\eta\geq 1,\mu,\delta\geq 0,\zeta=\frac{2\eta+\mu}{2\eta+1}$, and $t\in(1/2,1]$.

***Definition* 2.2:** The function $f\in\mathcal{B}_\Sigma^\alpha(t,\eta,\mu,\delta)$ if and only if for all $z,w\in\mathbb{U}$ the two subordinations hold.

$$(1-\eta)\left(\frac{f(z)}{z}\right)^\mu+\eta f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1}+\zeta\delta zf''(z)\prec H_\alpha(t,z) \text{ and}$$

$$(1-\eta)\left(\frac{g(w)}{w}\right)^\mu+\eta g'(w)\left(\frac{g(w)}{w}\right)^{\mu-1}+\zeta\delta zg''(w)\prec H_\alpha(t,w).$$

By replacing the values of the parameters $\eta,\mu$ and $\delta$, we obtain several novel subclasses of the class $\mathcal{B}_\Sigma^\alpha(t,\eta,\mu,\delta)$, as outlined below.

***Definition* 2.3:** The function $f\in{}^1\mathcal{B}_\Sigma^\alpha(t,\eta,\mu):=\mathcal{B}_\Sigma^\alpha(t,\eta,\mu,0)$ if and only if for all $z,w\in\mathbb{U}$ the following hold.

$$(1-\eta)\left(\frac{f(z)}{z}\right)^\mu+\eta f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1}\prec H_\alpha(t,z) \text{ and}$$

$$(1-\eta)\left(\frac{g(w)}{w}\right)^{\mu}+\eta g'(w)\left(\frac{g(w)}{w}\right)^{\mu-1}\prec H_{\alpha}(t,w).$$

***Definition* 2.4:** The function $f\in{}^{2}\mathcal{B}_{\Sigma}^{\alpha}(t,\eta,\delta):=\mathcal{B}_{\Sigma}^{\alpha}(t,\eta,1,\delta)$ if and only if for all $z,w\in\mathbb{U}$ the two subordinations hold.

$$(1-\eta)\left(\frac{f(z)}{z}\right)+\eta f'(z)+\delta zf''(z)\prec H_{\alpha}(t,z) \text{ and}$$

$$(1-\eta)\left(\frac{g(w)}{w}\right)+\eta g'(w)+\delta zg''(w)\prec H_{\alpha}(t,w).$$

***Definition* 2.5:** The function $f\in{}^{3}\mathcal{B}_{\Sigma}^{\alpha}(t,\eta):=\mathcal{B}_{\Sigma}^{\alpha}(t,\eta,1,0)$ if and only if for all $z,w\in\mathbb{U}$ the two subordinations hold.

$$(1-\eta)\left(\frac{f(z)}{z}\right)+\eta f'(z)\prec H_{\alpha}(t,z) \text{ and}$$

$$(1-\eta)\left(\frac{g(w)}{w}\right)+\eta g'(w)\prec H_{\alpha}(t,w).$$

***Definition* 2.6:** The function $f\in{}^{4}\mathcal{B}_{\Sigma}^{\alpha}(t):=\mathcal{B}_{\Sigma}^{\alpha}(t,1,1,0)$ if and only if for all $z,w\in\mathbb{U}$ the two subordinations hold.

$$f'(z)\prec H_{\alpha}(t,z) \text{ and}$$

$$g'(w)\prec H_{\alpha}(t,w).$$

Next, we define the set $\mathcal{P}=\{p\in\mathcal{A}:p(z)=1+d_1z+d_2z^2+d_3z^3+\cdots\}$, and we present the following lemmas which will be used in our proof.

***Lemma* 2.7:** (Duren, 2001) Let $p\in\mathcal{P}$, then
$|d_k|\le 2, k=1,2,\ldots$ .

***Lemma* 2.8:** (Grenander and Szegö, 1958) Let $p\in\mathcal{P}$, then

$$2d_2=d_1^2+(4-d_1^2)t$$
$$4d_3=d_1^3+2d_1(4-d_1^2)t-d_2(4-d_1^2)t^2+2(4-d_1^2)(1-|t|^2)z$$

for some $z,t$ such that $|z|\le 1$ and $|t|\le 1$.

## 3. Preliminary Results

***Theorem* 3.1:** Let $f\in\mathcal{B}_{\Sigma}^{\alpha}(t,\eta,\mu,\delta)$, then

$$|a_2a_4-a_3^2|\le\begin{cases}\Psi(2^-,t) & ;\Theta_1\ge 0 \text{ and } \Theta_2\ge 0\\ \max\left\{\frac{4\alpha^2t^2}{(2\eta+\mu+6\zeta\delta)^2},\Psi(2^-,t)\right\} & ;\Theta_1>0 \text{ and } \Theta_2<0\\ \frac{4\alpha^2t^2}{(2\eta+\mu+6\zeta\delta)^2} & ;\Theta_1\le 0 \text{ and } \Theta_2\le 0\\ \max\{\Psi(c_0,t),\Psi(2^-,t)\} & ;\Theta_1<0 \text{ and } \Theta_2>0\end{cases}$$

where,

$$\Psi(2^-,t)=\frac{4\alpha^2t^2}{(2\eta+\mu+6\zeta\delta)^2}+\frac{\Theta_1+3\Theta_2}{6(\eta+\mu+2\zeta\delta)^4(3\eta+\mu+12\zeta\delta)(2\eta+\mu+6\zeta\delta)^2}$$

$$\Psi(c_0,t)=\frac{4\alpha^2t^2}{(2\eta+\mu+6\zeta\delta)^2}-\frac{3\Theta_2^2}{8\Theta_1(\eta+\mu+2\zeta\delta)^4(3\eta+\mu+12\zeta\delta)(2\eta+\mu+6\zeta\delta)^2},\ c_0=\sqrt{\frac{-6\Theta_2}{\Theta_1}}$$ and

$$\begin{aligned}\Theta_1:=\ \ &\Theta_1(\eta,\mu;t)=8\alpha^2t^2\,|\,[2(2+\alpha)t^2-3](1+\alpha)(\eta+\mu+2\zeta\delta)^3\\ &-2\alpha^2t^2[(\mu^2+3\mu+2)(3\eta+\mu)+72\zeta\delta]\,|\,(2\eta+\mu+6\zeta\delta)^2-24\alpha^2t[\alpha t^2(3\eta+\mu+12\zeta\delta)\\ &+(2(1+\alpha)t^2-1)(\eta+\mu+2\zeta\delta)(2\eta+\mu+6\zeta\delta)](\eta+\mu+2\zeta\delta)^2(2\eta+\mu+6\zeta\delta)\\ &-24t^2\alpha^2(\eta+\mu+2\zeta\delta)^3[\eta^2+2\zeta\delta(3\eta-\mu+6\zeta\delta)].\end{aligned}$$

$$\begin{aligned}\Theta_2:=\Theta_2(\eta,\mu;t)=\ &8\alpha^2t[\alpha t^2(2\eta+\mu+6\zeta\delta)(3\eta+\mu+12\zeta\delta)\\ +\ &[-1+2(1+\alpha)t^2+t](\eta+\mu+2\zeta\delta)(2\eta+\mu+6\zeta\delta)^2\\ -\ &2t(\eta+\mu+2\zeta\delta)^2(3\eta+\mu+12\zeta\delta)](\eta+\mu+2\zeta\delta)^2.\end{aligned}$$

***Proof*:** Let $f\in\mathcal{B}_\Sigma^\alpha(t,\eta,\mu,\delta)$. Then

$$(1-\eta)\left(\frac{f(z)}{z}\right)^\mu+\eta f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1}+\zeta\delta zf''(z)=H_\alpha(t,u(z))\ (z\in\mathbb{U}) \tag{8}$$

and

$$(1-\eta)\left(\frac{g(w)}{w}\right)^\mu+\eta g'(w)\left(\frac{g(w)}{w}\right)^{\mu-1}+\zeta\delta zg''(w)=H_\alpha(t,v(w))\ (w\in\mathbb{U}) \tag{9}$$

Let $p,q\in\mathcal{P}$ and given as:

$$p(z)=\frac{1+u(z)}{1-u(z)}=1+c_1z+c_2z^2+c_3z^3+\cdots=1+\sum_{n=1}^{\infty}c_nz^n \tag{10}$$

and

$$q(w)=\frac{1+v(w)}{1-v(w)}=1+d_1w+d_2w^2+d_3w^3+\cdots=1+\sum_{n=1}^{\infty}d_nw^n \tag{11}$$

From Equations (10) and (11), one can write $u(z)$ and $v(w)$ as follows:

$$u(z)=\frac{p(z)-1}{p(z)+1}=\frac{1}{2}\left[c_1z+\left(c_2-\frac{c_1^2}{2}\right)z^2+\left(c_3-c_1c_2+\frac{c_1^3}{4}\right)z^3+\cdots\right] \tag{12}$$

and

$$v(w)=\frac{q(w)-1}{q(w)+1}=\frac{1}{2}\left[d_1w+\left(d_2-\frac{d_1^2}{2}\right)w^2+\left(d_3-d_1d_2+\frac{d_1^3}{4}\right)w^3+\cdots\right] \tag{13}$$

It follows from Equations (12), (13), and (5) that

$$\begin{aligned}H(t,u(z))=\ \ &1+\frac{C_1^\alpha(t)}{2}c_1z+\left[\frac{C_1^\alpha(t)}{2}\left(c_2-\frac{c_1^2}{2}\right)+\frac{C_2^\alpha(t)}{4}c_1^2\right]z^2\\ &+\left[\frac{C_1^\alpha(t)}{2}\left(c_3+c_1c_2+\frac{c_1^3}{4}\right)+\frac{C_2^\alpha(t)}{2}c_1\left(c_2-\frac{c_1^2}{2}\right)+\frac{C_3^\alpha(t)}{8}c_1^3\right]z^3+\cdots\end{aligned} \tag{14}$$

and

$$\begin{aligned}H(t,v(w))=\ \ &+\frac{C_1^\alpha(t)}{2}d_1w+\left[\frac{C_1^\alpha(t)}{2}\left(d_2-\frac{d_1^2}{2}\right)+\frac{C_2^\alpha(t)}{4}d_1^2\right]w^2\\ &+\left[\frac{C_1^\alpha(t)}{2}\left(d_3+d_1d_2+\frac{d_1^3}{4}\right)+\frac{C_2^\alpha(t)}{2}d_1\left(c_2-\frac{c_1^2}{2}\right)+\frac{C_3^\alpha(t)}{8}d_1^3\right]w^3+\cdots\end{aligned} \tag{15}$$

Using Equations (8), (14), (9), and (15), respectively, we obtain

$$(\eta+\mu+2\zeta\delta)a_2=\frac{C_1^\alpha(t)}{2}c_1 \tag{16}$$

$$(2\eta+\mu)\left[a_3\left(1+\frac{6\delta}{2\eta+1}\right)+\frac{a_2^2}{2}(\mu-1)\right]=\frac{C_1^\alpha(t)}{2}\left(c_2-\frac{c_1^2}{2}\right)+\frac{C_2^\alpha(t)}{4}c_1^2 \tag{17}$$

$$(3\eta+\mu)\left[a_4\left(1+\frac{12(2\eta+\mu)\delta}{(2\eta+1)(3\eta+\mu)}\right)+(\mu-1)a_2a_3+\frac{a_2^3}{6}(\mu-1)(\mu-2)\right]=\frac{C_1^\alpha(t)}{2}\left(c_3+c_1c_2+\frac{c_1^3}{4}\right)+\frac{c_1C_2^\alpha(t)}{2}\left(c_2-\frac{c_1^2}{2}\right)+\frac{c_1^3C_3^\alpha(t)}{8} \tag{18}$$

$$-(\eta+\mu+2\zeta\delta)a_2=\frac{C_1^\alpha(t)}{2}d_1 \tag{19}$$

$$(2\eta+\mu)\left[-a_3\left(1+\frac{6\delta}{2\eta+1}\right)+a_2^2\left(\frac{\mu+3}{2}+\frac{12\delta}{2\eta+\mu}\right)\right]=\frac{C_1^\alpha(t)}{2}\left(d_2-\frac{d_1^2}{2}\right)+\frac{C_2^\alpha(t)}{4}d_1^2 \tag{20}$$

$$(3\eta+\mu)\left[-a_4\left(1+\frac{12(2\eta+\mu)\delta}{(2\eta+1)(3\eta+\mu)}\right)+(\mu-1)\,a_2a_3-a_2^3\left(\frac{(\mu+4)(\mu+5)}{6}+\frac{60(2\eta+\mu)\delta}{(2\eta+1)(3\eta+\mu)}\right)\right]=\frac{C_1^\alpha(t)}{2}\left(d_3+d_1d_2+\frac{d_1^3}{4}\right)+\frac{d_1C_2^\alpha(t)}{2}\left(c_2-\frac{c_1^2}{2}\right)+\frac{d_1^3C_3^\alpha(t)}{8} \tag{21}$$

From Equations (16) and (19), we obtain

$$a_2=\frac{C_1^\alpha(t)}{2(\eta+\mu+2\zeta\delta)}c_1 \tag{22}$$

$$c_1=-d_1 \tag{23}$$

Now, subtracting Equation (20) from Equation (17) and using Equation (22), we can conclude that

$$a_3=a_2^2+\frac{C_1^\alpha(t)}{4(2\eta+\mu+6\zeta\delta)}(c_2-d_2)=\frac{(C_1^\alpha(t))^2}{(\eta+\mu+2\zeta\delta)^2}c_1^2+\frac{C_1^\alpha(t)}{4(2\eta+\mu+6\zeta\delta)}(c_2-d_2) \tag{24}$$

Also, subtracting Equation (21) from Equation (18) and substituting the values of Equations (22) and (24), we get

$$a_4=\frac{5(C_1^\alpha(t))^2c_1(c_2-d_2)}{16(\eta+\mu+2\zeta\delta)(2\eta+\mu+6\zeta\delta)}+\frac{C_1^\alpha(t)(c_3-d_3)}{4(3\eta+\mu+12\zeta\delta)}+\frac{(C_2^\alpha(t)-C_1^\alpha(t))}{4(3\eta+\mu+12\zeta\delta)}c_1(c_2+d_2)+\left[\frac{-(3\eta+\mu+12\zeta\delta)(\mu^2+3\mu-4)(C_1^\alpha(t))^3+6(\eta+\mu+2\zeta\delta)^3(C_1^\alpha(t)-2C_2^\alpha(t)+C_3^\alpha(t))}{48(\eta+\mu+2\zeta\delta)^3(3\eta+\mu+12\zeta\delta)}\right]c_1^3 \tag{25}$$

Therefore, Equations (22), (24) and (25), yield

$$a_2a_4-a_3^2=\frac{(C_1^\alpha(t))^3c_1^2(c_2-d_2)}{32(\eta+\mu+2\zeta\delta)^2(2\eta+\mu+6\zeta\delta)}+\frac{(C_1^\alpha(t))^2c_1(c_3-d_3)}{8(\eta+\mu+2\zeta\delta)(3\eta+\mu+12\zeta\delta)}+\frac{[C_2^\alpha(t)-C_1^\alpha(t)]C_1^\alpha(t)c_1^2}{8(\eta+\mu+2\zeta\delta)(3\eta+\mu+12\zeta\delta)}(c_2+d_2)-\frac{(C_1^\alpha(t))^2}{4(2\eta+\mu+6\zeta\delta)^2}(c_2-d_2)^2+\left[\frac{6(\eta+\mu+2\zeta\delta)^3(C_1^\alpha(t)-2C_2^\alpha(t)+C_3^\alpha(t))-[(3\eta+\mu)(\mu^2+3\mu+2)+72\zeta\delta](C_1^\alpha(t))^3}{96(\eta+\mu+2\zeta\delta)^4(3\eta+\mu+12\zeta\delta)}\right]c_1^4C_1^\alpha(t) \tag{26}$$

Under the same assumption of Lemma 2.8, we obtain

$$2c_2=(4-c_1^2)x+c_1^2,\ \text{ and }\ 2d_2=(4-d_1^2)y+d_1^2 \tag{27}$$

Moreover,

$$4c_3=2(4-c_1^2)c_1x-(4-c_1^2)c_1x^2+2(4-c_1^2)(1-|x|^2)z\,|+c_1^3,\text{ and}$$
$$4d_3=2(4-d_1^2)d_1x-(4-d_1^2)d_1x^2+2(4-d_1^2)(1-|y|^2)w+d_1^3 \tag{28}$$

From Equations (23), (27) and (28), we deduce

$$c_2 - d_2 = \frac{4-c_1^2}{2}(x-y), c_2 + d_2 = c_1^2 + \frac{4-c_1^2}{2}(x+y) \tag{29}$$

$$c_3 - d_3 = \frac{c_1^3}{2} + \frac{(4-c_1^2)c_1}{2}(x+y) - \frac{(4-c_1^2)c_1}{4}(x^2+y^2) + \frac{4-c_1^2}{2}[(1-|x|^2)z - (1-|y|^2)w] \tag{30}$$

Now, using Lemma 2.8 and assuming that $c \in [0,2]$, such that $c_1 = c$. Moreover, making use of Equations (29) and (30) in Equation (26), and setting $|x| = \gamma_1, |y| = \gamma_2$, we can deduce that
$|a_2 a_4 - a_3^2| \le \mathcal{S}_1 + \mathcal{S}_2(\gamma_1+\gamma_2) + \mathcal{S}_3(\gamma_1^2+\gamma_2^2) + \mathcal{S}_4(\gamma_1+\gamma_2)^2 = F(\gamma_1,\gamma_2)$,

where,

$$\mathcal{S}_1 = \mathcal{S}_1(c,t) = \left[\frac{6(\eta+\mu+2\zeta\delta)^3 C_3^\alpha(t) - [(3\eta+\mu)(\mu^2+3\mu+2)+72\zeta\delta](C_1^\alpha(t))^3}{96(\eta+\mu+2\zeta\delta)^4(3\eta+\mu+12\zeta\delta)}\right] c^4 C_1^\alpha(t) + \frac{(C_1^\alpha(t))^2 c(4-c^2)}{8(\eta+\mu+2\zeta\delta)(3\eta+\mu+12\zeta\delta)} \ge 0$$

$$\mathcal{S}_2 = \mathcal{S}_2(c,t) = \frac{(C_1^\alpha(t))^3 c^2(4-c^2)}{64(\eta+\mu+2\zeta\delta)^2(2\eta+\mu+6\zeta\delta)} + \frac{C_1^\alpha(t) C_2^\alpha(t) c^2(4-c^2)}{16(\eta+\mu+2\zeta\delta)(3\eta+\mu+12\zeta\delta)} \ge 0$$

$$\mathcal{S}_3 = \mathcal{S}_3(c,t) = \frac{\left(C_1^\alpha(t)\right)^2 c(c-2)(4-c^2)}{32(\eta+\mu+2\zeta\delta)(3\eta+\mu+12\zeta\delta)} \le 0$$

$$\mathcal{S}_4 = \mathcal{S}_4(c,t) = \frac{\left(C_1^\alpha(t)\right)^2 (4-c^2)^2}{64(2\eta+\mu+6\zeta\delta)^2} \ge 0 \left(\frac{1}{2} < t < 1, 0 \le c \le 2\right).$$

Next, we will investigate the maximum of $F(\gamma_1,\gamma_2)$ in the closed square
$\Upsilon := \{(\gamma_1,\gamma_2): 0 \le \gamma_1 \le 1, 0 \le \gamma_2 \le 1\}$.

Given that $\mathcal{S}_3 < 0$ and $\mathcal{S}_3 + 2\mathcal{S}_4 > 0$ for all $t \in \left(\frac{1}{2},1\right)$ and $c \in (0,2)$, we obtain that
$F_{\gamma_1\gamma_1}F_{\gamma_2\gamma_2} - \left(F_{\gamma_1\gamma_2}\right)^2 < 0$, for all $(\gamma_1,\gamma_2) \in \Upsilon$.

Therefore, the function $F$ has no local maximum inside the square $\Upsilon$. It remains to check whether $F$ has a maximum on $\Upsilon$. To do that, let $\gamma_2 = 0$ and $0 \le \gamma_1 \le 1$ (or $\gamma_1 = 0$ and $0 \le \gamma_2 \le 1$). This implies that
$F(0,\gamma_2) = G(\gamma) = \mathcal{S}_1 + \mathcal{S}_2\gamma_2 + (\mathcal{S}_3+\mathcal{S}_4)^2\gamma_2$.

Based on the value of $\mathcal{S}_3 + \mathcal{S}_4$, and for $0 < \gamma_2 < 1$, and any $c \in [0,2)$ and for all $t \in \left(\frac{1}{2},1\right)$, we have the following cases:
**(i)** If $\mathcal{S}_3 + \mathcal{S}_4 \ge 0$, then it is easy to see that $G'(\gamma_2) = 2(\mathcal{S}_3+\mathcal{S}_4)\gamma_2 + \mathcal{S}_2 > 0$. Hence, $G(\gamma_2)$ is monotonically increasing. Therefore, for any $c \in [0,2)$ and $t \in \left(\frac{1}{2},1\right)$, the maximum of $G(\gamma)$ happens at $\gamma_2 = 1$ and
$\max(G(\gamma_2)) = G(1) = \mathcal{S}_1 + \mathcal{S}_2 + \mathcal{S}_3 + \mathcal{S}_4$.

**(ii)** If $\mathcal{S}_3 + \mathcal{S}_4 < 0$ and $\mathcal{S}_2 + 2(\mathcal{S}_3+\mathcal{S}_4) \ge 0$, then $\mathcal{S}_2 + 2(\mathcal{S}_3+\mathcal{S}_4) < 2(\mathcal{S}_3+\mathcal{S}_4)\gamma_2 + \mathcal{S}_2 < \mathcal{S}_2$. Hence, $G'(\gamma_2) > 0$. Therefore, for any $c \in [0,2)$ and $t \in \left(\frac{1}{2},1\right)$, the maximum of $G(\gamma)$ happens at $\gamma_1 = 1$ and
$\max(G(\gamma_2)) = G(1) = \mathcal{S}_1 + \mathcal{S}_2 + \mathcal{S}_3 + \mathcal{S}_4$.

Moreover, if $c = 2$, then we have

$$F(\gamma_1,\gamma_2) = \mathcal{S}_1|_{c=2}$$
$$= \frac{C_1^\alpha(t)\left|6C_3^\alpha(t)(\eta+\mu+2\zeta\delta)^3-[(3\eta+\mu)(\mu^2+3\mu+2)+72\zeta\delta](C_1^\alpha(t))^3\right|}{6(\eta+\mu+2\zeta\delta)^4(3\eta+\mu+12\zeta\delta)} \tag{31}$$

Equation (31) and the cases **(i)** and **(ii)** yield
$\max\big(G(\gamma_2)\big) = G(1) = \mathcal{S}_1 + \mathcal{S}_2 + \mathcal{S}_3 + \mathcal{S}_4.$

Now, let s $\gamma_2 = 1$ and $0 \le \gamma_1 \le 1$ (or $\gamma_1 = 1$ and $0 \le \gamma_2 \le 1$ ), yield
$F(1,\gamma_2) = H(\gamma_2) = (\mathcal{S}_3 + \mathcal{S}_4)\gamma_2^2 + (\mathcal{S}_3 + 2\mathcal{S}_4)\gamma_2 + \mathcal{S}_1 + \mathcal{S}_2 + \mathcal{S}_3 + \mathcal{S}_4.$

Applying the same argument above of $\mathcal{S}_3 + \mathcal{S}_4$, we get that
$\max\big(G(\gamma_2)\big) = H(1) = \mathcal{S}_1 + 2\mathcal{S}_2 + 2\mathcal{S}_3 + 4\mathcal{S}_4.$

In the light of the fact that $G(1) \le H(1)$ for $c \in [0,2]$ and $t \in \left(\frac{1}{2},1\right)$, we conclude that $\max\big(F(\gamma_1,\gamma_2)\big) = F(1,1)$ on $\Upsilon$. Hence, $F$ has a maximum value at the point (1,1) that lies on the boundary of the closed square $\Upsilon$. Then, we define the map $\Psi: [0,2] \to \mathbb{R}$ as follows:

$$\Psi(c,t) = \max F(\gamma_1,\gamma_2) = F(1,1) = \mathcal{S}_1 + 2\mathcal{S}_2 + 2\mathcal{S}_3 + 4\mathcal{S}_4, \ \text{for any value of } t \in [0,2] \tag{32}$$

Now, substituting $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3$, and $\mathcal{S}_4$ into the function $\Psi$ given by Equation (32), we get,

$$\Psi(c,t) = \frac{(C_1^\alpha(t))^2}{(2\eta+\mu+6\zeta\delta)^2} + \frac{\Theta_1 c^4 + 12\Theta_2 c^2}{96(\eta+\mu+2\zeta\delta)^4(3\eta+\mu+12\zeta\delta)(2\eta+\mu+6\zeta\delta)^2}$$

where,

$$\begin{aligned}\Theta_1 := \Theta_1(\lambda,\mu;t) = \ & C_1^\alpha(t)(2\eta+\mu+6\zeta\delta)^2 \mid 6C_3^\alpha(t)(\eta+\mu+2\zeta\delta)^3 \\ & -[(\mu^2+3\mu+2)(3\eta+\mu)+72\zeta\delta](C_1^\alpha(t))^3 \mid -3C_1^\alpha(t)[(C_1^\alpha(t))^2(3\eta+\mu+12\zeta\delta) \\ & +4C_2^\alpha(t)(\eta+\mu+2\zeta\delta)(2\eta+\mu+6\zeta\delta)](\eta+\mu+2\zeta\delta)^2(2\eta+\mu+6\zeta\delta) \\ & -6\big(C_1^\alpha(t)\big)^2\big(\eta^2+2\zeta\delta(3\eta-\mu+6\zeta\delta)\big)(\eta+\mu+2\zeta\delta)^3.\end{aligned}$$

$$\begin{aligned}\Theta_2 := \Theta_2(\lambda,\mu;t) = \ & [(C_1^\alpha(t))^3(2\eta+\mu+6\zeta\delta)(3\eta+\mu+12\zeta\delta) \\ & +2C_1^\alpha(t)(C_1^\alpha(t)+2C_2^\alpha(t))(\eta+\mu+2\zeta\delta)(2\eta+\mu+6\zeta\delta)^2 \\ & -4\big(C_1^\alpha(t)\big)^2(\eta+\mu+2\zeta\delta)^2(3\eta+\mu+12\zeta\delta)\Big]\,(\eta+\mu+2\zeta\delta)^2.\end{aligned}$$

Now, for fixed value of $t$, suppose that $\Psi(c,t)$ has a maxima inside $c \in [0,2]$. The derivative of $\Psi(c,t)$ with respect to $c$ is

$$\Psi'(c,t) = \frac{(\Theta_1 c^2 + 6\Theta_2)c}{24(\eta+\mu+2\zeta\delta)^4(3\eta+\mu+12\zeta\delta)(2\eta+\mu+6\zeta\delta)^2}$$

We will determine $\Psi'(c,t)$ sign based on $\Theta_1$ and $\Theta_2$ signs as follows:

(i) If $\Theta_1 \ge 0$ and $\Theta_2 \ge 0$, this implies $\Psi'(c,t) \ge 0$. Hence, the function $\Psi(c,t)$ is monotonically increasing on (0,2). Thus,

$$\max\{\Psi(c,t): c \in (0,2)\} = \Psi(2^-,t) \tag{33}$$

(ii) If $\Theta_1 > 0$ and $\Theta_2 < 0$, then $c_0 = \sqrt{\frac{-6\Theta_2}{\Theta_1}}$ is the equilibrium point of $\Psi(c,t)$. Now, since $c_0 \in (0,2)$ and $\Psi''(c,t) > 0$, then the local minimum of $\Psi(c,t)$ will occurs at $c_0$. Hence, $\Psi(c,t)$ has no local maximum.

(iii) If $\Theta_1 \leq 0$ and $\Theta_2 \leq 0$, then $\Psi'(c,t) \leq 0$. Hence, the function $\Psi(c,t)$ is monotonically decreasing on (0,2). Thus,

$$\max\{\Psi(c,t): c \in (0,2)\} = \Psi(0^+,t) = 4\mathcal{S}_4 = \frac{(C_1^{\alpha}(t))^2}{(2\eta+\mu+6\zeta\delta)^2} \tag{34}$$

(iv) If $\Theta_1 < 0$ and $\Theta_2 > 0$, then $c_0 = \sqrt{\frac{-6\Theta_2}{\Theta_1}}$ is the equilibrium point of $\Psi(c,t)$. Now, since $\Psi''(c,t) < 0$ and $c_0 \in (0,2)$, then the maxima of $\Psi(c,t)$ occurs at $c_0$. Thus,

$$\max\{\Psi(c,t): c \in (0,2)\} = \Psi(c_0,t) \tag{35}$$

where,

$$\Psi(c_0,t) = \frac{4\alpha^2 t^2}{(2\eta+\mu+6\zeta\delta)^2} - \frac{3\Theta_2^2}{8\Theta_1(\eta+\mu+2\zeta\delta)^4(3\eta+\mu+12\zeta\delta)(2\eta+\mu+6\zeta\delta)^2}$$

Therefore, from Equations (33) to (35), the proof of Theorem 3.1 is finished.

## 4. Consequences and Corollaries

Specializing the parameters in the main result, the following consequences are obtained.

***Corollary* 4.1:** Let $f \in {}^{1}\mathcal{B}_{\Sigma}^{\alpha}(t,\eta,\mu)$, then

$$|a_2a_4 - a_3^2| \leq \begin{cases} \Psi(2^-,t) & ;\Theta_3 \geq 0 \text{ and } \Theta_4 \geq 0 \\ \max\left\{\frac{4\alpha^2t^2}{(2\eta+\mu)^2}, \Psi(2^-,t)\right\} & ;\Theta_3 > 0 \text{ and } \Theta_4 < 0 \\ \frac{4\alpha^2t^2}{(2\eta+\mu)^2} & ;\Theta_3 \leq 0 \text{ and } \Theta_4 \leq 0 \\ \max\{\Psi(c_0,t), \Psi(2^-,t)\} & ;\Theta_3 < 0 \text{ and } \Theta_4 > 0 \end{cases}$$

where,

$$\Psi(2^-,t) = \frac{4\alpha^2t^2}{(2\eta+\mu)^2} + \frac{E\Theta_3 + 3\Theta_4}{6(\eta+\mu)^4(3\eta+\mu)(2\eta+\mu)^2}$$

$$\Psi(c_0,t) = \frac{4\alpha^2t^2}{(2\eta+\mu)^2} - \frac{3\Theta_4^2}{8\Theta_3(\eta+\mu)^4(3\eta+\mu)(2\eta+\mu)^2}, \; c_0 = \sqrt{\frac{-6\Theta_4}{\Theta_3}}$$

And

$$\begin{aligned}\Theta_3 \coloneqq \Theta_3(\eta,\mu;t) = {} & 8\alpha^2t^2|[2(1+\alpha)(2+\alpha)t^2 - 3(1+\alpha)](\eta+\mu)^3 - 2\alpha^2t^2[(\mu^2+3\mu+2)(3\eta+\mu)]|(2\eta+\mu)^2 \\ & -24\alpha^2t[\alpha t^2(3\eta+\mu) + (2(1+\alpha)t^2 - 1)(\eta+\mu)(2\eta+\mu)](\eta+\mu)^2(2\eta+\mu) \\ & -24t^2\alpha^2(\eta+\mu)^3[\eta + (3\eta-\mu)].\end{aligned}$$

$$\begin{aligned}\Theta_4 \colon= \Theta_4(\eta,\mu;t) = {} & 8\alpha^2t[\alpha t^2(2\eta+\mu)(3\eta+\mu) + [-1+2(1+\alpha)t^2+t](\eta+\mu)(2\eta+\mu)^2 \\ & -2t(\eta+\mu)^2(3\eta+\mu)](\eta+\mu)^2.\end{aligned}$$

***Corollary* 4.2:** Let $f \in {}^{2}\mathcal{B}_{\Sigma}^{\alpha}(t,\eta,\delta)$, then

$$\left|a_2 a_4 - a_3^2\right| \leq \begin{cases} \Psi(2^-,t) & ;\Theta_5 \geq 0 \text{ and } \Theta_6 \geq 0 \\ \max\left\{\frac{4\alpha^2 t^2}{(2\eta+1+6\zeta\delta)^2}, \Psi(2^-,t)\right\} & ;\Theta_5 > 0 \text{ and } \Theta_6 < 0 \\ \frac{4\alpha^2 t^2}{(2\eta+1+6\zeta\delta)^2} & ;\Theta_5 \leq 0 \text{ and } \Theta_6 \leq 0 \\ \max\{\Psi(c_0,t), \Psi(2^-,t)\} & ;\Theta_5 < 0 \text{ and } \Theta_6 > 0 \end{cases}$$

where,

$$\Psi(2^-,t) = \frac{4\alpha^2 t^2}{(2\eta+1+6\zeta\delta)^2} + \frac{\Theta_5 + 3\Theta_6}{6(\eta+1+2\zeta\delta)^4(3\eta+1+12\zeta\delta)(2\eta+1+6\zeta\delta)^2}$$

$$\Psi(c_0,t) = \frac{4\alpha^2 t^2}{(2\eta+1+6\zeta\delta)^2} - \frac{3\Theta_6^2}{8\Theta_5(\eta+1+2\zeta\delta)^4(3\eta+1+12\zeta\delta)(2\eta+1+6\zeta\delta)^2},\ c_0 = \sqrt{\frac{-6\Theta_6}{\Theta_5}}$$

and

$$\Theta_5 \coloneqq \Theta_5(\eta,1;t) = 8\alpha^2 t^2 |[2(1+\alpha)(2+\alpha)t^2 - 3(1+\alpha)](\eta+1+2\zeta\delta)^3 - 2\alpha^2 t^2[6(3\eta+1) + 72\zeta\delta]|(2\eta+1+6\zeta\delta)^2$$
$$-24\alpha^2 t[\alpha t^2(3\eta+1+12\zeta\delta) + (2(1+\alpha)t^2 - 1)(\eta+1+2\zeta\delta)(2\eta+1+6\zeta\delta)](\eta+1+2\zeta\delta)^2(2\eta+1+6\zeta\delta)$$
$$-24t^2\alpha^2(\eta+1+2\zeta\delta)^3[\eta^2 + 2\zeta\delta(3\eta-1+6\zeta\delta)].$$
$$\Theta_6 := \Theta_6(\eta,1;t) = 8\alpha^2 t[\alpha t^2(2\eta+1+6\zeta\delta)(3\eta+1+12\zeta\delta) + [-1+2(1+\alpha)t^2+t](\eta+1+2\zeta\delta)(2\eta+1+6\zeta\delta)^2$$
$$-2t(\eta+1+2\zeta\delta)^2(3\eta+1+12\zeta\delta)](\eta+1+2\zeta\delta)^2.$$

***Corollary* 4.3:** Let $f \in {}^3\mathcal{B}_\Sigma^\alpha(t,\eta)$, then

$$\left|a_2 a_4 - a_3^2\right| \leq \begin{cases} \Psi(2^-,t) & ;\Theta_7 \geq 0 \text{ and } \Theta_8 \geq 0 \\ \max\left\{\frac{4\alpha^2 t^2}{(2\eta+1)^2}, \Psi(2^-,t)\right\} & ;\Theta_7 > 0 \text{ and } \Theta_8 < 0 \\ \frac{4\alpha^2 t^2}{(2\eta+1)^2} & ;\Theta_7 \leq 0 \text{ and } \Theta_8 \leq 0 \\ \max\{\Psi(c_0,t), \Psi(2^-,t)\} & ;\Theta_7 < 0 \text{ and } \Theta_8 > 0 \end{cases}$$

where,

$$\Psi(2^-,t) = \frac{4\alpha^2 t^2}{(2\eta+1)^2} + \frac{\Theta_7 + 3\Theta_8}{6(\eta+1)^4(3\eta+1)(2\eta+1)^2}$$

$$\Psi(c_0,t) = \frac{4\alpha^2 t^2}{(2\eta+1)^2} - \frac{3\Theta_8^2}{8\Theta_7(\eta+1)^4(3\eta+1)(2\eta+1)^2},\ c_0 = \sqrt{\frac{-6\Theta_8}{\Theta_7}}$$

and

$$\Theta_7 := \Theta_7(\eta,1;t) = 8\alpha^2 t^2 |[2(1+\alpha)(2+\alpha)t^2 - 3(1+\alpha)](\eta+1)^3 - 2\alpha^2 t^2[6(3\eta+1)]|(2\eta+1)^2$$
$$-24\alpha^2 t[\alpha t^2(3\eta+1) + (2(1+\alpha)t^2 - 1)(\eta+1)(2\eta+1)](\eta+1)^2(2\eta+1)$$
$$-24t^2\alpha^2\eta^2(\eta+1)^3.$$

$$\Theta_8 := \Theta_8(\eta,1;t) = 8\alpha^2 t[\alpha t^2(2\eta+1)(3\eta+1) + [-1+2(1+\alpha)t^2+t](\eta+1)(2\eta+1)^2$$
$$-2t(\eta+1)^2(3\eta+1)](\eta+1)^2.$$

***Corollary* 4.4:** Let $f \in {}^4\mathcal{B}_\Sigma^\alpha(t)$, then

$$\left|a_2a_4-a_3^2\right|\le\begin{cases}\Psi(2^-,t) & ;\Theta_9\ge 0 \quad \text{and} \quad \Theta_{10}\ge 0\\ \max\left\{\frac{4\alpha^2t^2}{9},\Psi(2^-,t)\right\} & ;\Theta_9> 0 \quad \text{and} \quad \Theta_{10}< 0\\ \frac{4\alpha^2t^2}{9} & ;\Theta_9\le 0 \quad \text{and} \quad \Theta_{10}\le 0\\ \max\{\Psi(c_0,t),\Psi(2^-,t)\} & ;\Theta_9< 0 \quad \text{and} \quad \Theta_{10}> 0\end{cases}$$

where,

$$\Psi(2^-,t)=\frac{4\alpha^2t^2}{9}+\frac{\Theta_9+3\Theta_{10}}{3456}$$

$$\Psi(c_0,t)=\frac{4\alpha^2t^2}{9}-\frac{\Theta_{10}^2}{1536\Theta_9},\ c_0=\sqrt{\frac{-6\Theta_{10}}{\Theta_9}}$$

and

$$\Theta_9:=\Theta_9(1,1;t)=\ 2\alpha^2t^2|8[2(1+\alpha)(2+\alpha)t^2-3(1+\alpha)]-48\alpha^2t^2|$$
$$-288\alpha^2t[4\alpha t^2+6(2(1+\alpha)t^2-1)]-192t^2\alpha^2$$

$$\Theta_{10}:=\Theta_{10}(1,1;t)=32\alpha^2t[12\alpha t^2+18[-1+2(1+\alpha)t^2+t]-32t].$$

## 5. Conclusion

In this study, we have introduced a general subclass $\mathcal{B}_\Sigma^\alpha(t,\eta,\mu,\delta)$ of bi-univalent normalized-analytic functions, which utilizes Gegenbauer polynomials. Subsequently, the optimal maximum bound for the 2nd Hankel determinant of this subclass was derived. Furthermore, by selecting suitable values for the parameters $\delta,\eta$, and $\mu$, we have arrived at similar results for the subclasses ${}^1\mathcal{B}_\Sigma^\alpha(t,\eta,\mu)$, ${}^2\mathcal{B}_\Sigma^\alpha(t,\eta,\delta)$, ${}^3\mathcal{B}_\Sigma^\alpha(t,\eta)$, and ${}^4\mathcal{B}_\Sigma^\alpha(t)$.

These results will have significant implications for the subclasses for Legendre polynomials $\mathcal{B}_\Sigma^{1/2}(t,\eta,\mu,\delta)$ and second-kind Chebyshev polynomials $\mathcal{B}_\Sigma^1(t,\eta,\mu,\delta)$.

### Conflict of Interest

The authors confirm that there is no conflict of interest to declare for this publication.

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