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Asymptotic and Oscillatory Properties for Even-Order Nonlinear Neutral Differential Equations with Damping Term

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Abstract: This research focuses on studying the asymptotic and oscillatory behavior of a special class of even-order nonlinear neutral differential equations, including damping terms. The research aims to achieve qualitative progress in understanding the relationship between the solutions of these equations and their associated functions. Leveraging the symmetry between positive and negative solutions simplifies the derivation of criteria that ensure the oscillation of all solutions. Using precise techniques such as the Riccati method and comparison methods, innovative criteria are developed that guarantee the oscillation of all the solutions of the studied equations. The study provides new conditions and effective analytical tools that contribute to deepening the theoretical understanding and expanding the practical applications of these systems. Based on solid scientific foundations and previous studies, the research concludes with the presentation of examples that illustrate the practical impact of the results, highlighting the theoretical value of research in the field of neutral differential equations.

Keywords: oscillation; nonoscillation; nonlinear equations; neutral differential equations; noncanonical case

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1. Introduction

This study focuses on a nonlinear even-order neutral differential equation with a damping term, expressed as follows:

$$\left(\kappa(\mathrm{s})\left|\Psi^{(n-1)}(\mathrm{s})\right|^{\alpha-1}\Psi^{(n-1)}(\mathrm{s})\right)' + \mathfrak{a}(\mathrm{s})\left|\Psi^{(n-1)}(\mathrm{s})\right|^{\alpha-1}\Psi^{(n-1)}(\mathrm{s}) + \mathfrak{q}(\mathrm{s})|\varkappa(\mathfrak{h}(\mathrm{s}))|^{\beta-1}\varkappa(\mathfrak{h}(\mathrm{s})) = 0, \quad (1)$$

where $\Psi(\mathrm{s}) = \varkappa(\mathrm{s}) + \mathrm{g}(\mathrm{s})\varkappa(\mathfrak{z}(\mathrm{s}))$, $\mathrm{s} \geq \mathrm{s}_0$, $n \geq 4$, $\alpha > 0$, and $\beta > 0$. Our analysis is based on the following assumption:

(H_1) $\kappa \in C^1([\mathrm{s}_0,\infty))$, $\kappa'(\mathrm{s}) \geq 0$, $\mathrm{g}, \mathfrak{q}, \mathfrak{a} \in C([\mathrm{s}_0,\infty),\mathbb{R})$, $0 \leq \mathrm{g}(\mathrm{s}) < 1$, and $\mathfrak{q}(\mathrm{s})$ is not eventually zero on $[\mathrm{s}^*,\infty)$ for $\mathrm{s}^* \geq \mathrm{s}_0$;

(H_2) $\mathfrak{z} \in C([s_0,\infty),\mathbb{R})$, $\mathfrak{h} \in C^1([s_0,\infty),\mathbb{R})$, $\mathfrak{z}(s) \leq s$, $\mathfrak{h}(s) \leq s$, $\mathfrak{h}'(s) > 0$, and $\lim_{s\to\infty}\mathfrak{z}(s) = \lim_{s\to\infty}\mathfrak{h}(s) = \infty$.

Let $s_\varkappa = \min\{\mathfrak{z}(s),\mathfrak{h}(s)\}$. A function $\varkappa(s) \in C^{n-1}([s_\varkappa,\infty),\mathbb{R})$, $s_\varkappa \geqslant s_0$, is called a solution of Equation (1) if it has the property $\kappa(s)\left|\Psi^{n-1}(s)\right|^{\alpha-1}\Psi^{n-1}(s) \in C^1[s_\varkappa,\infty)$ and satisfies Equation (1) on $[s_\varkappa,\infty)$. We only consider the nontrivial solutions of Equation (1) which ensure

$$\sup\{|\varkappa(s)| : s \geqslant S\} > 0, \text{ for all } S \geq s_\varkappa.$$

A solution of Equation (1) is classified as oscillatory if it exhibits an infinite sequence of zeros over the interval $[s_\varkappa,\infty)$. Otherwise, it is categorized as non-oscillatory. The differential Equation (1) is said to be oscillatory if every solution of (1) is oscillatory.

Differential equations (DEs) are essential for modeling dynamic phenomena across various fields, from physics to economics. Neutral differential equations (NDEs), which account for systems where current behavior depends on both past variables and their derivatives, have gained significant attention in recent research. They are particularly valuable for studying systems with delayed effects, offering key insights for both theoretical and practical applications, as highlighted in [1–3].

Oscillation theory is a fundamental part of mathematics used in studying the behavior of solutions in dynamical systems, especially the analysis of oscillations or stability over time. The theory is widely used in mechanical, electrical, and biological fields to understand the stability of systems and predict their behavior. The symmetry property plays an important role in simplifying models and discovering fundamental relationships, facilitating the identification of oscillatory patterns. The theory also includes the analysis of periodic solutions and forced oscillations. Tools have been developed in this field to accurately determine the conditions that lead to oscillation, which enhances its applications in engineering, physics, and biology (see [4,5]).

The study of oscillation theory has experienced substantial advancements in recent years, particularly in the context of DEs involving delay, neutral terms, and damping effects. Among these, delayed differential equations have attracted significant scholarly attention, as evidenced by the contributions of researchers such as Džurina and Jadlovská [6], Grace et al. [7], and Masood et al. [8]. Similarly, neutral differential equations have been extensively investigated, as documented in the works of Li et al. [9] and Bohner et al. [10]. Furthermore, notable progress has been achieved in the oscillatory analysis of odd-order differential equations, as highlighted in the studies by Li and Thandapani [11], Baculíková and Džurina [12], and Masood et al. [13]. In parallel, considerable research has been directed towards understanding the oscillatory behavior of even-order differential equations, as explored in the works of Zhang and Ladde [14], Li and Rogovchenko [15], and Moaaz et al. [16]. Lastly, the dynamic properties of damping equations have also been extensively studied, with significant contributions by Bohner et al. [17] and Bartušek and Doslá [18].

Dzurina and Stavroulakis [19], Sun and Meng [20], Elbert [21], and Agarwal et al. [22] have conducted research on the differential equation

$$\left(\kappa(s)\left|\varkappa'(s)\right|^{\alpha-1}\varkappa'(s)\right)' + \mathfrak{q}(s)f(\varkappa(\mathfrak{h}(s))) = 0,$$

and its associated equations.

In their research, Grice and Akin [23] examined oscillations in nonlinear DEs with delays, the specific equation they considered is given as follows:

$$\varkappa^{(4)}(s) + \varkappa''(s) + \mathfrak{q}(s)f(\varkappa(\mathfrak{h}(s))) = 0,$$

where the condition $f(\varkappa) > k\varkappa^{\beta}(s)$ is satisfied for $\varkappa \neq 0$. Their findings indicated that the DE

$$\Psi''(s) + p(s)\Psi(s) = 0,$$

can exhibit either oscillatory or non-oscillatory behavior.

Several studies, including those carried out by Grace [24], and Padhi et al. [25], and Tiryaki et al. [26] have emphasized the exploration of oscillatory behaviors exhibited by solutions to the equation

$$\left(\kappa_2(s)\left(\kappa_1(s)\varkappa'(s)\right)'\right)' + \mathfrak{a}(s)\Psi'(s) + f(\varkappa(\mathfrak{h}(s))) = 0.$$

Graef et al. [27] examined the oscillatory patterns displayed by solutions of higher-order nonlinear DEs featuring a nonlinear neutral term, expressed as

$$\left(\kappa(s)\left(\left(\varkappa(s) - \mathrm{g}(s)\varkappa^{\beta}(\mathfrak{h}(s))\right)^{(n-1)}(s)\right)^{\alpha}\right)' = \mathfrak{q}(s)\varkappa^{\gamma}(\mathfrak{z}(s)) + \mathfrak{a}(s)\varkappa^{\mu}(\Psi(s)).$$

Wu et al. [28] established new oscillation criteria for a class of damped second-order NDEs

$$\left(\kappa(s)\left|\Psi'(s)\right|^{\alpha-1}\Psi'(s)\right)' + \mathfrak{a}(s)\left|\Psi'(s)\right|^{\alpha-1}\Psi'(s) + \mathfrak{q}(s)|\varkappa(\mathfrak{h}(s))|^{\beta-1}\varkappa(\mathfrak{h}(s)) = 0,$$

with noncanonical operators. Alatwi et al. [29] investigated the oscillatory behavior of solutions to fourth-order nonlinear NDEs

$$\left(\kappa(s)\left|\Psi'''(s)\right|^{\alpha-1}\Psi'''(s)\right)' + \mathfrak{a}(s)\left|\Psi'''(s)\right|^{\alpha-1}\Psi'''(s) + \mathfrak{q}(s)|\varkappa(\mathfrak{h}(s))|^{\beta-1}\varkappa(\mathfrak{h}(s)) = 0,$$

emphasizing improved relationships between solutions, their functions, and derivatives and establishing new criteria for oscillation.

Despite the critical importance of these models, understanding the oscillatory behavior of solutions, especially in higher-order and non-canonical cases, remains a significant challenge. Previous research has mainly focused on second-, third-, and fourth-order equations, leaving a gap in the study of more complex cases. This study aims to fill that gap by establishing novel criteria for determining the oscillation behavior of solutions to even-order NDEs, relaxing the stringent conditions of earlier works. By analyzing inequalities related to key variables, we derive criteria ensuring the non-existence of positive solutions, utilizing advanced mathematical tools such as the Riccati transformation and comparison theorems. The newly developed criteria provide enhanced precision and flexibility, making them applicable to a broader range of differential models. Our approach strengthens the interconnection between positive solutions, their associated functions, and derivatives, leading to more general, less restrictive conditions. This work extends the methodology in [28], which focuses on second-order equations, offering a more comprehensive framework for addressing higher-order and non-canonical cases.

2. Preliminary Results

Let us define:

$$\theta(s) := \kappa(s)\varphi(s),$$

where

$$\varphi(s) := \exp\left(\int_{s_0}^{s} \frac{\mathfrak{a}(\varsigma)}{\kappa(\varsigma)} d\varsigma\right).$$

We introduce the following functions:

$$\mu_0(\mathrm{s}) := \int_{\mathrm{s}}^{\infty} \theta^{-1/\alpha}(\varsigma)\mathrm{d}\varsigma,\ \mu_i(\mathrm{s}) := \int_{\mathrm{s}}^{l} \mu_{i-1}(\varsigma)\mathrm{d}\varsigma,\ i = 1,2,\ldots,n-2,$$

$$J^{[0]}(\mathrm{s}) := J(\mathrm{s}) \text{ and } J^{[i]}(\mathrm{s}) = J\left(J^{[i-1]}(\mathrm{s})\right),\ \text{for } i = 1,2,\ldots,n,$$

$$\mathrm{g}_1(\mathrm{s};m) := \sum_{i=0}^{m}\left(\prod_{k=0}^{2i}\mathrm{g}\left(\mathfrak{z}^{[k]}(\mathrm{s})\right)\right)\left[\frac{1}{\mathrm{g}(\mathfrak{z}^{[2i]}(\mathrm{s}))} - \frac{\mu_{n-2}\left(\mathfrak{z}^{[2i+1]}(\mathrm{s})\right)}{\mu_{n-2}(\mathfrak{z}^{[2i]}(\mathrm{s}))}\right],$$

$$\mathrm{g}_2(\mathrm{s};m) := \sum_{i=0}^{m}\left(\prod_{k=0}^{2i}\mathrm{g}\left(\mathfrak{z}^{[k]}(\mathrm{s})\right)\right)\left[\frac{1}{\mathrm{g}(\mathfrak{z}^{[2i]}(\mathrm{s}))} - 1\right]\left(\frac{\mathfrak{z}^{[2i]}(\mathrm{s})}{\mathrm{s}}\right)^{(n-2)/\epsilon_0},$$

$$\widetilde{\mathfrak{q}}(\mathrm{s}) := \varphi(\mathrm{s})\mathfrak{q}(\mathrm{s})\mathrm{g}_1^{\beta}(\mathfrak{h}(\mathrm{s});m),$$

$$\widehat{\mathfrak{q}}(\mathrm{s}) := \varphi(\mathrm{s})\mathfrak{q}(\mathrm{s})(1-\mathrm{g}(\mathfrak{h}(\mathrm{s})))^{\beta},$$

and

$$\widetilde{\mathfrak{q}}_1(\mathrm{s}) := \varphi(\mathrm{s})\mathfrak{q}(\mathrm{s})\mathrm{g}_2^{\beta}(\mathfrak{h}(\mathrm{s});m).$$

Lemma 1 ([30])**.** *Suppose that $\chi \in C^m([\mathrm{s}_0,\infty),\varsigma^+)$, $\chi^{(m)}(\mathrm{s})$ is of fixed sign and not identically zero on $[\mathrm{s}_0,\infty)$ and that there exists $\mathrm{s}_1 \geq \mathrm{s}_0$ such that $\chi^{(m-1)}(\mathrm{s})\chi^{(m)}(\mathrm{s}) \leq 0$ for all $\mathrm{s}_1 \geq \mathrm{s}_0$. If $\lim_{\mathrm{s}\to\infty}\chi(\mathrm{s}) \neq 0$; then, for every $\delta \in (0,1)$, there exists $\mathrm{s}_\epsilon \in [\mathrm{s}_1,\infty)$ such that*

$$\chi(\mathrm{s}) \geq \frac{\epsilon}{(m-1)!}\mathrm{s}^{m-1}\left|\chi^{(m-1)}(\mathrm{s})\right|,$$

for $\mathrm{s} \in [\mathrm{s}_\epsilon,\infty)$.

Lemma 2 ([31])**.** *Let $\chi \in \mathbf{C}^m([\mathrm{s}_0,\infty),(0,\infty))$, ${}^{(i)}(\mathrm{s}) > 0$ for $i = 1,2,\ldots,m$, and $\chi^{(m+1)}(\mathrm{s}) \leq 0$, eventually. Then, eventually, $\chi(\mathrm{s})/\chi'(\mathrm{s}) \geq \epsilon\mathrm{s}/m$ for every $\epsilon \in (0,1)$.*

Lemma 3 ([32])**.** *Let α present the ratio of two odd positive integers. Suppose $\kappa > 0$ and B are constants. Then, the following inequality holds:*

$$Bu - Au^{(\alpha+1)/\alpha} \leq \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}}\frac{B^{\alpha+1}}{A^{\alpha}}. \tag{2}$$

Lemma 4 ([30])**.** *Let $\varkappa$ be an eventually positive solution of (1). Then $\varkappa$ will eventually fulfill one of the following cases*

$$\begin{aligned}
&\mathrm{C}_1 \ : \ \Psi(\mathrm{s}) > 0,\ \Psi'(\mathrm{s}) > 0,\ \Psi^{(n-1)}(\mathrm{s}) > 0,\ \textit{and}\ \Psi^{(n)}(\mathrm{s}) < 0;\\
&\mathrm{C}_2 \ : \ \Psi(\mathrm{s}) > 0,\ \Psi'(\mathrm{s}) > 0,\ \Psi^{(n-2)}(\mathrm{s}) > 0,\ \textit{and}\ \Psi^{(n-1)}(\mathrm{s}) < 0;\\
&\mathrm{C}_3 \ : \ (-1)^i\Psi^{(i)}(\mathrm{s}) > 0\ \textit{for}\ i = 0,1,2,\ldots,n-1;
\end{aligned}$$

for $\mathrm{s} \geqslant \mathrm{s}_1 \geqslant \mathrm{s}_0$.

Ω_i represents the set of all solutions that become positive and satisfy the corresponding case (C_i) for $i = 1,2,3$.

Lemma 5 ([33]). *Let $\varkappa$ be a solution of (1). As a consequence, for sufficiently large values of* s*, the following inequality must be satisfied:*

$$\varkappa(\mathrm{s}) > \sum_{i=0}^{n} \left(\prod_{k=0}^{2i} \mathrm{g}\left(\mathfrak{z}^{[k]}(\mathrm{s})\right) \right) \left[\frac{\Psi\left(\mathfrak{z}^{[2i]}(\mathrm{s})\right)}{\mathrm{g}(\mathfrak{z}^{[2i]}(\mathrm{s}))} - \Psi\left(\mathfrak{z}^{[2i+1]}(\mathrm{s})\right) \right]. \tag{3}$$

3. Main Results

This section establishes the monotonic properties of the solutions to the NDE (1). By analyzing these properties, we gain valuable insights into the long-term behavior and stability of the solutions. Furthermore, we introduce a series of innovative conditions aimed at effectively eliminating positive solutions that satisfy Equation (1).

3.1. Category Ω_3

This section presents a set of lemmas that examine the asymptotic behavior of solutions classified under (C_3) category.

Lemma 6. *Let $\varkappa \in \Omega_3$. Assume that*

$$\mu_0(\mathrm{s}) := \int_{\mathrm{s}}^{\infty} \theta^{-1/\alpha}(\varsigma)\mathrm{d}\varsigma < \infty. \tag{4}$$

Then,
$(\mathrm{Q}_{1,1})$ $\Psi(\mathrm{s})/\mu_{n-2}(\mathrm{s})$ *is increasing;*
$(\mathrm{Q}_{1,2})$ $(-1)^{i+1}\Psi^{(n-i-2)}(\mathrm{s}) \leq \theta^{1/\alpha}(\mathrm{s})\Psi^{(n-1)}(\mathrm{s})\mu_i(\mathrm{s})$, *for* $i = 0, 1, 2, \ldots, n-2$.

Proof. Let $\varkappa \in \Omega_3$. Then, there exists a $\mathrm{s}_1 \geq \mathrm{s}_0$, such that $\varkappa(\mathfrak{z}(\mathrm{s})) > 0$ and $\varkappa(\mathfrak{h}(\mathrm{s})) > 0$ for $\mathrm{s} \geq \mathrm{s}_1$. By multiplying both sides of (1) by $\varphi(\mathrm{s})$, we obtain

$$\left(\theta(\mathrm{s})\left|\Psi^{(n-1)}(\mathrm{s})\right|^{\alpha-1}\Psi^{(n-1)}(\mathrm{s})\right)' + \varphi(\mathrm{s})\mathfrak{q}(\mathrm{s})\varkappa^{\beta}(\mathfrak{h}(\mathrm{s})) = 0,\ \mathrm{s} \geq \mathrm{s}_0. \tag{5}$$

Since $\Psi^{(n-1)}(\mathrm{s}) < 0$, from (5), we obtain

$$\left(\theta(\mathrm{s})\left(-\Psi^{(n-1)}(\mathrm{s})\right)^{\alpha}\right)' = \varphi(\mathrm{s})\mathfrak{q}(\mathrm{s})\varkappa^{\beta}(\mathfrak{h}(\mathrm{s})) \geq 0. \tag{6}$$

$(\mathrm{Q}_{1,1})$ It follows from (6) that

$$\theta^{1/\alpha}(\varsigma)\Psi^{(n-1)}(\varsigma) \leq \theta^{1/\alpha}(\mathrm{s})\Psi^{(n-1)}(\mathrm{s}),\ \varsigma \geq \mathrm{s} \geq \mathrm{s}_1.$$

Dividing the latter inequality by $\theta^{1/\alpha}(\varsigma)$, we obtain

$$\Psi^{(n-1)}(\varsigma) \leq \frac{\theta^{1/\alpha}(\mathrm{s})\Psi^{(n-1)}(\mathrm{s})}{\theta^{1/\alpha}(\varsigma)}.$$

Integrating this inequality from s to ∞ allows us to conclude that

$$-\Psi^{(n-2)}(\mathrm{s}) \leq \theta^{1/\alpha}(\mathrm{s})\Psi^{(n-1)}(\mathrm{s})\int_{\mathrm{s}}^{\infty} \theta^{-1/\alpha}(\varsigma)\mathrm{d}\varsigma = \theta^{1/\alpha}(\mathrm{s})\Psi^{(n-1)}(\mathrm{s})\mu_0(\mathrm{s}).$$

That is,

$$\Psi^{(n-2)}(\mathrm{s}) \geq -\theta^{1/\alpha}(\mathrm{s})\Psi^{(n-1)}(\mathrm{s})\mu_0(\mathrm{s}). \tag{7}$$

Hence,

$$\left(\frac{\Psi^{(n-2)}}{\mu_0}\right)'(s) = \frac{\theta^{1/\alpha}(s)\mu_0(s)\Psi^{(n-1)}(s) + \Psi^{(n-2)}(s)}{\theta^{1/\alpha}(s)\mu_0^2(s)} \geq 0.$$

Since $\Psi^{(n-2)}(s)/\mu_0(s)$ is increasing, then

$$-\Psi^{(n-3)}(s) \geq \int_s^\infty \frac{\Psi^{(n-2)}(\varsigma)}{\mu_0(\varsigma)}\mu_0(\varsigma)d\varsigma \geq \frac{\Psi^{(n-2)}(s)}{\mu_0(s)}\mu_1(s).$$

That is,

$$\Psi^{(n-3)}(s) \leq -\frac{\Psi^{(n-2)}(s)}{\mu_0(s)}\mu_1(s). \tag{8}$$

This implies

$$\left(\frac{\Psi^{(n-3)}}{\mu_1}\right)'(s) = \frac{\mu_1(s)\Psi^{(n-2)}(s) + \mu_0(s)\Psi^{(n-3)}(s)}{\mu_1^2(s)} \leq 0.$$

Repeating the same process $(n-4)$ times yields the following result

$$\left(\frac{\Psi'}{\mu_{n-3}}\right)'(s) \leq 0.$$

Since $\Psi'(s)/\mu_{n-3}(s)$ is decreasing, then

$$-\Psi(s) \leq \int_s^\infty \frac{\Psi'(\varsigma)}{\mu_{n-3}(\varsigma)}\mu_{n-3}(\varsigma)d\varsigma \leq \frac{\Psi'(s)}{\mu_{n-3}(s)}\mu_{n-2}(s).$$

That is,

$$\Psi(s) \geq -\frac{\Psi'(s)}{\mu_{n-3}(s)}\mu_{n-2}(s). \tag{9}$$

This implies

$$\left(\frac{\Psi}{\mu_{n-2}}\right)'(s) = \frac{\mu_{n-2}(s)\Psi'(s) + \mu_{n-3}(s)\Psi(s)}{\mu_{n-2}^2(s)} \geq 0.$$

In addition, when combining the above inequalities, we easily obtain the following relationship:

$$\frac{\Psi^{(n-2)}(s)}{\mu_0(s)} \leq -\frac{\Psi^{(n-3)}(s)}{\mu_1(s)} \leq \frac{\Psi^{(n-4)}(s)}{\mu_2(s)} \leq \ldots \leq -\frac{\Psi'(s)}{\mu_{n-3}(s)} \leq \frac{\Psi(s)}{\mu_{n-2}(s)}. \tag{10}$$

$(Q_{1,2})$ By examining the monotonicity of $\theta^{1/\alpha}\Psi^{(n-1)}$, we obtain that

$$\theta^{1/\alpha}(s)\Psi^{(n-1)}(s)\mu_0(s) \geq \int_s^\infty \frac{\theta^{1/\alpha}(\varsigma)\Psi^{(n-1)}(\varsigma)}{\theta^{1/\alpha}(\varsigma)}d\varsigma \geq -\Psi^{(n-2)}(s).$$

This can be equivalently expressed as

$$\Psi^{(n-2)}(s) \geq -\theta^{1/\alpha}(s)\Psi^{(n-1)}(s)\mu_0(s).$$

By integrating this inequality from s to ∞, we find

$$-\Psi^{(n-3)}(s) \geq -\int_s^\infty \theta^{1/\alpha}(\varsigma)\Psi^{(n-1)}(\varsigma)\mu_0(\varsigma)d\varsigma \geq -\theta^{1/\alpha}(s)\Psi^{(n-1)}(s)\mu_1(s).$$

This can be equivalently expressed as

$$\Psi^{(n-3)}(s) \leq \theta^{1/\alpha}(s)\Psi^{(n-1)}(s)\mu_1(s).$$

By integrating the final inequality over the interval $[\mathrm{s}, \infty)$, the result is expressed as

$$-\Psi^{(n-4)}(\mathrm{s}) \leq \int_{\mathrm{s}}^{\infty} \theta^{1/\alpha}(\varsigma)\Psi^{(n-1)}(\varsigma)\mu_1(\varsigma)\mathrm{d}\varsigma \leq \theta^{1/\alpha}(\mathrm{s})\Psi^{(n-1)}(\mathrm{s})\mu_2(\mathrm{s}).$$

Proceeding with repeated integrations of this inequality over the same interval, we establish that

$$(-1)^{i+1}\Psi^{(n-i-2)}(\mathrm{s}) \leq \theta^{1/\alpha}(\mathrm{s})\Psi^{(n-1)}(\mathrm{s})\mu_i(\mathrm{s}), \text{ for } i = 0, 1, 2, \ldots, n-2.$$

Thus, the proof is concluded. □

Lemma 7. *Let $\varkappa \in \Omega_3$. Assume that (4) holds. Then,*
$(\mathrm{Q}_{2,1})$ *$\varkappa(\mathrm{s}) > \mathrm{g}_1(\mathrm{s}, m)\Psi(\mathrm{s})$;*
$(\mathrm{Q}_{2,2})$ *$\left(\theta(\mathrm{s})\left(-\Psi^{(n-1)}(\mathrm{s})\right)^{\alpha}\right)' \geq \widetilde{\mathfrak{q}}(\mathrm{s})\Psi^{\beta}(\mathrm{s})$.*

Proof. $(\mathrm{Q}_{2,1})$ In view of (3), we have that

$$\varkappa(\mathrm{s}) > \sum_{i=0}^{m}\left(\prod_{k=0}^{2i}\mathrm{g}\left(\mathfrak{z}^{[k]}(\mathrm{s})\right)\right)\left[\frac{\Psi\left(\mathfrak{z}^{[2i]}(\mathrm{s})\right)}{\mathrm{g}\left(\mathfrak{z}^{[2i]}(\mathrm{s})\right)} - \Psi\left(\mathfrak{z}^{[2i+1]}(\mathrm{s})\right)\right]. \tag{11}$$

Since $\Psi(\mathrm{s})/\mu_{n-2}(\mathrm{s})$ is increasing and $\mathfrak{z}^{[2i]}(\mathrm{s}) \geq \mathfrak{z}^{[2i+1]}(\mathrm{s})$, then

$$\Psi\left(\mathfrak{z}^{[2i+1]}(\mathrm{s})\right) \leq \frac{\mu_{n-2}\left(\mathfrak{z}^{[2i+1]}(\mathrm{s})\right)}{\mu_{n-2}\left(\mathfrak{z}^{[2i]}(\mathrm{s})\right)}\Psi\left(\mathfrak{z}^{[2i]}(\mathrm{s})\right).$$

Substituting the previous inequality into (11), we obtain

$$\varkappa(\mathrm{s}) > \sum_{i=0}^{m}\left(\prod_{k=0}^{2i}\mathrm{g}\left(\mathfrak{z}^{[k]}(\mathrm{s})\right)\right)\left[\frac{1}{\mathrm{g}\left(\mathfrak{z}^{[2i]}(\mathrm{s})\right)} - \frac{\mu_{n-2}\left(\mathfrak{z}^{[2i+1]}(\mathrm{s})\right)}{\mu_{n-2}\left(\mathfrak{z}^{[2i]}(\mathrm{s})\right)}\right]\Psi\left(\mathfrak{z}^{[2i]}(\mathrm{s})\right).$$

Since $\Psi'(\mathrm{s}) < 0$, and $\mathfrak{z}^{[2i]}(\mathrm{s}) \leq \mathrm{s}$, then the previous inequality becomes

$$\begin{aligned}\varkappa(\mathrm{s}) &\geq \sum_{i=0}^{m}\left(\prod_{k=0}^{2i}\mathrm{g}\left(\mathfrak{z}^{[k]}(\mathrm{s})\right)\right)\left[\frac{1}{\mathrm{g}\left(\mathfrak{z}^{[2i]}(\mathrm{s})\right)} - \frac{\mu_{n-2}\left(\mathfrak{z}^{[2i+1]}(\mathrm{s})\right)}{\mu_{n-2}\left(\mathfrak{z}^{[2i]}(\mathrm{s})\right)}\right]\Psi(\mathrm{s}) \\ &= \mathrm{g}_1(\mathrm{s}; m)\Psi(\mathrm{s}).\end{aligned} \tag{12}$$

$(\mathrm{Q}_{2,2})$ By combining (12) and (6), with $\Psi^{(n-1)}(\mathrm{s}) < 0$, we thus deduce that

$$\begin{aligned}\left(\theta(\mathrm{s})\left(-\Psi^{(n-1)}(\mathrm{s})\right)^{\alpha}\right)' &= \varphi(\mathrm{s})\mathfrak{q}(\mathrm{s})\varkappa^{\beta}(\mathfrak{h}(\mathrm{s})) \geq \varphi(\mathrm{s})\mathfrak{q}(\mathrm{s})\mathrm{g}_1^{\beta}(\mathfrak{h}(\mathrm{s}); m)\Psi^{\beta}(\mathfrak{h}(\mathrm{s})) \\ &= \widetilde{\mathfrak{q}}(\mathrm{s})\Psi^{\beta}(\mathfrak{h}(\mathrm{s})) \geq \widetilde{\mathfrak{q}}(\mathrm{s})\Psi^{\beta}(\mathrm{s}).\end{aligned}$$

The proof is now finished. □

Lemma 8. *Let $\varkappa \in \Omega_3$. Assume that (4) holds. We define the function $\omega(\mathrm{s})$ as follows:*

$$\omega(\mathrm{s}) := \frac{\theta(\mathrm{s})\left(-\Psi^{(n-1)}(\mathrm{s})\right)^{\alpha}}{\left(\Psi^{(n-2)}(\mathrm{s})\right)^{\beta}}, \ \mathrm{s} \geq \mathrm{s}_1. \tag{13}$$

Then, we have the following conditions:
$(Q_{3,1})$ $\omega(s)\mu_0^{\gamma}(s)$ *is bounded;*
$(Q_{3,2})$ $\omega'(s) \geq \widetilde{q}(s)\mu_{n-2}^{\beta}(s)/\mu_0^{\beta}(s) + \beta m\theta^{-1/\alpha}(s)\omega^{(\gamma+1)/\gamma}(s)$,
where m is a positive constant and $\gamma = \max\{\alpha, \beta\}$.

Proof. $(Q_{3,1})$ By Lemma 6, we have $\left(\theta(s)\left(-\Psi^{(n-1)}(s)\right)^{\alpha}\right)' \geq 0$, which implies that $\theta(s)\left(-\Psi^{(n-1)}(s)\right)^{\alpha}$ is non-decreasing. From (7), we obtain

$$\left(\Psi^{(n-2)}(s)\right)^{\alpha} \geq \theta(s)\left(-\Psi^{(n-1)}(s)\right)^{\alpha}\mu_0^{\alpha}(s) = \left(\Psi^{(n-2)}(s)\right)^{\beta}\omega(s)\mu_0^{\alpha}(s). \tag{14}$$

It follows that

$$\left(\Psi^{(n-2)}(s)\right)^{\alpha-\beta} \geq \omega(s)\mu_0^{\alpha}(s),\ s \geq s_1. \tag{15}$$

If $\alpha > \beta$, by applying $\Psi^{(n-1)}(s) < 0$ as indicated in (15), we establish that the positive function $\omega\mu_0^{\alpha}$ remains bounded.
Conversely, if $\beta \geq \alpha$, and using the result from (7) again, we derive the following:

$$\left(\Psi^{(n-2)}(s)\right)^{\beta} \geq \left(\theta^{1/\alpha}(s)\left(-\Psi^{(n-1)}(s)\right)\right)^{\beta-\alpha+\alpha}\mu_0^{\beta}(s), \tag{16}$$

which implies that

$$\left[\theta^{1/\alpha}(s)\left(-\Psi^{(n-1)}(s)\right)\right]^{\alpha-\beta} \geq \frac{\theta(s)\left(-\Psi^{(n-1)}(s)\right)^{\alpha}}{\left(\Psi^{(n-2)}(s)\right)^{\beta}}\mu_0^{\beta}(s) = \omega(s)\mu_0^{\beta}(s).$$

Since $\left[\theta^{1/\alpha}(s)\left(-\Psi^{(n-1)}(s)\right)\right]^{\alpha-\beta}$ is decreasing, then $\omega(s)\mu_0^{\beta}(s)$ is bounded. Therefore, the function $\omega(s)\mu_0^{\gamma}(s)$ is bounded, where $\gamma = \max\{\alpha, \beta\}$.
$(Q_{3,2})$ In view of the definitions of $\omega(s)$ and $(Q_{2,2})$, we have

$$\begin{aligned}\omega'(s) &= \frac{\left(\theta(s)\left(-\Psi^{(n-1)}(s)\right)^{\alpha}\right)'}{\left(\Psi^{(n-2)}(s)\right)^{\beta}} + \beta\frac{\theta(s)\left(-\Psi^{(n-1)}(s)\right)^{\alpha+1}}{\left(\Psi^{(n-2)}(s)\right)^{\beta+1}} \\ &\geq \widetilde{q}(s)\frac{\Psi^{\beta}(s)}{\left(\Psi^{(n-2)}(s)\right)^{\beta}} + \frac{\beta}{\theta^{1/\alpha}(s)}\left[\Psi^{(n-2)}(s)\right]^{(\beta-\alpha)/\alpha}\omega^{(\alpha+1)/\alpha}(s).\end{aligned} \tag{17}$$

Using (10), we obtain

$$\frac{\Psi(s)}{\Psi^{(n-2)}(s)} \geq \frac{\mu_{n-2}(s)}{\mu_0(s)}. \tag{18}$$

Substituting (18) into (17), we obtain

$$\omega'(s) \geq \widetilde{q}(s)\left(\frac{\mu_{n-2}(s)}{\mu_0(s)}\right)^{\beta} + \beta\theta^{-1/\alpha}(s)\left[\Psi^{(n-2)}(s)\right]^{(\beta-\alpha)/\alpha}\omega^{(\alpha+1)/\alpha}(s).$$

If $\alpha > \beta$, and considering that $\Psi^{(n-1)} < 0$ for $s \geq s_1$, the function $\left(\Psi^{(n-2)}\right)^{(\beta-\alpha)/\alpha}$ is increasing. Letting

$$m_1 = \left(\Psi^{(n-2)}(s_0)\right)^{(\beta-\alpha)/\alpha}\ (\text{if } \alpha = \beta,\ \text{then } m_1 = 1),$$

the inequality becomes

$$\omega'(\mathrm{s}) \geq \widetilde{\mathfrak{q}}(\mathrm{s})\left(\frac{\mu_{n-2}(\mathrm{s})}{\mu_0(\mathrm{s})}\right)^{\beta} + \beta m_1 \theta^{-1/\alpha}(\mathrm{s})\omega^{(\alpha+1)/\alpha}(\mathrm{s}),\ \mathrm{s} \geq \mathrm{s}_0. \tag{19}$$

Now, if $\beta \geq \alpha$, the inequality takes the form

$$\omega'(\mathrm{s}) \geq \widetilde{\mathfrak{q}}(\mathrm{s})\left(\frac{\mu_{n-2}(\mathrm{s})}{\mu_0(\mathrm{s})}\right)^{\beta} + \beta\theta^{-1/\beta}(\mathrm{s})\left(-\Psi^{(n-1)}(\mathrm{s})\right)^{(\beta-\alpha)/\beta}\omega^{(\beta+1)/\beta}(\mathrm{s}). \tag{20}$$

Since $\left(\theta^{1/\alpha}(\mathrm{s})\left(-\Psi^{(n-1)}(\mathrm{s})\right)\right)^{(\beta-\alpha)/\beta}$ is increasing, inequality (20) implies

$$\begin{aligned} \omega'(\mathrm{s}) \ &\geq\ \widetilde{\mathfrak{q}}(\mathrm{s})\left(\frac{\mu_{n-2}(\mathrm{s})}{\mu_0(\mathrm{s})}\right)^{\beta} + \beta\theta^{-1/\alpha}(\mathrm{s})\left(\theta^{1/\alpha}(\mathrm{s})\left(-\Psi^{(n-1)}(\mathrm{s})\right)\right)^{(\beta-\alpha)/\beta}\omega^{(\beta+1)/\beta}(\mathrm{s}) \\ &\geq\ \widetilde{\mathfrak{q}}(\mathrm{s})\left(\frac{\mu_{n-2}(\mathrm{s})}{\mu_0(\mathrm{s})}\right)^{\beta} + \beta m_2\theta^{-1/\alpha}(\mathrm{s})\omega^{(\beta+1)/\beta}(\mathrm{s}),\ \mathrm{s} \geq \mathrm{s}_1 \geq \mathrm{s}_0, \end{aligned} \tag{21}$$

where

$$m_2 = \left(\theta^{1/\alpha}(\mathrm{s}_1)\left(-\Psi'''(\mathrm{s}_1)\right)\right)^{1-\alpha/\beta}\ (\text{if } \alpha = \beta,\ \text{then } m_2 = 1).$$

By combining (19) and (21), we find

$$\omega'(\mathrm{s}) \geq \widetilde{\mathfrak{q}}(\mathrm{s})\left(\frac{\mu_2(\mathrm{s})}{\mu_0(\mathrm{s})}\right)^{\beta} + \beta m\theta^{-1/\alpha}(\mathrm{s})\omega^{(\gamma+1)/\gamma}(\mathrm{s}),\ \mathrm{s} \geq \mathrm{s}_1, \tag{22}$$

where

$$\gamma = \max\{\alpha, \beta\},$$

and

$$m = \begin{cases} 1, & \alpha = \beta; \\ const > 0, & \alpha \neq \beta. \end{cases}$$

The proof is now finished. □

Lemma 9. *Assume that (4) and* $\mathrm{g}(\mathrm{s}) < \mu_{n-2}(\mathrm{s})/\mu_{n-2}(\mathfrak{z}(\mathrm{s}))$ *hold. If*

$$\limsup_{\mathrm{s}\to\infty} \int_{\mathrm{s}_0}^{\mathrm{s}} \left(\mu_0^{\gamma}(\varsigma)\widetilde{\mathfrak{q}}(\varsigma)\left(\frac{\mu_{n-2}(\varsigma)}{\mu_0(\varsigma)}\right)^{\beta} - \frac{L}{\mu_0(\varsigma)\theta^{1/\alpha}(\varsigma)}\right)\mathrm{d}\varsigma = \infty, \tag{23}$$

then $\Omega_3 = \varnothing$.

Proof. Suppose the contrary where $\varkappa \in \Omega_3$; i.e., there exists a $\mathrm{s}_1 \geq \mathrm{s}_0$, such that $\varkappa(\mathfrak{z}(\mathrm{s})) > 0$ and $\varkappa(\mathfrak{h}(\mathrm{s}))$ for all $\mathrm{s} \geq \mathrm{s}_1$. Considering the fact that $\Psi(\mathrm{s}) \geq \varkappa(\mathrm{s}) > 0$ for $\mathrm{s} \geq \mathrm{s}_1$ and (5), we have

$$\left(\theta(\mathrm{s})\left|\Psi^{(n-1)}(\mathrm{s})\right|^{\alpha-1}\Psi^{(n-1)}(\mathrm{s})\right)' = -\varphi(\mathrm{s})\mathfrak{q}(\mathrm{s})\varkappa^{\beta}(\mathfrak{h}(\mathrm{s})) \leq 0,$$

which implies that $\theta(\mathrm{s})\left|\Psi^{(n-1)}(\mathrm{s})\right|^{\alpha-1}\Psi^{(n-1)}(\mathrm{s})$ is non-increasing. Since $\Psi^{(n-1)}(\mathrm{s}) < 0$, then by using Lemma 7, we obtain

$$\left(\theta(\mathrm{s})\left(-\Psi^{(n-1)}(\mathrm{s})\right)^{\alpha}\right)' - \widetilde{\mathfrak{q}}(\mathrm{s})\Psi^{\beta}(\mathrm{s}) \geq 0,\ \mathrm{s} \geq \mathrm{s}_1.$$

Let $\omega(\mathrm{s})$ be defined by (13) for $\mathrm{s} \geq \mathrm{s}_2 \geq \mathrm{s}_1$. It then follows that $\omega(\mathrm{s}) > 0$ for all $\mathrm{s} \geq \mathrm{s}_2$. From Lemma 8, we obtain

$$\omega'(\mathrm{s}) \geq \widetilde{\mathfrak{q}}(\mathrm{s})\left(\frac{\mu_{n-2}(\mathrm{s})}{\mu_0(\mathrm{s})}\right)^{\beta} + \beta m \theta^{-1/\alpha}(\mathrm{s})\omega^{(\gamma+1)/\gamma}(\mathrm{s}),\ \mathrm{s} \geq \mathrm{s}_2. \tag{24}$$

Multiplying (24) by $\mu_0^{\gamma}(\mathrm{s})$ and integrating the resulting inequality from $\mathrm{s}_3 \geq \mathrm{s}_2$ to s, we have

$$\begin{aligned} &\int_{\mathrm{s}_3}^{\mathrm{s}} \mu_0^{\gamma}(\varsigma)\widetilde{\mathfrak{q}}(\varsigma)\left(\frac{\mu_{n-2}(\varsigma)}{\mu_0(\varsigma)}\right)^{\beta} d\varsigma \\ \leq\ & \int_{\mathrm{s}_3}^{\mathrm{s}} \mu_0^{\gamma-1}(\varsigma)\theta^{-1/\alpha}(\varsigma)\left[\gamma\omega(\varsigma) - \beta m \mu_0(\varsigma)\omega^{(\gamma+1)/\gamma}(\varsigma)\right] d\varsigma + \mu_0^{\gamma}(\mathrm{s})\omega(\mathrm{s}). \end{aligned} \tag{25}$$

Using Lemma 3, where $B = \gamma$, $A = \beta m_1 \mu_0(\mathrm{s})$, and $u = \omega(\mathrm{s})$, we obtain

$$\begin{aligned} \gamma u(\mathrm{s}) - \beta m \mu_0(\mathrm{s}) u^{(\gamma+1)/\gamma}(\mathrm{s}) &\leq \frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}} \frac{\gamma^{\gamma+1}}{\beta^{\gamma} m^{\gamma} \mu_0^{\gamma}(\mathrm{s})} \\ &= \left(\frac{\gamma}{\gamma+1}\right)^{\gamma+1}\left(\frac{\gamma}{\beta m}\right)^{\gamma}\frac{1}{\mu_0^{\gamma}(\mathrm{s})} \\ &= L\frac{1}{\mu_0^{\gamma}(\mathrm{s})}, \end{aligned}$$

which, with (25), gives

$$\int_{\mathrm{s}_3}^{\mathrm{s}} \left(\mu_0^{\gamma}(\varsigma)\widetilde{\mathfrak{q}}(\varsigma)\left(\frac{\mu_{n-2}(\varsigma)}{\mu_0(\varsigma)}\right)^{\beta} - \frac{L}{\mu_0(\varsigma)\theta^{1/\alpha}(\varsigma)}\right) d\varsigma \leq \mu_0^{\gamma}(\mathrm{s})\omega(\mathrm{s}),$$

where $L = \begin{cases} \left(\frac{\gamma}{\gamma+1}\right)^{\gamma+1}\left(\frac{\gamma}{\beta m}\right)^{\gamma}, & \alpha \neq \beta, \\ \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}, & \alpha = \beta. \end{cases}$

From Lemma 8, we see that $\mu_0^{\gamma}(\mathrm{s})\omega(\mathrm{s})$ is bounded. Letting $\mathrm{s} \to \infty$ in the above inequality, we obtain a contradiction with (23). The proof is now complete. □

Lemma 10. *Assume that (4) and* $\mathrm{g}(\mathrm{s}) < \mu_{n-2}(\mathrm{s})/\mu_{n-2}(\mathfrak{z}(\mathrm{s}))$ *hold. If*

$$\liminf_{\mathrm{s}\to\infty} \mu_0^{\gamma+1}(\mathrm{s})\theta^{1/\alpha}(\mathrm{s})\widetilde{\mathfrak{q}}(\mathrm{s})\left(\frac{\mu_{n-2}(\mathrm{s})}{\mu_0(\mathrm{s})}\right)^{\beta} > L, \tag{26}$$

then $\Omega_3 = \varnothing$.

Proof. Suppose that (26) holds. Then for any $\varepsilon > 0$, there exists a sufficiently large $\mathrm{s}_1 \geq \mathrm{s}_0$, such that

$$\mu_0^{\gamma}(\mathrm{s})\widetilde{\mathfrak{q}}(\mathrm{s})\left(\frac{\mu_{n-2}(\varsigma)}{\mu_0(\varsigma)}\right)^{\beta} > \frac{L-\varepsilon}{\mu_0(\mathrm{s})\theta^{1/\alpha}(\mathrm{s})}.$$

Integrating this inequality from s_1 to s, we then obtain

$$\begin{aligned} \int_{\mathrm{s}_1}^{\mathrm{s}} \left(\mu_0^{\gamma}(\varsigma)\widetilde{\mathfrak{q}}(\varsigma)\left(\frac{\mu_{n-2}(\varsigma)}{\mu_0(\varsigma)}\right)^{\beta} - \frac{L}{\mu_0(\varsigma)\theta^{1/\alpha}(\varsigma)}\right) d\varsigma &> -\varepsilon\int_{\mathrm{s}_1}^{\mathrm{s}} \frac{1}{\mu_0(\mathrm{s})\theta^{1/\alpha}(\mathrm{s})} d\varsigma \\ &= \varepsilon\left(\ln\frac{1}{\mu_0(\mathrm{s})} - \ln\frac{1}{\mu_0(\mathrm{s}_1)}\right). \end{aligned}$$

By allowing $\mathrm{s} \to \infty$ n the inequality presented above, we establish that (23) is satisfied. Thus, the proof is concluded. □

3.2. Category Ω_1

In this subsection, we introduce a collection of lemmas focused on the asymptotic properties of solutions belonging to the (C_1) class.

Lemma 11. *Let $\varkappa \in \Omega_1$. Assume that (4) holds. Then,*

$$\left(\theta(\mathrm{s})\left(\Psi^{(n-1)}(\mathrm{s})\right)^{\alpha}\right)' + \widehat{\mathfrak{q}}(\mathrm{s})\Psi^{\beta}(\mathfrak{h}(\mathrm{s})) \leq 0. \tag{27}$$

Proof. Let $\varkappa \in \Omega_1$. Then there exists a $\mathrm{s}_1 \geq \mathrm{s}_0$, such that $\varkappa(\mathfrak{z}(\mathrm{s})) > 0$ and $\varkappa(\mathfrak{h}(\mathrm{s})) > 0$ for $\mathrm{s} \geq \mathrm{s}_1$. By multiplying both sides of (1) by φ, we obtain

$$\left(\theta(\mathrm{s})\left|\Psi^{(n-1)}(\mathrm{s})\right|^{\alpha-1}\Psi^{(n-1)}(\mathrm{s})\right)' = -\varphi(\mathrm{s})\mathfrak{q}(\mathrm{s})\varkappa^{\beta}(\mathfrak{h}(\mathrm{s})),\ \mathrm{s} \geq \mathrm{s}_0.$$

Since $\Psi^{(n-1)}(\mathrm{s}) > 0$, this inequality simplifies to

$$\left(\theta(\mathrm{s})\left(\Psi^{(n-1)}(\mathrm{s})\right)^{\alpha}\right)' = -\varphi(\mathrm{s})\mathfrak{q}(\mathrm{s})\varkappa^{\beta}(\mathfrak{h}(\mathrm{s})) \leq 0. \tag{28}$$

This implies that $\theta(\mathrm{s})\left(\Psi^{(n-1)}(\mathrm{s})\right)^{\alpha}$ is non-increasing. By using the definition of $\Psi(\mathrm{s})$, we can deduce

$$\varkappa(\mathrm{s}) = \Psi(\mathrm{s}) - \mathrm{g}(\mathrm{s})\varkappa(\mathfrak{z}(\mathrm{s})) \geq \Psi(\mathrm{s}) - \mathrm{g}(\mathrm{s})\Psi(\mathfrak{z}(\mathrm{s})).$$

Since $\Psi'(\mathrm{s}) > 0$, and $\mathfrak{z}(\mathrm{s}) \leq \mathrm{s}$, we can conclude that $\Psi(\mathrm{s}) \geq \Psi(\mathfrak{z}(\mathrm{s}))$, which implies

$$\varkappa(\mathrm{s}) \geq (1 - \mathrm{g}(\mathrm{s}))\Psi(\mathrm{s}).$$

Consequently, it follows that

$$\varkappa(\mathfrak{h}(\mathrm{s})) \geq (1 - \mathrm{g}(\mathfrak{h}(\mathrm{s})))\Psi(\mathfrak{h}(\mathrm{s})). \tag{29}$$

Using (29) with (28), we have

$$\begin{aligned} \left(\theta(\mathrm{s})\left(\Psi'''(\mathrm{s})\right)^{\alpha}\right)' &= -\varphi(\mathrm{s})\mathfrak{q}(\mathrm{s})\varkappa^{\beta}(\mathfrak{h}(\mathrm{s})) \\ &\leq -\varphi(\mathrm{s})\mathfrak{q}(\mathrm{s})(1 - \mathrm{g}(\mathfrak{h}(\mathrm{s})))^{\beta}\Psi^{\beta}(\mathfrak{h}(\mathrm{s})) = -\widehat{\mathfrak{q}}(\mathrm{s})\Psi^{\beta}(\mathfrak{h}(\mathrm{s})). \end{aligned}$$

The proof is now complete. □

Lemma 12. *Assume that (4) holds. If there exists a positive non-decreasing function $\varrho(\mathrm{s}) \in C^1([\mathrm{s}_0, \infty), (0, \infty))$, such that*

$$\limsup_{\mathrm{s}\to\infty} \int_{\mathrm{s}_0}^{\mathrm{s}} \left(\varrho(\varsigma)\widehat{\mathfrak{q}}(\varsigma) - \frac{((n-2)!)^{\nu}}{(v+1)^{v+1}} \frac{\theta(\delta(\varsigma))(\varrho'(\varsigma))^{v+1}}{(K\lambda_1\varrho(\varsigma)\mathfrak{h}'(\varsigma)\mathfrak{h}^{n-2}(\varsigma))^{v}}\right) d\varsigma = \infty, \tag{30}$$

then $\Omega_1 = \varnothing$.

Proof. Suppose the contrary, where $\varkappa \in \Omega_1$. In other words, there exists a $\mathrm{s}_1 \geq \mathrm{s}_0$ such that $\varkappa(\mathfrak{z}(\mathrm{s})) > 0$ and $\varkappa(\mathfrak{h}(\mathrm{s}))$ for all $\mathrm{s} \geq \mathrm{s}_1$. Define a function $\varpi(\mathrm{s})$ by

$$\varpi(\mathrm{s}) := \varrho(\mathrm{s})\frac{\theta(\mathrm{s})\left(\Psi^{(n-1)}(\mathrm{s})\right)^{\alpha}}{\Psi^{\beta}(\mathfrak{h}(\mathrm{s}))},\ \mathrm{s} \geq \mathrm{s}_1. \tag{31}$$

Then, $\varpi(\mathrm{s}) > 0$ and

$$\begin{aligned}\varpi'(\mathrm{s}) &= \varrho'(\mathrm{s})\frac{\theta(\mathrm{s})\left(\Psi^{(n-1)}(\mathrm{s})\right)^{\alpha}}{\Psi^{\beta}(\mathfrak{h}(\mathrm{s}))} + \varrho(\mathrm{s})\frac{\left(\theta(\mathrm{s})\left(\Psi^{(n-1)}(\mathrm{s})\right)^{\alpha}\right)'}{\Psi^{\beta}(\mathfrak{h}(\mathrm{s}))} \\ &\quad -\beta\mathfrak{h}'(\mathrm{s})\varrho(\mathrm{s})\frac{\theta(\mathrm{s})\left(\Psi^{(n-1)}(\mathrm{s})\right)^{\alpha}}{\Psi^{\beta+1}(\mathfrak{h}(\mathrm{s}))}\Psi'(\mathfrak{h}(\mathrm{s})). \qquad (32)\end{aligned}$$

By using Lemma 1, we obtain

$$\Psi'(\mathrm{s}) \geq \frac{\lambda_1}{(n-2)!}\mathrm{s}^{n-2}\Psi^{(n-1)}(\mathrm{s}),$$

or equivalently

$$\Psi'(\mathfrak{h}(\mathrm{s})) \geq \frac{\lambda_1}{(n-2)!}\mathfrak{h}^{n-2}(\mathrm{s})\Psi^{(n-1)}(\mathfrak{h}(\mathrm{s})). \qquad (33)$$

By using (27), (31), and (33) in (32), we have

$$\begin{aligned}\varpi'(\mathrm{s}) &\leq -\varrho(\mathrm{s})\widehat{\mathfrak{q}}(\mathrm{s}) + \frac{\varrho'(\mathrm{s})}{\varrho(\mathrm{s})}\varpi(\mathrm{s}) \\ &\quad -\frac{\lambda_1}{(n-2)!}\beta\mathfrak{h}'(\mathrm{s})\mathfrak{h}^{n-2}(\mathrm{s})\varrho(\mathrm{s})\frac{\theta(\mathrm{s})\left(\Psi^{(n-1)}(\mathrm{s})\right)^{\alpha}}{\Psi^{\beta+1}(\mathfrak{h}(\mathrm{s}))}\Psi^{(n-1)}(\mathfrak{h}(\mathrm{s})). \qquad (34)\end{aligned}$$

For this inequality, we first treat the case $\alpha < \beta$. Note that $\theta(\mathrm{s})\left(\Psi^{(n-1)}(\mathrm{s})\right)^{\alpha}$ is a positive nonincreasing function; then,

$$\theta^{1/\alpha}(\mathrm{s})\Psi^{(n-1)}(\mathrm{s}) \leq \theta^{1/\alpha}(\mathfrak{h}(\mathrm{s}))\Psi^{(n-1)}(\mathfrak{h}(\mathrm{s})).$$

In view of (34), we obtain

$$\begin{aligned}\varpi'(\mathrm{s}) &\leq -\varrho(\mathrm{s})\widehat{\mathfrak{q}}(\mathrm{s}) + \frac{\varrho'(\mathrm{s})}{\varrho(\mathrm{s})}\varpi(\mathrm{s}) \\ &\quad -\frac{\lambda_1}{(n-2)!}\beta\frac{\mathfrak{h}'(\mathrm{s})\mathfrak{h}^{n-2}(\mathrm{s})}{(\varrho(\mathrm{s})\theta(\mathfrak{h}(\mathrm{s})))^{1/\alpha}}\Psi^{(\beta-\alpha)/\alpha}(\mathfrak{h}(\mathrm{s}))\varpi^{(\alpha+1)/\alpha}(\mathrm{s}).\end{aligned}$$

Since Ψ is an increasing function, thus there exist the constants $K_1 > 0$ and $\mathrm{s}_2 \geq \mathrm{s}_1$ such that

$$\Psi^{(\beta-\alpha)/\alpha}(\mathfrak{h}(\mathrm{s})) \geq K_1,\ \mathrm{s} \geq \mathrm{s}_2. \qquad (35)$$

Hence, we obtain

$$\varpi'(\mathrm{s}) \leq -\varrho(\mathrm{s})\widehat{\mathfrak{q}}(\mathrm{s}) + \frac{\varrho'(\mathrm{s})}{\varrho(\mathrm{s})}\varpi(\mathrm{s}) - \frac{\lambda_1\alpha K_1}{(n-2)!}\frac{\mathfrak{h}'(\mathrm{s})\mathfrak{h}^{n-2}(\mathrm{s})}{(\varrho(\mathrm{s})\theta(\mathfrak{h}(\mathrm{s})))^{1/\alpha}}\varpi^{(\alpha+1)/\alpha}(\mathrm{s}). \qquad (36)$$

Note that if $\alpha = \beta$, then $K_1 = 1$; thus, (36) still holds.
Now, if $\alpha > \beta$, because $\kappa'(\mathrm{s}) \geq 0$, we have $\theta'(\mathrm{s}) \geq 0$. Recall that $\left(\theta(\mathrm{s})\left(\Psi^{(n-1)}(\mathrm{s})\right)^{\alpha}\right)' \leq 0$, and hence $\Psi^{(n)}(\mathrm{s}) \leq 0$, which implies that

$$\left(\Psi^{(n-1)}(\mathrm{s})\right)^{(\beta-\alpha)/\beta} \geq K_2,\ \mathrm{s} \geq \mathrm{s}_3. \qquad (37)$$

By combining (34) and (37), we have

$$\begin{aligned}\varpi'(\mathrm{s}) &\le -\varrho(\mathrm{s})\widehat{\mathfrak{q}}(\mathrm{s}) + \frac{\varrho'(\mathrm{s})}{\varrho(\mathrm{s})}\varpi(\mathrm{s}) - \frac{\lambda_1\beta}{(n-2)!}\frac{\mathfrak{h}'(\mathrm{s})\mathfrak{h}^{n-2}(\mathrm{s})}{(\varrho(\mathrm{s})\theta(\mathrm{s}))^{1/\beta}}\left(\Psi^{(n-1)}(\mathfrak{h}(\mathrm{s}))\right)^{(\beta-\alpha)/\beta}\varpi^{(\beta+1)/\beta}(\mathrm{s}) \\ &\le -\varrho(\mathrm{s})\widehat{\mathfrak{q}}(\mathrm{s}) + \frac{\varrho'(\mathrm{s})}{\varrho(\mathrm{s})}\varpi(\mathrm{s}) - \frac{\lambda_1\beta}{(n-2)!}K_2\frac{\mathfrak{h}'(\mathrm{s})\mathfrak{h}^{n-2}(\mathrm{s})}{(\varrho(\mathrm{s})\theta(\mathrm{s}))^{1/\beta}}\varpi^{(\beta+1)/\beta}(\mathrm{s}),\end{aligned}$$

which, together with (36), implies that

$$\varpi'(\mathrm{s}) \le -\varrho(\mathrm{s})\widehat{\mathfrak{q}}(\mathrm{s}) + \frac{\varrho'(\mathrm{s})}{\varrho(\mathrm{s})}\varpi(\mathrm{s}) - \frac{\lambda_1\nu}{(n-2)!}K\frac{\mathfrak{h}'(\mathrm{s})\mathfrak{h}^{n-2}(\mathrm{s})}{(\varrho(\mathrm{s})\theta(\delta(\mathrm{s})))^{1/\nu}}\varpi^{(\nu+1)/\nu}(\mathrm{s}),\ \mathrm{s} \ge \mathrm{s}_3, \tag{38}$$

where $v = \min\{\alpha, \beta\}$, $K = \min\{K_1, K_2\}$, and

$$\delta(\mathrm{s}) = \begin{cases} \mathrm{s}, & \alpha > \beta, \\ \mathfrak{h}(\mathrm{s}), & \alpha \le \beta. \end{cases}$$

Using Lemma 3, where $B = \varrho'(\mathrm{s})/\varrho(\mathrm{s})$, $\kappa = \lambda_1\nu K\mathfrak{h}'(\mathrm{s})\mathfrak{h}^{n-2}(\mathrm{s})/(n-2)!(\varrho(\mathrm{s})\theta(\theta(\mathrm{s})))^{1/v}$, and $u = \varpi(\mathrm{s})$, we obtain

$$\frac{\varrho'(\mathrm{s})}{\varrho(\mathrm{s})}\varpi(\mathrm{s}) - \frac{\lambda_0\nu}{2}K\frac{\mathfrak{h}'(\mathrm{s})\mathfrak{h}^{n-2}(\mathrm{s})}{(\varrho(\mathrm{s})\theta(\delta(\mathrm{s})))^{1/v}}\varpi^{(v+1)/v}(\mathrm{s}) \le \frac{((n-2)!)^{\nu}}{(v+1)^{v+1}}\frac{\theta(\delta(\mathrm{s}))(\varrho'(\mathrm{s}))^{v+1}}{(K\lambda_1\varrho(\mathrm{s})\mathfrak{h}'(\mathrm{s})\mathfrak{h}^{n-2}(\mathrm{s}))^{v}},$$

which, with (38), gives

$$\varpi'(\mathrm{s}) \le -\varrho(\mathrm{s})\widehat{\mathfrak{q}}(\mathrm{s}) + \frac{((n-2)!)^{\nu}}{(v+1)^{v+1}}\frac{\theta(\delta(\mathrm{s}))(\varrho'(\mathrm{s}))^{v+1}}{(K\lambda_1\varrho(\mathrm{s})\mathfrak{h}'(\mathrm{s})\mathfrak{h}^{n-2}(\mathrm{s}))^{v}}.$$

By integrating the above inequality from $\mathrm{s}_4 \ge \mathrm{s}_3$ to s, we find

$$\varpi(\mathrm{s}) \le \varpi(\mathrm{s}_4) - \int_{\mathrm{s}_4}^{\mathrm{s}}\left(\varrho(\varsigma)\widehat{\mathfrak{q}}(\varsigma) - \frac{((n-2)!)^{\nu}}{(v+1)^{v+1}}\frac{\theta(\delta(\varsigma))(\varrho'(\varsigma))^{v+1}}{(K\lambda_1\varrho(\varsigma)\mathfrak{h}'(\varsigma)\mathfrak{h}^{n-2}(\varsigma))^{v}}\right)\mathrm{d}\varsigma.$$

Letting $\mathrm{s} \to \infty$ in the above inequality, we then obtain a contradiction with (30). The proof is now complete. □

Lemma 13. *Suppose that (4) holds. If there is a positive non-decreasing function $\varrho \in C^1([\mathrm{s}_0,\infty),(0,\infty))$, such that*

$$\limsup_{\mathrm{s}\to\infty}\int_{\mathrm{s}_0}^{\mathrm{s}}\widehat{\mathfrak{q}}(\mathrm{s})\mathrm{d}\varsigma = \infty, \tag{39}$$

then $\Omega_1 = \varnothing$.

Proof. Condition (39) follows by substituting $\varrho(\mathrm{s}) = 1$ into (30). □

3.3. Category Ω_2

In this subsection, we introduce a collection of lemmas focused on the asymptotic properties of solutions belonging to the (C_2) class.

Lemma 14. *Let $\varkappa \in \Omega_2$. Assume that (4) holds. Then, eventually,*
$(\mathrm{Q}_{4,1})$ $\Psi(\mathrm{s}) \ge \epsilon_0\mathrm{s}\Psi'(\mathrm{s})$;
$(\mathrm{Q}_{4,2})$ $\Psi^{(n-2)}(\mathrm{s}) \ge -\theta^{1/\alpha}(\mathrm{s})\mu_0(\mathrm{s})\Psi^{(n-1)}(\mathrm{s})$;
$(\mathrm{Q}_{4,3})$ $\Psi^{(n-2)}(\mathrm{s})/\mu_0(\mathrm{s})$ *is increasing*.

Proof. By applying Lemma 2 with $m = n - 2$ and $\chi(\mathrm{s}) = \Psi(\mathrm{s})$, we derive

$$\Psi(\mathrm{s}) \geq \frac{\epsilon_0}{n-2}\mathrm{s}\Psi'(\mathrm{s}).$$

Given that $\left(\theta^{1/\alpha}(\mathrm{s})\Psi^{(n-1)}(\mathrm{s})\right)' < 0$, we obtain

$$\Psi^{(n-2)}(\mathrm{s}) \geq -\int_{\mathrm{s}}^{\infty}\Psi^{(n-1)}(\varsigma)\mathrm{d}\varsigma \geq -\int_{\mathrm{s}}^{\infty}\frac{\theta^{1/\alpha}(\varsigma)\Psi^{(n-1)}(\varsigma)}{\theta^{1/\alpha}(\varsigma)}\mathrm{d}\varsigma \geq -\theta^{1/\alpha}(\mathrm{s})\mu_0(\mathrm{s})\Psi^{(n-1)}(\mathrm{s}).$$

Thus,

$$\left(\frac{\Psi^{(n-2)}}{\mu_0}\right)'(\mathrm{s}) = \frac{\theta^{1/\alpha}(\mathrm{s})\mu_0(\varsigma)\Psi^{(n-1)}(\mathrm{s}) + \Psi^{(n-2)}(\mathrm{s})}{\theta^{1/\alpha}(\mathrm{s})\mu_0^2(\varsigma)} \geq 0.$$

This concludes the proof. □

Lemma 15. *Let $\varkappa \in \Omega_2$. Assume that (4) holds. Then, eventually*
$(\mathrm{Q}_{5,1})$ $\varkappa(\mathrm{s}) \geq \mathrm{g}_2(\mathrm{s};m)\Psi(\mathrm{s})$;
$(\mathrm{Q}_{5,2})$ $\left(\theta(\mathrm{s})\left(-\Psi^{(n-1)}(\mathrm{s})\right)^{\alpha}\right)' - \widetilde{\mathfrak{q}}_1(\mathrm{s})\Psi^{\alpha}(\mathfrak{h}(\mathrm{s})) \geq 0.$

Proof. $(\mathrm{Q}_{5,1})$ From Lemma 5, we have that (3) holds. Based on the properties of solutions in the class Ω_2, we conclude that $\Psi\left(\mathfrak{z}^{[2i]}(\mathrm{s})\right) \geq \Psi\left(\mathfrak{z}^{[2i+1]}(\mathrm{s})\right)$ for $i = 1, 2, \ldots, m$. Thus, (3) becomes

$$\varkappa(\mathrm{s}) > \sum_{i=0}^{m}\left(\prod_{k=0}^{2i}\mathrm{g}\left(\mathfrak{z}^{[k]}(\mathrm{s})\right)\right)\left[\frac{1}{\mathrm{g}\left(\mathfrak{z}^{[2i]}(\mathrm{s})\right)} - 1\right]\Psi\left(\mathfrak{z}^{[2i]}(\mathrm{s})\right). \tag{40}$$

Using $(\mathrm{Q}_{4,1})$, we obtain

$$\Psi\left(\mathfrak{z}^{[2i]}(\mathrm{s})\right) \geq \left(\frac{\mathfrak{z}^{[2i]}(\mathrm{s})}{\mathrm{s}}\right)^{(n-2)/\epsilon_0}\Psi(\mathrm{s}).$$

Which, with (40), gives

$$\begin{aligned}\varkappa(\mathrm{s}) &> \sum_{i=0}^{m}\left(\prod_{k=0}^{2i}\mathrm{g}\left(\mathfrak{z}^{[k]}(\mathrm{s})\right)\right)\left[\frac{1}{\mathrm{g}\left(\mathfrak{z}^{[2i]}(\mathrm{s})\right)} - 1\right]\left(\frac{\mathfrak{z}^{[2i]}(\mathrm{s})}{\mathrm{s}}\right)^{(n-2)/\epsilon_0}\Psi(\mathrm{s}) \\ &= \mathrm{g}_2(\mathrm{s};m)\Psi(\mathrm{s}).\end{aligned}$$

$(\mathrm{Q}_{5,2})$ Since $\Psi^{(n-1)}(\mathrm{s}) < 0$, then from (5), we obtain

$$\left(\theta(\mathrm{s})\left(-\Psi^{(n-1)}(\mathrm{s})\right)^{\alpha}\right)' = \varphi(\mathrm{s})\mathfrak{q}(\mathrm{s})\varkappa^{\beta}(\mathfrak{h}(\mathrm{s})) \geq 0. \tag{41}$$

By using $(\mathrm{Q}_{5,1})$, we can deduce

$$\begin{aligned}\left(\theta(\mathrm{s})\left(-\Psi^{(n-1)}(\mathrm{s})\right)^{\alpha}\right)' &= \varphi(\mathrm{s})\mathfrak{q}(\mathrm{s})\varkappa^{\beta}(\mathfrak{h}(\mathrm{s})) \\ &\geq \varphi(\mathrm{s})\mathfrak{q}(\mathrm{s})\mathrm{g}_2^{\beta}(\mathfrak{h}(\mathrm{s});m)\Psi^{\beta}(\mathfrak{h}(\mathrm{s})) = \widetilde{\mathfrak{q}}_1(\mathrm{s})\Psi^{\beta}(\mathfrak{h}(\mathrm{s})).\end{aligned}$$

The proof is now complete. □

Lemma 16. *Assume that* $\alpha \geq 1$. *There is then a positive function* $\widetilde{\varrho}(\mathrm{s}) \in C^1([\mathrm{s}_0,\infty),(0,\infty))$ *such that*

$$\limsup_{\mathrm{s}\to\infty}\int_{\mathrm{s}_1}^{\mathrm{s}}\left(\vartheta(\varsigma)-\frac{\theta(\varsigma)\widetilde{\varrho}(\varsigma)}{(\alpha+1)^{\alpha+1}}\left(\frac{\widetilde{\varrho}'(\varsigma)}{\widetilde{\varrho}(\varsigma)}+\frac{(1+\alpha)}{\theta^{1/\alpha}(\varsigma)\mu_0(\varsigma)}\right)^{\alpha+1}\right)\mathrm{d}\varsigma=\infty, \tag{42}$$

holds for some $\lambda_2 \in (0,1)$ *and any positive constants* M_1 *and* M_2, *where*

$$\vartheta(\mathrm{s}) := \widetilde{\varrho}(\mathrm{s})\widetilde{\mathfrak{q}}_1(\mathrm{s})\zeta(\mathrm{s})\left(\frac{\lambda_2}{(n-2)!}\mathfrak{h}^{n-2}(\mathrm{s})\right)^{\beta}-\frac{(\alpha-1)\widetilde{\varrho}(\mathrm{s})}{\theta^{1/\alpha}(\mathrm{s})\mu_0^{\alpha+1}(\mathrm{s})},$$

then $\Omega_2 = \varnothing$.

Proof. Suppose the contrary, where $\varkappa \in \Omega_2$. In other words, there exists a $\mathrm{s}_1 \geq \mathrm{s}_0$ such that $\varkappa(\mathfrak{z}(\mathrm{s})) > 0$ and $\varkappa(\mathfrak{h}(\mathrm{s}))$ for all $\mathrm{s} \geq \mathrm{s}_1$. Since $\Psi^{(n-1)} < 0$, then (5) becomes

$$\left(\theta(\mathrm{s})\left(-\Psi^{(n-1)}(\mathrm{s})\right)^{\alpha}\right)' = \varphi(\mathrm{s})\mathfrak{q}(\mathrm{s})\varkappa^{\beta}(\mathfrak{h}(\mathrm{s})) \geq 0.$$

From $(Q_{5,1})$, we deduce that

$$\begin{aligned}\left(\theta(\mathrm{s})\left(-\Psi^{(n-1)}(\mathrm{s})\right)^{\alpha}\right)' &= \varphi(\mathrm{s})\mathfrak{q}(\mathrm{s})\varkappa^{\beta}(\mathfrak{h}(\mathrm{s}))\\ &\geq \varphi(\mathrm{s})\mathfrak{q}(\mathrm{s})\mathrm{g}_2^{\beta}(\mathfrak{h}(\mathrm{s});m)\Psi^{\beta}(\mathfrak{h}(\mathrm{s}))\\ &= \widetilde{\mathfrak{q}}_1(\mathrm{s})\Psi^{\beta}(\mathfrak{h}(\mathrm{s})),\end{aligned}$$

which means that

$$\left(\theta(\mathrm{s})\left(-\Psi^{(n-1)}(\mathrm{s})\right)^{\alpha}\right)' \geq \widetilde{\mathfrak{q}}_1(\mathrm{s})\Psi^{\beta}(\mathfrak{h}(\mathrm{s})). \tag{43}$$

Since $\theta(\mathrm{s})\left(-\Psi^{(n-1)}(\mathrm{s})\right)^{\alpha}$ is increasing, this means that $\theta^{1/\alpha}(\mathrm{s})\Psi^{(n-1)}(\mathrm{s})$ is decreasing. Therefore,

$$\begin{aligned}\Psi^{(n-2)}(l)-\Psi^{(n-2)}(\mathrm{s}) &= \int_{\mathrm{s}}^{l}\frac{1}{\theta^{1/\alpha}(\varsigma)}\theta^{1/\alpha}(\varsigma)\Psi^{(n-1)}(\varsigma)\mathrm{d}\varsigma\\ &\leq \theta^{1/\alpha}(\mathrm{s})\Psi^{(n-1)}(\mathrm{s})\int_{\mathrm{s}}^{l}\frac{1}{\theta^{1/\alpha}(\varsigma)}\mathrm{d}\varsigma.\end{aligned}$$

Putting $l \to \infty$, we have,

$$-\Psi^{(n-2)}(\mathrm{s}) \leq \theta^{1/\alpha}(\mathrm{s})\Psi(\mathrm{s})\mu_0(\mathrm{s}).$$

That is,

$$\left(\Psi^{(n-2)}(\mathrm{s})\right)^{\alpha} \geq \theta(\mathrm{s})\left(-\Psi^{(n-1)}(\mathrm{s})\right)^{\alpha}\mu_0^{\alpha}(\mathrm{s}). \tag{44}$$

Let us define $E(\mathrm{s})$ as

$$E(\mathrm{s}) := \widetilde{\varrho}(\mathrm{s})\left(-\frac{\theta(\mathrm{s})\left(-\Psi^{(n-1)}(\mathrm{s})\right)^{\alpha}}{\left(\Psi^{(n-2)}(\mathrm{s})\right)^{\alpha}}+\frac{1}{\mu_0^{\alpha}(\mathrm{s})}\right). \tag{45}$$

From (44), we have $E(\mathrm{s}) > 0$, for $\mathrm{s} \geq \mathrm{s}_1$. Therefore, we have

$$\begin{aligned} E'(\mathrm{s}) \leq & \frac{\widetilde{\varrho}'(\mathrm{s})}{\widetilde{\varrho}(\mathrm{s})}E(\mathrm{s}) - \widetilde{\varrho}(\mathrm{s})\mathfrak{q}_2(\mathrm{s})\frac{\Psi^{\beta}(\mathfrak{h}(\mathrm{s}))}{\left(\Psi^{(n-2)}(\mathrm{s})\right)^{\alpha}} \\ & -\alpha\frac{\widetilde{\varrho}(\mathrm{s})\theta(\mathrm{s})\left(-\Psi^{(n-1)}(\mathrm{s})\right)^{\alpha+1}}{\left(\Psi^{(n-2)}(\mathrm{s})\right)^{\alpha+1}} + \frac{\alpha\widetilde{\varrho}(\mathrm{s})}{\mu_0^{\alpha+1}(\mathrm{s})\theta^{1/\alpha}(\mathrm{s})}. \end{aligned}$$

Using (45), we deduce that

$$\begin{aligned} E'(\mathrm{s}) \leq & \frac{\widetilde{\varrho}'(\mathrm{s})}{\widetilde{\varrho}(\mathrm{s})}E(\mathrm{s}) - \widetilde{\varrho}(\mathrm{s})\widetilde{\mathfrak{q}}_1(\mathrm{s})\frac{\Psi^{\beta}(\mathfrak{h}(\mathrm{s}))}{\left(\Psi^{(n-2)}(\mathrm{s})\right)^{\alpha}} \\ & -\alpha\frac{\widetilde{\varrho}(\mathrm{s})}{\theta^{1/\alpha}(\mathrm{s})}\left(\frac{E(\mathrm{s})}{\widetilde{\varrho}(\mathrm{s})} - \frac{1}{\mu_0^{\alpha}(\mathrm{s})}\right)^{(\alpha+1)/\alpha} + \frac{\alpha\widetilde{\varrho}(\mathrm{s})}{\mu_0^{\alpha+1}(\mathrm{s})\theta^{1/\alpha}(\mathrm{s})} \\ = & \frac{\widetilde{\varrho}'(\mathrm{s})}{\widetilde{\varrho}(\mathrm{s})}E(\mathrm{s}) - \widetilde{\varrho}(\mathrm{s})\widetilde{\mathfrak{q}}_1(\mathrm{s})\frac{\Psi^{\beta}(\mathfrak{h}(\mathrm{s}))}{\left(\Psi^{(n-2)}(\mathfrak{h}(\mathrm{s}))\right)^{\beta}}\frac{\left(\Psi^{(n-2)}(\mathfrak{h}(\mathrm{s}))\right)^{\alpha}}{\left(\Psi^{(n-2)}(\mathrm{s})\right)^{\alpha}}\left(\Psi^{(n-2)}(\mathfrak{h}(\mathrm{s}))\right)^{\beta-\alpha} \\ & -\alpha\frac{\widetilde{\varrho}(\mathrm{s})}{\theta^{1/\alpha}(\mathrm{s})}\left(\frac{E(\mathrm{s})}{\widetilde{\varrho}(\mathrm{s})} - \frac{1}{\mu_0^{\alpha}(\mathrm{s})}\right)^{(\alpha+1)/\alpha} + \frac{\alpha\widetilde{\varrho}(\mathrm{s})}{\mu_0^{\alpha+1}(\mathrm{s})\theta^{1/\alpha}(\mathrm{s})}. \end{aligned} \tag{46}$$

Using Lemma 1, we obtain

$$\Psi(\mathrm{s}) \geq \frac{\lambda_2}{(n-2)!}\mathrm{s}^{n-2}\Psi^{(n-2)}(\mathrm{s}). \tag{47}$$

Since $\Psi^{(n-1)} < 0$, then

$$\frac{\Psi^{(n-2)}(\mathfrak{h}(\mathrm{s}))}{\Psi^{(n-2)}(\mathrm{s})} \geq 1. \tag{48}$$

By using (47) and (48) in (46), it becomes clear that

$$\begin{aligned} E'(\mathrm{s}) \leq & \frac{\widetilde{\varrho}'(\mathrm{s})}{\widetilde{\varrho}(\mathrm{s})}E(\mathrm{s}) - \widetilde{\varrho}(\mathrm{s})\widetilde{\mathfrak{q}}_1(\mathrm{s})\left(\frac{\lambda_2}{(n-2)!}\mathfrak{h}^{n-2}(\mathrm{s})\right)^{\beta}\left(\Psi^{(n-2)}(\mathfrak{h}(\mathrm{s}))\right)^{\beta-\alpha} \\ & -\frac{\alpha\widetilde{\varrho}(\mathrm{s})}{\theta^{1/\alpha}(\mathrm{s})}\left(\frac{E(\mathrm{s})}{\widetilde{\varrho}(\mathrm{s})} - \frac{1}{\mu_0^{\alpha}(\mathrm{s})}\right)^{(\alpha+1)/\alpha} + \frac{\alpha\widetilde{\varrho}(\mathrm{s})}{\mu_0^{\alpha+1}(\mathrm{s})\theta^{1/\alpha}(\mathrm{s})}. \end{aligned} \tag{49}$$

If we consider the scenario where $\alpha < \beta$, applying the increasing nature of $\theta\left(-\Psi^{(n-1)}\right)^{\alpha}$ for $\mathrm{s} \geq \mathrm{s}_1$, we obtain

$$\theta(\mathrm{s})\left(-\Psi^{(n-1)}(\mathrm{s})\right)^{\alpha} \geq \theta(\mathrm{s}_1)\left(-\Psi^{(n-1)}(\mathrm{s}_1)\right)^{\alpha} = M_1.$$

That is,

$$\theta^{1/\alpha}(\mathrm{s})\Psi^{(n-1)}(\mathrm{s}) \leq \theta^{1/\alpha}(\mathrm{s}_1)\Psi^{(n-1)}(\mathrm{s}_1) = -M_1^{1/\alpha} < 0,$$

then,

$$\theta^{1/\alpha}(\mathrm{s})\Psi^{(n-1)}(\mathrm{s}) \leq -M_1^{1/\alpha}.$$

If we divide this inequality by $\theta^{1/\alpha}$ and integrating the resulting inequality from s to l, we obtain

$$\Psi^{(n-2)}(l) \leq \Psi^{(n-2)}(\mathrm{s}) - M_1^{1/\alpha}\int_{\mathrm{s}}^{l}\frac{1}{\theta^{1/\alpha}(\mathrm{s})}\mathrm{d}\varsigma.$$

Letting $l \to \infty$ and using (4), we obtain

$$0 \le \Psi^{(n-2)}(\mathrm{s}) - M_1^{1/\alpha}\mu_0(\mathrm{s}),$$

which yields

$$\Psi^{(n-2)}(\mathrm{s}) \ge M_1^{1/\alpha}\mu_0(\mathrm{s}).$$

Thus, we conclude that

$$\left(\Psi^{(n-2)}(\mathrm{s})\right)^{\beta-\alpha} \ge M_1^{(\beta-\alpha)/\alpha}\mu_0^{\beta-\alpha}(\mathrm{s}). \tag{50}$$

By using (50) in (49), we obtain

$$\begin{aligned} E'(\mathrm{s}) \quad &\le \quad \frac{\widetilde{\varrho}'(\mathrm{s})}{\widetilde{\varrho}(\mathrm{s})}E(\mathrm{s}) - \widetilde{\varrho}(\mathrm{s})\widetilde{\mathfrak{q}}_1(\mathrm{s})\left(\frac{\lambda_2}{(n-2)!}\mathfrak{h}^{n-2}(\mathrm{s})\right)^{\beta} M_1^{(\beta-\alpha)/\alpha}\mu_0^{\beta-\alpha}(\mathrm{s}) \\ &\quad -\frac{\alpha\widetilde{\varrho}(\mathrm{s})}{\theta^{1/\alpha}(\mathrm{s})}\left(\frac{E(\mathrm{s})}{\widetilde{\varrho}(\mathrm{s})} - \frac{1}{\mu_0^{\alpha}(\mathrm{s})}\right)^{(\alpha+1)/\alpha} + \frac{\alpha\widetilde{\varrho}(\mathrm{s})}{\mu_0^{\alpha+1}(\mathrm{s})\theta^{1/\alpha}(\mathrm{s})}. \end{aligned} \tag{51}$$

In the scenario where $\alpha = \beta$, it is clear that $\left(\Psi^{(n-2)}(\mathrm{s})\right)^{\beta-\alpha} = 1$; thus, (51) still holds. In the case where $\alpha > \beta$, given that $\Psi^{(n-2)}(\mathrm{s})$ is non-increasing and positive, we can find a constant $M_2 > 0$ such that $\Psi^{(n-2)}(\mathrm{s}) \le M_2$. This leads to the conclusion that

$$\left(\Psi^{(n-2)}(\mathrm{s})\right)^{\beta-\alpha} \ge M_2^{\beta-\alpha}. \tag{52}$$

By using (52) in (49), we have

$$\begin{aligned} E'(\mathrm{s}) \quad &\le \quad \frac{\widetilde{\varrho}'(\mathrm{s})}{\widetilde{\varrho}(\mathrm{s})}E(\mathrm{s}) - \widetilde{\varrho}(\mathrm{s})\mathfrak{q}_2(\mathrm{s})\left(\frac{\lambda_2}{(n-2)!}\mathfrak{h}^{n-2}(\mathrm{s})\right)^{\beta} M_2^{\beta-\alpha} \\ &\quad -\frac{\alpha\widetilde{\varrho}(\mathrm{s})}{\theta^{1/\alpha}(\mathrm{s})}\left(\frac{E(\mathrm{s})}{\widetilde{\varrho}(\mathrm{s})} - \frac{1}{\mu_0^{\alpha}(\mathrm{s})}\right)^{(\alpha+1)/\alpha} + \frac{\alpha\widetilde{\varrho}(\mathrm{s})}{\mu_0^{\alpha+1}(\mathrm{s})\theta^{1/\alpha}(\mathrm{s})}. \end{aligned} \tag{53}$$

which, together with (51), implies that

$$\begin{aligned} E'(\mathrm{s}) \quad &\le \quad \frac{\widetilde{\varrho}'(\mathrm{s})}{\widetilde{\varrho}(\mathrm{s})}E(\mathrm{s}) - \widetilde{\varrho}(\mathrm{s})\widetilde{\mathfrak{q}}_1(\mathrm{s})\left(\frac{\lambda_2}{(n-2)!}\mathfrak{h}^{n-2}(\mathrm{s})\right)^{\beta} \xi(\mathrm{s}) \\ &\quad -\frac{\alpha\widetilde{\varrho}(\mathrm{s})}{\theta^{1/\alpha}(\mathrm{s})}\left(\frac{E(\mathrm{s})}{\widetilde{\varrho}(\mathrm{s})} - \frac{1}{\mu_0^{\alpha}(\mathrm{s})}\right)^{(\alpha+1)/\alpha} + \frac{\alpha\widetilde{\varrho}(\mathrm{s})}{\mu_0^{\alpha+1}(\mathrm{s})\theta^{1/\alpha}(\mathrm{s})}. \end{aligned} \tag{54}$$

where $\xi(\mathrm{s}) = \begin{cases} 1 & \text{if } \alpha = \beta, \\ M_1^{\beta-\alpha}\mu_0^{\beta-\alpha} & \text{if } \alpha < \beta, \\ M_2^{\beta-\alpha} & \text{if } \alpha > \beta. \end{cases}$

By using the inequality

$$A^{(\alpha+1)/\alpha} - (A-B)^{(\alpha+1)/\alpha} \le \frac{B^{1/\alpha}}{\alpha}[(1+\alpha)A - B], \ AB > 0,$$

with $A = E(\mathrm{s})/\widetilde{\varrho}(\mathrm{s})$ and $B = 1/\mu_0^{\alpha}(\mathrm{s})$, we obtain

$$\begin{aligned} E'(\mathrm{s}) \quad &\le \quad \frac{\widetilde{\varrho}'(\mathrm{s})}{\widetilde{\varrho}(\mathrm{s})}E(\mathrm{s}) - \widetilde{\varrho}(\mathrm{s})\widetilde{\mathfrak{q}}_1(\mathrm{s})\left(\frac{\lambda_2}{(n-2)!}\mathfrak{h}^{n-2}(\mathrm{s})\right)^{\beta} \zeta(\mathrm{s}) + \frac{\alpha\widetilde{\varrho}(\mathrm{s})}{\mu_0^{\alpha+1}(\mathrm{s})\theta^{1/\alpha}(\mathrm{s})} \\ &\quad -\frac{\alpha\widetilde{\varrho}(\mathrm{s})}{\theta^{1/\alpha}(\mathrm{s})}\left(\left(\frac{E(\mathrm{s})}{\widetilde{\varrho}(\mathrm{s})}\right)^{(\alpha+1)/\alpha} - \frac{1}{\alpha\mu_0(\mathrm{s})}\left[(1+\alpha)\frac{E(\mathrm{s})}{\widetilde{\varrho}(\mathrm{s})} - \frac{1}{\mu_0^{\alpha}(\mathrm{s})}\right]\right), \end{aligned}$$

which is

$$E'(s) \leq \left(\frac{\widetilde{\varrho}'(s)}{\widetilde{\varrho}(s)} + \frac{(1+\alpha)}{\theta^{1/\alpha}(s)\mu_0(s)}\right)E(s) - \widetilde{\varrho}(s)\widetilde{\mathfrak{q}}_1(s)\left(\frac{\lambda_2}{(n-2)!}\mathfrak{h}^{n-2}(s)\right)^{\beta}\zeta(s) - \frac{\alpha E^{(\alpha+1)/\alpha}(s)}{\theta^{1/\alpha}(s)\widetilde{\varrho}^{1/\alpha}(s)} - \frac{\widetilde{\varrho}(s)}{\theta^{1/\alpha}(s)\mu_0^{\alpha+1}(s)} + \frac{\alpha\widetilde{\varrho}(s)}{\theta^{1/\alpha}(s)\mu_0^{\alpha+1}(s)}. \quad (55)$$

Using Lemma 3 where $B = \varrho'(s)/\varrho(s) + (1+\alpha)/\theta^{1/\alpha}(s)\mu_0(s)$, $A = \alpha/\theta^{1/\alpha}(s)\widetilde{\varrho}^{1/\alpha}(s)$ and $u = \varphi(s)$, we obtain

$$E'(s) \leq -\widetilde{\varrho}(s)\widetilde{\mathfrak{q}}_1(s)\xi(s)\left(\frac{\lambda_2}{(n-2)!}\mathfrak{h}^{n-2}(s)\right)^{\beta} + \frac{(\alpha-1)\widetilde{\varrho}(s)}{\theta^{1/\alpha}(s)\mu_0^{\alpha+1}(s)} + \frac{\theta(s)\widetilde{\varrho}(s)}{(\alpha+1)^{\alpha+1}}\left(\frac{\widetilde{\varrho}'(s)}{\widetilde{\varrho}(s)} + \frac{(1+\alpha)}{\theta^{1/\alpha}(s)\mu_0(s)}\right)^{\alpha+1}. \quad (56)$$

Integrating (56) from s_1 to s, we have

$$\int_{s_1}^{s}\left(\vartheta(\varsigma) - \frac{\theta(\varsigma)\widetilde{\varrho}(\varsigma)}{(\alpha+1)^{\alpha+1}}\left(\frac{\widetilde{\varrho}'(\varsigma)}{\widetilde{\varrho}(\varsigma)} + \frac{(1+\alpha)}{\theta^{1/\alpha}(\varsigma)\mu_0(\varsigma)}\right)^{\alpha+1}\right)d\varsigma \leq E(s_1),$$

which contradicts (42). The proof is now complete. □

Lemma 17. *Assume that $\alpha \geq 1$. If*

$$\limsup_{s\to\infty}\int_{s_0}^{s}\left(\widetilde{\mathfrak{q}}_1(\varsigma)\xi(\varsigma)\left(\frac{\lambda_2}{(n-2)!}\mathfrak{h}^{n-2}(\varsigma)\right)^{\beta} - \frac{\alpha}{\theta^{1/\alpha}(\varsigma)\mu_0^{\alpha+1}(\varsigma)}\right)d\varsigma = \infty, \quad (57)$$

holds for some $\lambda_2 \in (0,1)$ and any positive constants M_1 and M_2, then $\Omega_2 = \varnothing$.

Proof. Condition (57) follows by substituting $\widetilde{\varrho}(s) = 1$ into (42). □

4. Theorems on the Oscillatory Behavior of Solutions

In this section, we build upon the results from the previous discussion to establish new criteria for analyzing the oscillatory behavior of all solutions of the Equation (1). By integrating the earlier derived conditions that exclude positive solutions in cases (C_1) (C_2) and (C_3), we derive two theorems that provide essential tools for identifying the oscillatory nature of the given equation.

Theorem 1. *Let $\alpha \geq 1$. Suppose that conditions (23), (30), and (42) are satisfied. Then, the Equation (1) exhibits oscillatory behavior.*

Proof. Let $\varkappa$ be a solution of Equation (1) that becomes positive for sufficiently large s. By Lemma 4, the behavior of Ψ and its derivatives falls into one of three possible scenarios. Utilizing Lemmas 9, 12, and 16, we conclude that, under the constraints given by (23), (30), and (42), no positive solutions of Equation (1) can exist that also satisfy conditions (C_1), (C_2) and (C_3). Thus, the proof is established. □

Theorem 2. *Let $\alpha \geq 1$. Suppose that conditions (26), (39), and (57) are satisfied. Then, the Equation (1) exhibits oscillatory behavior.*

Proof. Let $\varkappa$ be a solution of Equation (1) that becomes positive for sufficiently large s. By Lemma 4, the behavior of Ψ and its derivatives falls into one of three possible scenarios.

Utilizing Lemmas 10, 13, and 17, we conclude that, under the constraints given by (26), (39), and (57), no positive solutions of Equation (1) can exist that also satisfy conditions (C_1), (C_2) and (C_3). Thus, the proof is established. □

Example 1. *Consider the non-linear differential equation (NDE):*

$$\left(s^{\alpha}\left|\Psi'''(s)\right|^{\alpha-1}\Psi'''(s)\right)' + \frac{3\alpha}{s^{1-\alpha}}\left|\Psi'''(s)\right|^{\alpha-1}\Psi'''(s) + \frac{q_0}{s^{2\alpha+1}}|\varkappa(\mathfrak{h}_0 s)|^{\alpha-1}\varkappa(\mathfrak{h}_0 s) = 0, \quad (58)$$

where $\Psi(s) = \varkappa(s) + g_0\varkappa(\mathfrak{z}_0 s)$, $s \geq 1$, $\alpha \geq 1$, $0 \leq g_0 < 1$, $\mathfrak{z}_0$, $\mathfrak{h}_0 \in (0,1)$ *and* $q_0 > 0$. *By analyzing this equation in relation to (1), we can deduce that* $n = 4$, $\beta = \alpha = 1$, $\kappa(s) = s^{\alpha}$, $q(s) = q_0/s^{2\alpha+1}$, $g(s) = g_0$, $\mathfrak{h}(s) = \mathfrak{h}_0 s$ *and* $\mathfrak{z}(s) = \mathfrak{z}_0 s$. *It is easy to find that*

$$\varphi(s) = s^{3\alpha},\ \theta(s) = s^{4\alpha},$$

$$\mu_0(s) = \frac{1}{3s^3},\ \mu_1(s) = \frac{1}{6s^2},\ \mu_2(s) = \frac{1}{6s},$$

$$g_1(s;m) = \left(1 - \frac{g_0}{\mathfrak{z}_0}\right)\sum_{i=0}^{m} g_0^{2i},\ g_2(s;m) = (1-g_0)\sum_{i=0}^{m} g_0^{2i}\mathfrak{z}_0^{4i/\epsilon_0},$$

and

$$\tilde{q}(s) = q_0 s^{\alpha-1}g_1^{\alpha},\ \hat{q}(s) = q_0 s^{\alpha-1}(1-g_0)^{\alpha},\ \tilde{q}_1(s) = q_0 s^{\alpha-1}g_2^{\alpha}.$$

Condition (23) holds when

$$q_0 > 3L\left(\frac{6}{g_1}\right)^{\alpha}. \quad (59)$$

Condition (30) with $\varrho(s) = s^{\alpha}$ *is satisfied when*

$$q_0 > \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}\left(\frac{2\mathfrak{h}_0}{K\lambda_1(1-g_0)}\right)^{\alpha}. \quad (60)$$

Condition (42) with $\tilde{\varrho}(s) = s^{\alpha}$ *holds when*

$$q_0 > \left(\frac{2}{\lambda_2 c_2 \mathfrak{h}_0^2}\right)^{\alpha}\left[\left(\frac{\alpha+3(1+\alpha)}{\alpha+1}\right)^{\alpha+1} + (\alpha-1)3^{\alpha+1}\right]. \quad (61)$$

Applying Theorem 1, it follows that Equation (58) exhibits oscillatory behavior provided that the conditions (59), (60), and (61) are satisfied. This result can be confirmed by substituting appropriate values into Equation (58).

Example 2. *Consider the non-linear differential equation (NDE):*

$$\left(s(\varkappa(s) + 0.5\varkappa(0.8s))'''\right)' + 3(\varkappa(s) + 0.5\varkappa(0.8s))''' + \frac{255}{s^3}\varkappa(0.7s) = 0, \quad (62)$$

Clearly,

$$\alpha = \beta = 1,\ \kappa(s) = s,\ q(s) = 255/s^3,\ c(s) = 1/2,\ \mathfrak{h}(s) = 0.7s,\ \text{and}\ \mathfrak{z}(s) = 0.8s.$$

Consequently, we can easily deduce:

$$\varphi(s) = s^3,\ \theta(s) = s^4,$$

$$\mu_0(s) = \frac{1}{3s^3},\ \mu_1(s) = \frac{1}{6s^2},\ \mu_2(s) = \frac{1}{6s},$$

$$g_1(s;10) = \left(1 - \frac{0.5}{0.9}\right)\sum_{i=0}^{10}(0.5)^{2i} \cong 0.59259,$$

$$g_2(s;10) = \sum_{i=0}^{10}(0.5)^{2i+1}(0.9)^{4i/\epsilon_0} \cong 0.603\,20, \textit{ where } \epsilon_0 = 0.9,$$

and

$$\widetilde{q}(s) = 0.59259q_0,\ \widehat{q}(s) = 0.5q_0,\ \widetilde{q}_1(s) = 0.603\,20q_0.$$

Condition (26) leads to

$$\liminf_{s\to\infty} \frac{1}{3^2 s^6} s^4 (0.59259q_0) \frac{3s^3}{6s} = 0.03292q_0 > \frac{1}{4},$$

which is satisfied when

$$q_0 > 7.5942$$

Condition (39) leads to

$$\limsup_{s\to\infty} \int_{s_0}^{s} \widehat{q}(s)d\varsigma = \limsup_{s\to\infty} \int_{s_0}^{s} 0.5q_0 d\varsigma = \infty.$$

Condition (57) with $\lambda_2 = 0.5$ *leads to*

$$\limsup_{s\to\infty} \int_{s_1}^{s} \left(0.603\,20q_0 \left(\frac{0.5}{2}(0.7)^2\varsigma^2\right) - \frac{9\varsigma^6}{\varsigma^4}\right)d\varsigma = \limsup_{s\to\infty} \int_{s_1}^{s} (0.07389q_0 - 9)\varsigma^2 d\varsigma = \infty,$$

which is satisfied when

$$q_0 > 121.8.$$

Thus, when $q_0 > 121.8$, *conditions (26), (39), and (57) are met. Applying Theorem 2, it follows that (62) exhibits oscillatory behavior.*

5. Conclusions

In this paper, we study the asymptotic and oscillatory behavior of a class of even-order nonlinear neutral differential equations, incorporating damping terms to deepen the understanding of solution behavior. The relationship between the solutions and their associated functions is refined, providing new insights into the monotonic properties of these solutions, and establishing precise conditions and criteria that guarantee the presence of oscillation. The study introduces improved criteria based on the Riccati technique and the comparison method, offering effective analytical tools for examining these dynamic systems. This research enriches the existing literature by broadening the scope of current criteria and presenting new approaches for analyzing NDEs. However, the study is confined to even-order equations and does not extend to odd-order cases. Consequently, future work could benefit from applying the methodology outlined in this paper to odd-order neutral equations, opening new avenues for a more comprehensive understanding of these systems. Additionally, it is interesting to note that we have derived oscillation criteria without the need for the condition $\alpha \geq 1$.

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