International Journal of Mathematics and
Computer Science, **19**(2024), no. 4, 1077–1092

Bi-Univalent Functions Using Bell Distribution Associated With Meixner-Pollaczek Polynomials

Abdullah Alsoboh[1], **Ala Amourah**[2], **Jamal Salah**[3]

[1]Department of Mathematics
Faculty of Science
Philadelphia University
19392 Amman, Jordan

[2]Mathematics Education Program
Faculty of Education and Arts
Sohar University
Sohar 3111, Oman
and
Jadara University Research Center
Jadara University
Irbid, Jordan

[3]College of Applied and Health Sciences
A'Sharqiyah University
Post Box no. 42, Post Code no. 400
Ibra, Oman

email: aalsoboh@philadelphia.edu.jo, AAmourah@su.edu.om,
damous73@yahoo.com

(Received March 1, 2024, Accepted April 3, 2024,
Published June 1, 2024)

Abstract

The purpose of this paper is to introduce new classes of bi-univalent functions using Bell Distribution. These classes are defined using

Key words and phrases: Bell distribution, Meixner-Pollaczek polynomials, bi-univalent functions, analytic functions, Fekete-Szegö problem functions, Fekete-Szegö problem, Subordination, q-calculus.
AMS (MOS) Subject Classifications: 30C45.
The corresponding authors are Abdullah Alsoboh and Jamal Salah.
ISSN 1814-0432, 2024, http://ijmcs.future-in-tech.net

Meixner-Pollaczek polynomials. We derive estimates for the Taylor-Maclaurin coefficients, specifically $|a_2|$ and $|a_3|$. Moreover, we explore Fekete-Szegö functional problems for functions belonging to these subclasses. Furthermore, by focusing on specific parameters in our main results, we uncover several new discoveries.

1 Preliminaries

Orthogonal polynomials are a specific type of polynomials that satisfy a unique orthogonality condition when considering a specific weight function over a given interval. These polynomials have been extensively studied in various areas of mathematics such as approximation theory, numerical analysis, and mathematical physics. One of their key features is that they form a basis for the space of square-integrable functions with respect to the weight function. This allows for efficient representation and approximation of functions using polynomial expansions. Several well-known families of orthogonal polynomials exist including Legendre polynomials, Chebyshev polynomials, Meixner-Pollaczek polynomials, and Jacobi polynomials. Each of these families has its own weight function and orthogonality properties, tailored to specific applications (see [1, 2]).

Meixner-Pollaczek polynomials are an important type of orthogonal polynomials that have been applied in various areas of mathematics, particularly in probability theory and mathematical physics. These polynomials are named after mathematicians Wolfgang Meixner and Erwin Pollaczek and are known for their orthogonality property in relation to a specific weight function on the real line. Meixner-Pollaczek polynomials are commonly used in the study of stochastic processes such as random walks and queuing systems. They are often used as solutions to difference equations or differential equations with a discrete spectrum. These polynomials have interesting combinatorial properties and have been extensively researched due to their connections to special functions including hypergeometric functions and q-series. The versatility and analytical properties of Meixner-Pollaczek polynomials make them indispensable tools in the analysis of probabilistic models and the investigation of spectral properties of differential operators see([3], [4]).

Let $\mathcal{A}$ be the class of all analytic functions Θ defined in the disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ and normalized by the conditions $\Theta(0) = 0$ and $\Theta'(0) = 1$. Therefore, every $\Theta \in \mathcal{A}$ has a Taylor-Maclaurin series expansion of the form:

$$\Theta(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (z \in \mathbb{U}). \tag{1.1}$$

Furthermore, let $\mathcal{S}$ represent the set of all functions $\Theta \in \mathcal{A}$ that are univalent in $\mathbb{U}$.

The field of geometric function theory stands to gain significant advantages from the robust tools offered by differential subordination of analytic functions. The initial differential subordination problem was introduced by Miller and Mocanu [5], with further references provided in [6]. The comprehensive developments in this area have been documented in Miller and Mocanu's book [7], including publication dates.

By the Koebe one-quarter theorem [7], every function Θ in the set $\mathcal{S}$ has an inverse Θ^{-1} defined by

$$\Theta^{-1}(\Theta(z)) = z \qquad (z \in \mathbb{U})$$

and

$$\Theta(\Theta^{-1}(w)) = w \qquad (|w| < r_0(\Theta);\ r_0(\Theta) \geq \frac{1}{4})$$

where

$$\Theta^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots . \tag{1.2}$$

A function is considered bi-univalent in $\mathbb{U}$ if both $\Theta(z)$ and $\Theta^{-1}(z)$ are univalent in $\mathbb{U}$.

Let Σ represent the class of bi-univalent functions in $\mathbb{U}$, as defined by equation (1.1). The class Σ includes various examples of functions such as

$$\frac{z}{1-z}, \qquad \log\frac{1}{1-z}, \qquad \log\sqrt{\frac{1+z}{1-z}}.$$

However, the familiar Koebe function is not included in the set Σ. Moreover, there are other well-known examples of functions in $\mathbb{U}$, such as

$$\frac{2z - z^2}{2} \text{ and } \frac{z}{1-z^2},$$

that are not members of Σ.

In 1933, Fekete and Szegö [8] established a precise upper limit for the functional $\eta a_2^2 - a_3$, where η is a real number ($0 \leq \eta \leq 1$), applied to a univalent function Θ. This led to the formulation of the classical Fekete-Szegö problem or inequality, which aims to determine the optimal bounds for this functional across all compact families of functions Θ belonging to $\mathcal{A}$, regardless of the complex value of η.

2 Bell Distribution and Meixner-Pollaczek polynomials

In 2018, Castellares et al. introduced the Bell distribution [9], which is suitable for count data with over-dispersion. The Bell distribution is an improvement over the Bell numbers [10, 11]. The probability density function of a discrete random variable X, which follows the Bell distribution, is expressed as:

$$\mathcal{P}(X=m)=\frac{\vartheta^m e^{e^{\left(-\vartheta^2\right)+1}}\mathcal{B}_m}{m!}; \qquad m=1,2,3,\cdots. \tag{2.1}$$

where $\mathcal{B}_m=\frac{1}{e}\sum\limits_{k=0}^{\infty}\frac{k^m}{k!}$ are the Bell numbers, $m\geq 2$, and $\vartheta>0$.

Example of the Bell numbers are $\mathcal{B}_2=2, \mathcal{B}_3=5, \mathcal{B}_4=15$ and $\mathcal{B}_5=52$.

Now, we introduce a new power series whose coefficients represent the probabilities of the Bell distribution

$$\mathcal{B}(\vartheta,z)=z+\sum_{n=2}^{\infty}\frac{\vartheta^{n-1}\mathcal{B}_n}{(n-1)!\,e^{\vartheta^2-1}}z^n, \qquad (z\in\mathbb{U}), \tag{2.2}$$

where $\vartheta>0$.

Next, we consider the linear operator $\mathbb{P}_\vartheta:\mathcal{A}\to\mathcal{A}$ defined by the convolution (or Hadamard product)

$$\begin{aligned}\mathbb{P}_\vartheta f(z)&=\mathcal{B}(\vartheta,z)*f(z)\\&=z+\sum_{n=2}^{\infty}\frac{\vartheta^{n-1}e^{1-\vartheta^2}\mathcal{B}_n}{(n-1)!}a_nz^n, \qquad (z\in\mathbb{U}),\\&=z+\frac{2\vartheta}{e^{\vartheta^2-1}}a_2z^2+\frac{5\vartheta^2}{2e^{\vartheta^2-1}}a_3z^3+\frac{15\vartheta^3}{3!e^{\vartheta^2-1}}a_4z^4+\cdots,.\end{aligned} \tag{2.3}$$

The Meixner–Pollaczek polynomials $\mathcal{P}_n^{(\lambda)}(x;\Phi)$ (see [12]) of a real variable x as coefficients of

$$\mathcal{G}^\lambda(\varkappa,\Phi;z)=\frac{1}{\left(1-ze^{i\Phi}\right)^{\lambda-i\varkappa}\left(1-ze^{i\Phi}\right)^{\lambda+i\varkappa}}=\sum_{n=0}^{\infty}\mathcal{P}_n^{(\lambda)}(\varkappa;\Phi)z^n, \tag{2.4}$$

where

$$\mathcal{P}_n^{(\lambda)}(\varkappa;\Phi) = \frac{(2\lambda)_n}{n!} e^{in\Phi} \left(\frac{e^{2i\Phi}}{e^{2i\Phi}-1}\right)^n {}_2F_1\left(\begin{matrix}-n, \lambda+i\varkappa\\ 2\lambda\end{matrix}\middle| 1-\frac{1}{e^{2i\Phi}}\right), \tag{2.5}$$

are orthogonal with respect to the continuous weight

$$\omega(x;\Phi) = \left|\Gamma(\lambda+i\varkappa)\right|^2 e^{(2\Phi-\pi)\varkappa}, \tag{2.6}$$

On the interval $(-\infty,\infty)$, for $n\in\mathbb{N}$, $\lambda>0$, and $0<\Phi<\pi$, note that the complex Gamma function in Equation (2.5) takes the form [13]

$$\left|\Gamma(\lambda+i\varkappa)\right|^2 = \Gamma(\lambda+i\varkappa)\Gamma(\lambda-i\varkappa).$$

Special cases:
1) $\lim\limits_{\Phi\to\frac{\pi}{2}} \mathcal{P}_n^{(\frac{\alpha+1}{2})}(\frac{-\varkappa}{2\Phi};\Phi)$ is called Laguerre polynomial $L_n^\alpha(x)$.
2) $\lim\limits_{\lambda\to\infty} n!\lambda^{\frac{-n}{2}}\mathcal{P}_n^{(\lambda)}(\frac{-\varkappa\sqrt{\lambda}-\lambda\cos\Phi}{\sin\Phi};\Phi)$ is called Hermite polynomial $H_n(x)$.

The Meixner-Pollaczek polynomials can be represented by a three-term recurrence relation

$$\mathcal{P}_n^{(\lambda)}(\varkappa;\Phi) = \left(\varkappa+\alpha_n^{(\lambda,\Phi)}\right)\mathcal{P}_{n-1}^{(\lambda)}(\varkappa;\Phi) - C_n^{(\lambda,\Phi)}\mathcal{P}_{n-2}^{(\lambda)}(\varkappa;\Phi), \tag{2.7}$$

where

$$\alpha_n^{(\lambda,\Phi)} := \frac{\lambda+n-1}{\tan\Phi}; \quad \text{and} \quad C_n^{(\lambda,\Phi)} := \frac{(n-1)(2\lambda+n-2)}{4\sin^2\Phi}, \tag{2.8}$$

with $\mathcal{P}_{-1}^{(\lambda)}(\varkappa)=0$, $\mathcal{P}_0^{(\lambda)}(\varkappa)=1$ and $\alpha_n^{(\lambda,\frac{\pi}{2})} = \lim\limits_{\Phi\to\frac{\pi}{2}} \alpha_n^{(\lambda,\Phi)} = 0$.

The first few polynomials $\mathcal{P}_n^{(\lambda)}(x;\delta)$ are obtained from Equation (13) as follows (see [14]):

$$\begin{aligned} &\mathcal{P}_0^{(\lambda)}(\varkappa;\delta) = 1\\ &\mathcal{P}_1^{(\lambda)}(\varkappa;\delta) = \varkappa+\delta\lambda\\ &\mathcal{P}_2^{(\lambda)}(\varkappa;\delta) = \varkappa^2+\big(\delta\lambda+\lambda+1\big)\varkappa-2\delta^2\lambda+\delta\lambda^2+\delta\lambda-2\lambda \end{aligned} \tag{2.9}$$

Recently, a group of researchers began investigating subclasses of bi-univalent functions associated with orthogonal polynomials. Estimates for the initial coefficients of these functions have been identified. However, the challenge of determining precise bounds for the coefficients $|a_n|$, $(n = 3, 4, 5, \cdots)$ remains unresolved, as noted in various sources ([15]-[27]).

Several researchers have investigated specific subclasses of analytic functions by using various probability distributions, such as the Pascal, Poisson, and Borel distributions (see, for example, [28]-[33]). As far as we know, there have been no previous studies investigating a bi-univalent class of functions using the Bell Distribution series in combination with the Meixner-Pollaczek polynomials through the subordination principle. The main objective of this study is to investigate the properties of bi-univalent functions in relation to Meixner-Pollaczek polynomials. The investigation starts with the following definitions.

3 Definition and Examples

In this section, we will define and examine a new subclass of bi-univalent functions within the unit disk. This will be done by applying the principle of subordination. To establish this new class, we will make use of the Bell Distribution and subordination through Meixner-Pollaczek polynomials.

Definition 3.1. *Let λ be a positive number. A function $\Theta \in \Sigma$ given by (1.1) is said to be in the class $\mathfrak{G}_\Sigma\left(\vartheta, \varrho, \mathcal{G}^\lambda(\varkappa, \Phi; z)\right)$ if the following subordinations are satisfied:*

$$(1-\varrho)\frac{\mathbb{P}_\vartheta f(z)}{z} + \varrho(\mathbb{P}_\vartheta f(z))' \prec \mathcal{G}^\lambda(\varkappa, \Phi; z) \tag{3.1}$$

$$(1-\varrho)\frac{\mathbb{P}_\vartheta g(w)}{w} + \varrho(\mathbb{P}_\vartheta g(w))' \prec \mathcal{G}^\lambda(\varkappa, \Phi; w), \tag{3.2}$$

The function $g(w)$, defined by (1.2), is given when x is in the interval $[-1, 1]$, $\varrho \geq 0$ and $0 < \Phi < \pi$. The Meixner-Pollaczek polynomials $\mathcal{G}^\lambda(\varkappa, \Phi; z)$ are given by (2.4).

Example 3.2. Let λ be a positive number. A function $\Theta \in \Sigma$ given by (1.1) is said to be in the class $\mathfrak{G}_\Sigma\left(\vartheta, 0, \mathcal{G}^\lambda(\varkappa, \Phi; z)\right)$ if the following subordinations are satisfied:

$$\frac{\mathbb{P}_{\vartheta} f(z)}{z} \prec \mathcal{G}^{\lambda}(\varkappa, \Phi; z) \tag{3.3}$$

$$\frac{\mathbb{P}_{\vartheta} g(w)}{w} \prec \mathcal{G}^{\lambda}(\varkappa, \Phi; w), \tag{3.4}$$

The function $g(w)$, defined by (1.2), is given when x is in the interval $[-1, 1]$ and $0 < \Phi < \pi$. The Meixner-Pollaczek polynomials $\mathcal{G}^{\lambda}(\varkappa, \Phi; z)$ are given by (2.4).

Example 3.3. Let λ be a positive number. A function $\Theta \in \Sigma$ given by (1.1) is said to be in the class $\mathfrak{G}_{\Sigma}\left(\vartheta, 1, \mathcal{G}^{\lambda}(\varkappa, \Phi; z)\right)$ if the following subordinations are satisfied:

$$(\mathbb{P}_{\vartheta} f(z))' \prec \mathcal{G}^{\lambda}(\varkappa, \Phi; z) \tag{3.5}$$

$$(\mathbb{P}_{\vartheta} g(w))' \prec \mathcal{G}^{\lambda}(\varkappa, \Phi; w), \tag{3.6}$$

The function $g(w)$, defined by (1.2), is given when x is in the interval $[-1, 1]$ and $0 < \Phi < \pi$. The Meixner-Pollaczek polynomials $\mathcal{G}^{\lambda}(\varkappa, \Phi; z)$ are given by (2.4).

4 Coefficient bounds of the class $\mathfrak{G}_{\Sigma}(\varkappa, \delta, \lambda)$

First, let's provide the coefficient estimates for the class $\mathfrak{G}_{\Sigma}\left(\vartheta, \varrho, \mathcal{G}^{\lambda}(\varkappa, \Phi; z)\right)$ as defined in Definition 3.3.

Theorem 4.1. *Let $\Theta \in \Sigma$ given by (1.1) belong to the class $\mathfrak{G}_{\Sigma}\left(\vartheta, \varrho, \mathcal{G}^{\lambda}(\varkappa, \Phi; z)\right)$. Then*

$$|a_2| \leq \frac{\frac{e^{\vartheta^2-1}}{\vartheta}\left|\varkappa + \delta\lambda\right|\sqrt{2\left(\varkappa + \delta\lambda\right)}}{\sqrt{\left|\left(5(1+2\varrho)e^{\vartheta^2-1} - 8(1+\varrho)^2\right)\varkappa^2 + \left(10\delta\lambda(1+2\varrho)e^{\vartheta^2-1} - 8(\delta\lambda+\lambda+1)(1+\varrho)^2\right)\varkappa + 8(1+\varrho)^2\left(2\delta^2\lambda - \delta\lambda^2 - \delta\lambda + 2\lambda\right)\right|}}.$$

and

$$|a_3| \leq \frac{\left(e^{\vartheta^2-1}\right)^2\left(\varkappa + \delta\lambda\right)^2}{4(1+\varrho)^2\vartheta^2} + \frac{2e^{\vartheta^2-1}|\varkappa + \delta\lambda|}{5(1+2\varrho)\vartheta^2}.$$

Proof. Let $\Theta \in \mathfrak{G}_\Sigma\left(\vartheta, \varrho, \mathcal{G}^\lambda(\varkappa, \Phi; z)\right)$. According to Definition 3.3, there exist analytic functions w and v such that $w(0) = v(0) = 0$ and $|w(z)| < 1$. If $|v(w)| < 1$ for all $z, w \in \mathbb{U}$, we can express this as follows:

$$(1-\varrho)\frac{\mathbb{P}_\vartheta f(z)}{z} + \varrho(\mathbb{P}_\vartheta f(z))' = \mathcal{G}^\lambda(\varkappa, \Phi; w(z)) \tag{4.1}$$

and

$$(1-\varrho)\frac{\mathbb{P}_\vartheta g(w)}{w} + \varrho(\mathbb{P}_\vartheta g(w))' = \mathcal{G}^\lambda(\varkappa, \Phi; v(w)), \tag{4.2}$$

From the equalities (4.1) and (4.2), we obtain

$$(1-\varrho)\frac{\mathbb{P}_\vartheta f(z)}{z} + \varrho(\mathbb{P}_\vartheta f(z))' = 1 + \mathcal{P}_1^{(\lambda)}(\varkappa;\delta)c_1 z + \left[\mathcal{P}_1^{(\lambda)}(\varkappa;\delta)c_2 + \mathcal{P}_2^{(\lambda)}(\varkappa;\delta)c_1^2\right] z^2 + \cdots \tag{4.3}$$

and

$$(1-\varrho)\frac{\mathbb{P}_\vartheta g(w)}{w} + \varrho(\mathbb{P}_\vartheta g(w))' = 1 + \mathcal{P}_1^{(\lambda)}(\varkappa;\delta)d_1 w + \left[\mathcal{P}_1^{(\lambda)}(\varkappa;\delta)d_2 + \mathcal{P}_2^{(\lambda)}(\varkappa;\delta)d_1^2\right])w^2 + \cdots. \tag{4.4}$$

It is widely known that if

$$|w(z)| = \left|c_1 z + c_2 z^2 + c_3 z^3 + \cdots\right| < 1, \quad (z \in \mathbb{U})$$

and

$$|v(w)| = \left|d_1 w + d_2 w^2 + d_3 w^3 + \cdots\right| < 1, \quad (w \in \mathbb{U}),$$

then

$$|c_j| \leq 1 \text{ and } |d_j| \leq 1 \text{ for all } j \in \mathbb{N}. \tag{4.5}$$

Thus, upon comparing the corresponding coefficients in (4.3) and (4.4), we have

$$\frac{2(1+\varrho)\vartheta}{e^{\vartheta^2-1}} a_2 = \mathcal{P}_1^{(\lambda)}(\varkappa;\delta)c_1, \tag{4.6}$$

$$\frac{5(1+2\varrho)\vartheta^2}{2e^{\vartheta^2-1}} a_3 = \mathcal{P}_1^{(\lambda)}(\varkappa;\delta)c_2 + \mathcal{P}_2^{(\lambda)}(\varkappa;\delta)c_1^2, \tag{4.7}$$

and

$$-\frac{2(1+\varrho)\vartheta}{e^{\vartheta^2-1}} a_2 = \mathcal{P}_1^{(\lambda)}(\varkappa;\delta)d_1, \tag{4.8}$$

$$\frac{5(1+2\varrho)\vartheta^2}{2e^{\vartheta^2-1}}\left(2a_2^2 - a_3\right) = \mathcal{P}_1^{(\lambda)}(\varkappa;\delta)d_2 + \mathcal{P}_2^{(\lambda)}(\varkappa;\delta)d_1^2, \tag{4.9}$$

From (4.6) and (4.8), it follows that

$$c_1 = -d_1 \tag{4.10}$$

and

$$2\left(\frac{2(1+\varrho)\vartheta}{e^{\vartheta 2-1}}\right)^2 a_2^2 = \left[\mathcal{P}_1^{(\lambda)}(\varkappa;\delta)\right]^2 \left(c_1^2+d_1^2\right)$$

$$c_1^2+d_1^2 = \frac{8(1+\varrho)^2\vartheta^2}{\left(e^{\vartheta 2-1}\right)^2\left[\mathcal{P}_1^{(\lambda)}(\varkappa;\delta)\right]^2}\, a_2^2. \tag{4.11}$$

If we add (4.7) and (4.9), we get

$$\frac{5(1+2\varrho)\vartheta^2}{e^{\vartheta 2-1}} a_2^2 = \mathcal{P}_1^{(\lambda)}(\varkappa;\delta)\left(c_2+d_2\right) + \mathcal{P}_2^{(\lambda)}(\varkappa;\delta)\left(c_1^2+d_1^2\right). \tag{4.12}$$

By substituting the value of $(c_1^2+d_1^2)$ from equation (4.11) into the right-hand side of equation (4.12), we can deduce that.

$$\left(5(1+2\varrho) - \frac{8(1+\varrho)^2\mathcal{P}_2^{(\lambda)}(\varkappa;\delta)}{\left(e^{\vartheta 2-1}\right)\left[\mathcal{P}_1^{(\lambda)}(\varkappa;\delta)\right]^2}\right)\frac{\vartheta^2}{e^{\vartheta 2-1}}\, a_2^2 = \mathcal{P}_1^{(\lambda)}(\varkappa;\delta)\left(c_2+d_2\right)$$

$$a_2^2 = \frac{\left(e^{\vartheta 2-1}\right)^2\left[\mathcal{P}_1^{(\lambda)}(\varkappa;\delta)\right]^3}{\vartheta^2\left(5(1+2\varrho)\left(e^{\vartheta 2-1}\right)\left[\mathcal{P}_1^{(\lambda)}(\varkappa;\delta)\right]^2 - 8(1+\varrho)^2\mathcal{P}_2^{(\lambda)}(\varkappa;\delta)\right)}\left(c_2+d_2\right) \tag{4.13}$$

Moreover, computations using (2.9), (4.5) and (4.13), yield that

$$|a_2| \le \frac{\frac{e^{\vartheta 2-1}}{\vartheta}\left|\varkappa+\delta\lambda\right|\sqrt{2\left(\varkappa+\delta\lambda\right)}}{\sqrt{\begin{array}{c}\left|\left(5(1+2\varrho)e^{\vartheta 2-1} - 8(1+\varrho)^2\right)\varkappa^2 + \left(10\delta\lambda(1+2\varrho)e^{\vartheta 2-1} - 8(\delta\lambda+\lambda+1)(1+\varrho)^2\right)\varkappa \right. \\ \left. +8(1+\varrho)^2\left(2\delta^2\lambda - \delta\lambda^2 - \delta\lambda + 2\lambda\right)\right|.\end{array}}}$$

Furthermore, if we subtract (4.9) from (4.7), we obtain

$$\frac{5(1+2\varrho)\vartheta^2}{e^{\vartheta 2-1}}\left(a_3-a_2^2\right) = \mathcal{P}_1^{(\lambda)}(\varkappa;\delta)\left(c_2-d_2\right) + \mathcal{P}_2^{(\lambda)}(\varkappa;\delta)\left(c_1^2-d_1^2\right). \tag{4.14}$$

Then, in view of (4.5) and (4.11), Eq. (4.14) becomes

$$a_3 = \frac{\left(e^{\vartheta^2-1}\right)^2 \left[\mathcal{P}_1^{(\lambda)}(\varkappa;\delta)\right]^2}{8(1+\varrho)^2\vartheta^2}\left(c_1^2+d_1^2\right) + \frac{e^{\vartheta^2-1}\mathcal{P}_1^{(\lambda)}(\varkappa;\delta)}{5(1+2\varrho)\vartheta^2}\left(c_2-d_2\right)$$

Thus, applying (2.9) and (4.5), we conclude that

$$|a_3| \leq \frac{\left(e^{\vartheta^2-1}\right)^2 (\varkappa+\delta\lambda)^2}{4(1+\varrho)^2\vartheta^2} + \frac{2e^{\vartheta^2-1}|\varkappa+\delta\lambda|}{5(1+2\varrho)\vartheta^2}.$$

This completes the proof of Theorem. □

By using the values of a_2^2 and a_3, we can show the Fekete-Szegö inequality for functions in the class $\mathfrak{G}_\Sigma\left(\vartheta,\varrho,\mathcal{G}^\lambda(\varkappa,\Phi;z)\right)$.

Theorem 4.2. *Let $\Theta\in\Sigma$ given by (1.1) belong to the class $\mathfrak{G}_\Sigma\left(\vartheta,\varrho,\mathcal{G}^\lambda(\varkappa,\Phi;z)\right)$. Then, we have*

$$\left|a_3-\mu a_2^2\right| \leq \begin{cases} \frac{2e^{\vartheta^2-1}|\varkappa+\delta\lambda|}{5(1+2\varrho)\vartheta^2}, & |1-\mu|\leq \mathcal{Q}(\varrho,\varkappa,\lambda,\delta,\vartheta), \\ \\ 2\left|\varkappa+\delta\lambda\right|\left|\mathcal{K}(\mu)\right|, & |1-\mu|\geq \mathcal{Q}(\varrho,\varkappa,\lambda,\delta,\vartheta), \end{cases}$$

where

$$\mathcal{Q}(\varrho,\varkappa,\lambda,\delta,\vartheta) = \left|1-\frac{8(1+\varrho)^2\left(\varkappa^2+\left(\delta\lambda+\lambda+1\right)\varkappa-2\delta^2\lambda+\delta\lambda^2+\delta\lambda-2\lambda\right)}{5(1+2\varrho)e^{\vartheta^2-1}\left(\varkappa+\delta\lambda\right)^2}\right|,$$

and

$$\mathcal{K}(\mu) = \frac{e^{\vartheta^2-1}\left[\mathcal{P}_1^{(\lambda)}(\varkappa;\delta)\right]^2(1-\mu)}{\vartheta^2\left(5(1+2\varrho)\left(e^{\vartheta^2-1}\right)\left[\mathcal{P}_1^{(\lambda)}(\varkappa;\delta)\right]^2 - 8(1+\varrho)^2\mathcal{P}_2^{(\lambda)}(\varkappa;\delta)\right)}.$$

Proof. From (4.13) and (4.14)

$$\begin{aligned} a_3-\mu a_2^2 &= \frac{e^{\vartheta^2-1}\mathcal{P}_1^{(\lambda)}(\varkappa;\delta)}{5(1+2\varrho)\vartheta^2}\left(c_2-d_2\right) \\ &\quad + \frac{(1-\mu)\left(e^{\vartheta^2-1}\right)^2\left[\mathcal{P}_1^{(\lambda)}(\varkappa;\delta)\right]^3\left(c_2+d_2\right)}{\vartheta^2\left(5(1+2\varrho)\left(e^{\vartheta^2-1}\right)\left[\mathcal{P}_1^{(\lambda)}(\varkappa;\delta)\right]^2 - 8(1+\varrho)^2\mathcal{P}_2^{(\lambda)}(\varkappa;\delta)\right)} \\ &= \mathcal{P}_1^{(\lambda)}(\varkappa;\delta)\left(\left[\mathcal{K}(\mu)+\frac{e^{\vartheta^2-1}}{5(1+2\varrho)\vartheta^2}\right]c_2 + \left[\mathcal{K}(\mu)-\frac{e^{\vartheta^2-1}}{5(1+2\varrho)\vartheta^2}\right]d_2\right), \end{aligned}$$

where

$$\mathcal{K}(\mu)=\frac{\left(e^{\vartheta^2-1}\right)^2\left[\mathcal{P}_1^{(\lambda)}(\varkappa;\delta)\right]^2(1-\mu)}{\vartheta^2\left(5(1+2\varrho)\left(e^{\vartheta^2-1}\right)\left[\mathcal{P}_1^{(\lambda)}(\varkappa;\delta)\right]^2-8(1+\varrho)^2\mathcal{P}_2^{(\lambda)}(\varkappa;\delta)\right)},$$

Then, in view of (2.9), we conclude that

$$\left|a_3-\mu a_2^2\right|\le\begin{cases}\frac{2e^{\vartheta^2-1}\left|\mathcal{P}_1^{(\lambda)}(\varkappa;\delta)\right|}{5(1+2\varrho)\vartheta^2}, & |\mathcal{K}(\mu)|\le\frac{e^{\vartheta^2-1}}{5(1+2\varrho)\vartheta^2},\\ 2\left|\mathcal{P}_1^{(\lambda)}(\varkappa;\delta)\right||\mathcal{K}(\mu)|, & |\mathcal{K}(\mu)|\ge\frac{e^{\vartheta^2-1}}{5(1+2\varrho)\vartheta^2}.\end{cases}$$

Which completes the proof of Theorem 4.2. □

5 Corollaries and Consequences

Theorems 4.1 and 4.2 yield a result that closely aligns with Examples 3.2 and 3.3.

Corollary 5.1. *Let $\Theta\in\Sigma$ given by (1.1) belong to the class $\mathfrak{G}_\Sigma\left(\vartheta,0,\mathcal{G}^\lambda(\varkappa,\Phi;z)\right)$. Then*

$$|a_2|\le\frac{\frac{e^{\vartheta^2-1}}{\vartheta}\left|\varkappa+\delta\lambda\right|\sqrt{2\left(\varkappa+\delta\lambda\right)}}{\sqrt{\begin{array}{r}\left|\left(5e^{\vartheta^2-1}-8\right)\varkappa^2+\left(10\delta\lambda e^{\vartheta^2-1}-8(\delta\lambda+\lambda+1)\right)\varkappa\right.\\ \left.+8\left(2\delta^2\lambda-\delta\lambda^2-\delta\lambda+2\lambda\right)\right|.\end{array}}}$$

$$|a_3|\le\frac{\left(e^{\vartheta^2-1}\right)^2\left(\varkappa+\delta\lambda\right)^2}{4\vartheta^2}+\frac{2e^{\vartheta^2-1}|\varkappa+\delta\lambda|}{5\vartheta^2}.$$

and

$$\left|a_3-\mu a_2^2\right|\le\begin{cases}\frac{2e^{\vartheta^2-1}|\varkappa+\delta\lambda|}{5\vartheta^2}, & |1-\mu|\le\mathcal{Q}(0,\varkappa,\lambda,\delta,\vartheta),\\ 2\left|\varkappa+\delta\lambda\right||\mathcal{K}(\mu)|, & |1-\mu|\ge\mathcal{Q}(0,\varkappa,\lambda,\delta,\vartheta),\end{cases}$$

where

$$\mathcal{Q}(0,\varkappa,\lambda,\delta,\vartheta)=\left|1-\frac{8\left(\varkappa^2+\left(\delta\lambda+\lambda+1\right)\varkappa-2\delta^2\lambda+\delta\lambda^2+\delta\lambda-2\lambda\right)}{5e^{\vartheta^2-1}\left(\varkappa+\delta\lambda\right)^2}\right|,$$

and

$$\mathcal{K}(\mu)=\frac{e^{\vartheta 2-1}\left[\mathcal{P}_1^{(\lambda)}(\varkappa;\delta)\right]^2(1-\mu)}{\vartheta^2\left(5\left(e^{\vartheta 2-1}\right)\left[\mathcal{P}_1^{(\lambda)}(\varkappa;\delta)\right]^2-8\mathcal{P}_2^{(\lambda)}(\varkappa;\delta)\right)}.$$

Corollary 5.2. *Let $\Theta\in\Sigma$ given by (1.1) belong to the class $\mathfrak{G}_\Sigma\left(\vartheta,1,\mathcal{G}^\lambda(\varkappa,\Phi;z)\right)$. Then*

$$|a_2|\le\frac{\frac{e^{\vartheta 2-1}}{\vartheta}\left|\varkappa+\delta\lambda\right|\sqrt{2\left(\varkappa+\delta\lambda\right)}}{\sqrt{\begin{array}{c}\left|\left(15e^{\vartheta 2-1}-32\right)\varkappa^2+\left(30\delta\lambda e^{\vartheta 2-1}-32(\delta\lambda+\lambda+1)\right)\varkappa\right.\\ \left.+32\left(2\delta^2\lambda-\delta\lambda^2-\delta\lambda+2\lambda\right)\right|.\end{array}}}$$

$$|a_3|\le\frac{\left(e^{\vartheta 2-1}\right)^2(\varkappa+\delta\lambda)^2}{16\vartheta^2}+\frac{2e^{\vartheta 2-1}|\varkappa+\delta\lambda|}{15\vartheta^2}.$$

and

$$\left|a_3-\mu a_2^2\right|\le\begin{cases}\frac{2e^{\vartheta 2-1}|\varkappa+\delta\lambda|}{15\vartheta^2}, & |1-\mu|\le\mathcal{Q}(1,\varkappa,\lambda,\delta,\vartheta),\\ \\ 2\left|\varkappa+\delta\lambda\right|\left|\mathcal{K}(\mu)\right|, & |1-\mu|\ge\mathcal{Q}(1,\varkappa,\lambda,\delta,\vartheta),\end{cases}$$

where

$$\mathcal{Q}(1,\varkappa,\lambda,\delta,\vartheta)=\left|1-\frac{32\left(\varkappa^2+\left(\delta\lambda+\lambda+1\right)\varkappa-2\delta^2\lambda+\delta\lambda^2+\delta\lambda-2\lambda\right)}{15e^{\vartheta 2-1}\left(\varkappa+\delta\lambda\right)^2}\right|,$$

and

$$\mathcal{K}(\mu)=\frac{e^{\vartheta 2-1}\left[\mathcal{P}_1^{(\lambda)}(\varkappa;\delta)\right]^2(1-\mu)}{\vartheta^2\left(15\left(e^{\vartheta 2-1}\right)\left[\mathcal{P}_1^{(\lambda)}(\varkappa;\delta)\right]^2-32\mathcal{P}_2^{(\lambda)}(\varkappa;\delta)\right)}.$$

6 Concluding Remarks

In this paper, we have introduced and investigated the coefficient problems of a new subclass of bi-univalent functions. This subclass is referred to as $\mathfrak{G}_\Sigma\left(\vartheta,\varrho,\mathcal{G}^\lambda(\varkappa,\Phi;z)\right)$ and is closely related to the Bell distribution and

Meixner-Pollaczek polynomials. We have obtained estimates for the Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$, as well as the Fekete-Szegö functional problems for functions belonging to this new subclass.

The research paper explores the connection between the Bell distribution and Meixner-Pollaczek polynomials with specific families. This finding could potentially stimulate further research in other areas, such as the estimates on bound of $|a_n|$ for $n \geq 4; n \in \mathbb{N}$ for the classes that have been introduced here.

References

[1] W. Gautschi, Orthogonal polynomials, Computation and approximation, OUP Oxford, 2004.

[2] B. Doman, *The classical orthogonal polynomials*, World Scientific, 2015.

[3] J. Meixner, Orthogonale Polynomsysteme mit einer besonderen Gestalt der erzeugenden Funktion, Journal of the London Mathematical Society, **1,** no. 1, (1934), 6–13.

[4] R. Koekoek, P. A. Lesky, R. F. Swarttouw, Hypergeometric orthogonal polynomials, Springer: Berlin/Heidelberg, 2010, 183–253.

[5] S. S. Miller, P. T. Mocanu, *Second Order Differential Inequalities in the Complex Plane*, J. Math. Anal. Appl., 1978, *65*, 289–305.

[6] S. S. Miller, P. T. Mocanu, *Differential Subordinations and Univalent Functions*, Mich. Math. J., 1981, *28*, 157–172.

[7] S. S. Miller, P. T. Mocanu, *Differential Subordinations, Theory and Applications*, CRC Press: New York, 2000.

[8] M. Fekete, G. Szegö, *Eine Bemerkung Ãber ungerade schlichte Funktionen*, J. Lond. Math. Soc., **1,** no. 2, (1933), 85-89.

[9] F. Castellares, S. L. Ferrari, A. J. Lemonte, *On the Bell distribution and its associated regression model for count data*, Applied Mathematical Modelling, **56,** (2018), 172–185.

[10] E. T. Bell, *Exponential polynomials*, Annals of Mathematics, **35,** (1934), 258–277.

[11] E. T. Bell, *Exponential numbers*, The American Mathematical Monthly, **41,** no. 7, (1934), 411–419.

[12] R. Koekoek, P. A. Lesky, R. F. Swarttouw, *Hypergeometric Orthogonal Polynomials and Their q–Analogues* Springer: Berlin/Heidelberg, Germany, 2010.

[13] F. W. J. Olver, D. W. Boisvert, C. W. Clark, *DLMF Handbook of Mathematical Functions*, Cambridge University Press: Cambridge, 2010.

[14] A. S. Kelil, A. R. Appadu, *On Certain Properties and Applications of the Perturbed Meixner–Pollaczek Weight,* Mathematics, **9,** no. 9, (2021), 28 pages.

[15] A. Amourah, B. A. Frasin, T. Abdeljawad, *Fekete-Szegö inequality for analytic and bi-univalent functions subordinate to Gegenbauer polynomials*. J. Funct. Spaces, 2021, Article ID 5574673, 7 pages.

[16] B. A. Frasin, M. K. Aouf, *New subclasses of bi-univalent functions*, Appl. Math. Lett., **24,** (2011), 1569–1573.

[17] B. A. Frasin, S. R. Swamy, J. Nirmala, *Some special families of holomorphic and Al-Oboudi type bi-univalent functions related to k-Fibonacci numbers involving modified Sigmoid activation function*, Afr. Mat., (2020).
https://doi.org/10.1007/s13370-020-00850-w.

[18] T. Al-Hawary, A. Amourah, A. Alsoboh, O. Alsalhi, A New Comprehensive Subclass of Analytic Bi-Univalent Functions Related to Gegenbauer Polynomials, *Symmetry, 15*, (2023), 576.

[19] A. Amourah, A. Alsoboh, O. Ogilat, G. M. Gharib, R. Saadeh, M. Al Soudi, A Generalization of Gegenbauer Polynomials and Bi-Univalent Functions, *Axioms, 12*, (2023), 128.
https://doi.org/10.3390/axioms12020128.

[20] F. Yousef, B. A. Frasin, T. Al-Hawary, *Fekete-Szegö inequality for analytic and bi-univalent functions subordinate to Chebyshev polynomials,* Filomat, **32,** no. 9, (2018), 3229–3236.

[21] F. Yousef, T. Al-Hawary, G. Murugusundaramoorthy, *Fekete-Szegö functional problems for some subclasses of bi-univalent functions defined*

by Frasin differential operator, Afrika Matematika, **30,** nos. 3-4, (2019), 495–503.

[22] F. Yousef, S. Alroud, M. Illafe, *New subclasses of analytic and bi-univalent functions endowed with coefficient estimate problems*, Analysis and Mathematical Physics, **11,** no. 2, (2021), 1–12.

[23] S. Bulut, *Coefficient estimates for a class of analytic and bi-univalent functions*, Novi Sad J. Math., **43,** (2013), 59–65.

[24] S. Bulut, N. Magesh, C. Abirami, *A comprehensive class of analytic bi-univalent functions by means of Chebyshev polynomials*, J. Fractional Calc. & Appl., **8,** no. 2, (2017), 32–39.

[25] S. Bulut, N. Magesh, V. K. Balaji, *Initial bounds for analytic and bi-univalent functions by means of Chebyshev polynomials*, Analysis, **11,** no. 1, (2017), 83–89.

[26] M. Al-Kaseasbeh, A. Alamoush, A. Amourah, A. Aljarah, J. Jerash, *Subclasses of Spiralike Functions Involving Convoluted Differential Operator*, Int. J. Open Problems Complex Analysis, **12,** no. 3.

[27] I. Aldawish, T. Al-Hawary, B.A. Frasin, *Subclasses of bi-univalent functions defined by Frasin differential operator*, Mathematics, **8,** no. 5, (2020), 783.

[28] A. Amourah, M. Alomary, F. Yousef, A. Alsoboh, *Consolidation of a certain discrete probability distribution with a subclass of bi-univalent functions involving Gegenbauer polynomials*, Mathematical Problems in Engineering, (2022), Article ID 6354994, 6 pages.

[29] A. Alsoboh, A. Amourah, M. Darus, R. I. A. Sharefeen, Applications of Neutrosophic $q-$Poisson distribution Series for Subclass of Analytic Functions and Bi-Univalent Functions, *Mathematics, 11*, (2023), 868. https://doi.org/10.3390/math11040868.

[30] A. Amourah, B. A. Frasin, T. M. Seoudy, An Application of Miller–Ross-Type Poisson Distribution on Certain Subclasses of Bi-Univalent Functions Subordinate to Gegenbauer Polynomials, *Mathematics*, *10*, 2022, 2462. https://doi.org/10.3390/math10142462.

[31] A. Amourah, B. A. Frasin, M. Ahmad, F. Yousef, Exploiting the Pascal Distribution Series and Gegenbauer Polynomials to Construct and Study a New Subclass of Analytic Bi-Univalent Functions, *Symmetry*, *14*, (2022), 147. https://doi.org/10.3390/sym14010147

[32] A. Amourah, O. Alnajar, M. Darus, A. Shdouh, O. Ogilat, Estimates for the Coefficients of Subclasses Defined by the Bell Distribution of Bi-Univalent Functions Subordinate to Gegenbauer Polynomials, *Mathematics, 11.8*, (2023), 1799.

[33] I. Aldawish, B. A. Frasin, A. Amourah, Bell Distribution Series Defined on Subclasses of Bi-Univalent Functions That Are Subordinate to Horadam Polynomials, Axioms, 12.4, (2023), 362.