On The Weak Fuzzy Complex Differential Equations and Some Types of the 1st Order 1st degree WFC-ODEs

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Abstract

The objective of this paper is to introduce the concept of Weak Fuzzy Complex differential equations. We have defined the general solution of the n-th order Weak Fuzzy Complex ordinary differential equation. That we have used a special isomorphism transformation function to write the WFC-ODE as two Real ODEs and solved them with respect to their own variables. Then, by the inverse of the transformation function, we have got the general solution in F (J) as a structure of two general solutions in R. Therefore, we have shown some types of first-order first-degree separable, exact, and linear WFC-ODEs. Also, we have found their general solutions with examples to demonstrate them.

Keywords: Weak Fuzzy Complex (WFC) Numbers; Weak Fuzzy Complex Functions; Differential Equations (DE)

1. Introduction

Real number extensions have always been the focus of attention of researchers from all over the world. In the year 2023, Weak Fuzzy Complex numbers appeared for the first time in [9], where they defined them as a new generalization of classical real numbers by applying fuzzy operators [3][5] studied WFC vector spaces and matrices. The necessary and sufficient conditions for Weak Fuzzy Complex Pythagoras triples and Weak Fuzzy Complex Pythagoras quadruples have been handled in [2][7], as applications of Weak Fuzzy Complex numbers in Diophantine equations. Also, [1] studied WFC linear Diophantine equations in two WFC variables. In addition, a special transformation function that has an important role in working with variables from a Real number set instead of a Weak Fuzzy Complex set was defined in [10], where the foundation of functions in WFC variables was studied. Actually, one of the important results of using this type of numbers, modeling the solutions of some vectorial equations defined by norms in 3-dimensional Euclidean space A-Curves [4]. However, Python introduced Weak Fuzzy Complex numbers in [11].

On the other side, we know that DEs are important for modeling time dependent processes in many disciplines, e.g. in engineering, physics, chemistry, and economics. For instance, in classical mechanics, if the position, velocity, acceleration, and various forces acting on a body are given, Newton's second law of motion allows us to express these variables as a differential equation and, by solving it, to compute the position of the body as a function of time. According to the important role of differential equations in a lot of fields, we study them in WFC variables.

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In section 2 of this paper, we mention the main concepts. Then, section 3 shows the definition of the WFC differential equations and section 4 focuses on some types of first-order WFC-ODEs with some related examples.

2. Preliminaries

The foundational concepts that are important to get ODEs in WFC variables are presented in this section.

Definition 1 ([9]) The *set of Weak Fuzzy Complex numbers* was defined as follows, where 'J' is the Weak Fuzzy Complex operator ($J \notin R$):

$F_J = \{x_0 + x_1 J\ ;\ x_0 , x_1 \in R , J^2 = t \in]0,1[\ \}$

Properties of the Weak Fuzzy Complex numbers: ([9])

Let $X = x_0 + x_1 J, Y = y_0 + y_1 J \in F_J$, where $x_0 , x_1, y_0, y_1 \in R$

- Addition: $X + Y = (x_0 + y_0) + (x_1 + y_1)J$.
- Multiplication $X \times Y = (x_0 y_0 + x_1 y_1 t) + (x_0 y_1 + x_1 y_0)J$.
- The conjugate of X is: $\bar{X} = x_0 + x_1 J$

Definition 2 ([9]) Let φ be *the transformation function* from F_J to $R \times R$, which we define as follows:

$\varphi: F_J \mapsto R \times R.$

$\varphi(x_0 + x_1 J) = (x_0 + x_1 (-\sqrt{t}), x_0 + x_1(+\sqrt{t})) = (x_0 - x_1 \sqrt{t}, x_0 + x_1\sqrt{t})$;

where $J^2 = t \in]0,1[\Rightarrow J = \pm\sqrt{t}$ and $x_0, x_1, y_0, y_1 \in R$ (This map is an isomorphism).

Definition 3 Let $\varphi: F_J \mapsto R \times R$ such that: $\varphi(X) = (a, b)$, the inverse function of φ is defined as follows:

$\varphi^{-1}: R \times R \mapsto F_J$

$\varphi^{-1}(a, b) = \frac{1}{2}[a + b] + \frac{1}{2\sqrt{t}} J[b - a].$

Definition 4 On the ring of Weak Fuzzy Complex numbers F_J, we define *the Weak Fuzzy Complex function* in one variable as follows:

$f: F_J \mapsto F_J,$

$f = f(X),$

$X = x_0 + x_1 J\ ; x_0 , x_1 \in R$

Remark 1 Let $f: F_J \mapsto F_J$ be a Weak Fuzzy Complex function in one variable $X = x_0 + x_1 J;\ J^2 = t$. Hence, f can be written by using two classical functions in two variables as follows:

$g: R^2 \mapsto R$, s.t. $g = g(x_0 , x_1)$

$h: R^2 \mapsto R$, s.t. $h = h(x_0 , x_1)$

where $f(X) = g(x_0 , x_1) + h(x_0 , x_1) J.$

Definition 5 Let $\varphi: F_J \mapsto R \times R$, $f: F_J \mapsto F_J$ such that:

$\varphi(x_0 + x_1 J) = (x_0 - x_1 \sqrt{t}, x_0 + x_1\sqrt{t})\ ; J^2 = t \in]0,1[,$

$f = f(X), X = x_0 + x_1 J\ ; x_0 , x_1 \in R.$

the canonical formula as follows:

$f(X) = \varphi^{-1} \circ \varphi\big(f(X)\big) = g(x_0 , x_1) + J h(x_0 , x_1)\ .$

Remark 2 For $f: F_J \mapsto F_J$ s.t. $f = f(X)$, then

$\varphi(f(X)) = (f_1(x_0 - x_1 \sqrt{t}\,), f_2(\,x_0 + x_1\sqrt{t}\,));\ f_1, f_2: R \mapsto R.$

On the other hand, we have:

$$g(x_0 , x_1) = \frac{1}{2}[f_1(x_0 - x_1 \sqrt{t}\,) + f_2(\,x_0 + x_1\sqrt{t}\,)],$$

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$h(x_0\,,x_1) = \dfrac{1}{2\sqrt{t}}J[f_2(x_0 + x_1\sqrt{t}\,) - f_1(\,x_0 - x_1\sqrt{t}\,)].$

Definition 6 Let $X = x_0 + x_1 J, Y = y_0 + y_1 J \in \boldsymbol{F}_J$, we say that $X \leq Y$, if and only if:

$$\begin{cases} x_0 - x_1\sqrt{t} \leq y_0 - y_1\sqrt{t} \\ x_0 + x_1\sqrt{t} \leq y_0 + y_1\sqrt{t} \end{cases}$$

Definition 7 Let $A = a_0 + a_1 J, B = b_0 + b_1 J \in \boldsymbol{F}_J$, we define *the interval* $[A, B]$ if and only if $A \leq B$, according to the definition of the partial order relation ($\leq$).

- If $A \nleq B$, then $[A, B] = \phi$.
- We can understand $[A, B]$ as follows:

$[A, B] = \{C = c_0 + c_1 J \in \boldsymbol{F}_J;\ A \leq C \leq B\}.$

Definition 8 Let $X \in \boldsymbol{F}_J$, we say that $X = x_0 + x_1 J$ is not an *invertible WFC element* if one of its $\varphi(X)$ components at least is zero s.t.

$x_0 - x_1\sqrt{t} = 0$

or $x_0 + x_1\sqrt{t} = 0$

and that depends on the value of $\sqrt{t}$.

Definition 9 Let f be a Weak Fuzzy Complex function in one variable defined on $\boldsymbol{F}_J$, where

$\varphi(f(X)) = (f_1(x_0 - x_1\sqrt{t}\,), f_2(\,x_0 + x_1\sqrt{t}\,)) \in \boldsymbol{R} \times \boldsymbol{R};$

then we say that

1) f is not *invertible* if at least one of its $\varphi(f(X))$ components is zero s.t.

$f_1\left(x_0 - x_1\sqrt{t}\right) = 0$

or $f_2\left(x_0 + x_1\sqrt{t}\right) = 0$

s.t. f is *invertible* on $\boldsymbol{F}_J \backslash \{X; \varphi(f(X)) = (f_1, f_2) \in \boldsymbol{R} \times \boldsymbol{R} \backslash \{(0,0), (0, k), (k, 0); k \in \boldsymbol{R}\}$

$$= \boldsymbol{F}_J \backslash \{X; f(X) \in \boldsymbol{F}_J \backslash \{(\tfrac{1}{2} \mp \tfrac{1}{2\sqrt{t}}J)k, 0; k \in \boldsymbol{R}\}\}$$

$$= \boldsymbol{F}_J \backslash \{X; f(X) \notin \{(\frac{1}{2} \mp \frac{1}{2\sqrt{t}}J)k, 0; k \in \boldsymbol{R}\}\}$$

where $\varphi^{-1}(0, k) = \varphi^{-1}(0,1)k = (\frac{1}{2} + \frac{1}{2\sqrt{t}}J)k$ and $\varphi^{-1}(k, 0) = \varphi^{-1}(1,0)k = (\frac{1}{2} - \frac{1}{2\sqrt{t}}J)k$.

2) a *singular point* of the WFC function is a point where the function is not defined or not invertible.

Definition 10 Let $f: \boldsymbol{F}_J \mapsto \boldsymbol{F}_J$ be a Weak Fuzzy Complex function in one variable, where

$\varphi(f(X)) = (f_1(x_0 - x_1\sqrt{t}\,), f_2(\,x_0 + x_1\sqrt{t}\,)); f_1, f_2: \boldsymbol{R} \mapsto \boldsymbol{R}$, then we say:

1) f is *continuous* on $\boldsymbol{F}_J$ if and only if f_1, f_2 are continuous on $\boldsymbol{R}$.

2) f is *differentiable* on $\boldsymbol{F}_J$ if and only if f_1, f_2 are differentiable on $\boldsymbol{R}$, with respect to their own variables.

3) f is *integrable* on $\boldsymbol{F}_J$ if and only if f_1, f_2 are integrable on $\boldsymbol{R}$.

Definition 11 Let $f: \boldsymbol{F}_J \mapsto \boldsymbol{F}_J$ be a *differentiable/ integrable* function on $\boldsymbol{F}_J$. We define

1) $f'(X) = \varphi^{-1}\left(f'_1(x_0 - x_1\sqrt{t}\,), f'_2(\,x_0 + x_1\sqrt{t}\,)\right)$

2) $\int f(X).\,dX = \varphi^{-1}\left(\int f_1.\,d(x_0 - x_1\sqrt{t}), \int f_2.\,d(\,x_0 + x_1\sqrt{t}\,)\right).$

Definition 12 Let $f: \boldsymbol{F}_J \mapsto \boldsymbol{F}_J$ be a differentiable Weak Fuzzy Complex function on $\boldsymbol{F}_J$. We can *derive* f *n-time* as follows:

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$f^{(n)}(X) = \varphi^{-1}\left(f_1{}^{(n)}(x_0 - x_1\sqrt{t}), f_2{}^{(n)}(x_0 + x_1\sqrt{t})\right);\ n \in \boldsymbol{N}.$

3. The Differential Equations in F_J

The derivative represents a rate of change quantity concerning the change in another quantity. We know that the differential equation is an equation that contains at least one derivative of the dependent variable with respect to the other variables (i.e., the independent variable). If the independent variable is a single, then we call the related equation an “Ordinary Differential Equation” (ODE) which arises most commonly in the study of dynamical systems and electrical networks. ODE is much easier to be treated than “Partial Differential Equation” (PDE), whose unknown function depends on two or more independent variables. In this article, we will be interested in studying ODEs on (WFC-ODEs) [6, 8, 12, 13,14,15,16,17,18,19,20,21,22,23-36].

Definition 13 Let Y be a dependent variable, and $Y = f(X)$ represents an unknown weak fuzzy complex function where the independent variable $X = x_0 + x_1 J$ is a weak fuzzy complex variable. We define *the weak fuzzy complex differential equation of n-th order* (the highest derivative appearing in the differential equation) as the following:

$$\mathcal{F}(X, Y, Y', Y'', \dots, Y^{(n)}) = 0 \qquad \textbf{(1)}$$

where $Y^{(n)} = \frac{d^n Y}{dX^n}$ and $Y = y_0 + y_1 J = \varphi^{-1}(y_0 - y_1\sqrt{t}, y_0 + y_1\sqrt{t}) = \varphi^{-1}(f_1(x_0 - x_1\sqrt{t}), f_2(x_0 + x_1\sqrt{t}))$.

Remark 3 We can consider the differential operator is as follows:

$\frac{d^n}{dX^n} = \varphi^{-1}\left(\frac{d^n}{d(x_0 - x_1\sqrt{t})^n}, \frac{d^n}{d(x_0 + x_1\sqrt{t})^n}\right)$ where $X = \varphi^{-1}(x_0 - x_1\sqrt{t}, x_0 + x_1\sqrt{t})$.

Remark 4 The WFC-ODE with the WFC independent variable $X = x_0 + x_1 J$ is equivalent to two ODEs, the first one is with the independent real variable $x_0 - x_1\sqrt{t}$ and the second ODE is with the independent real variable $x_0 + x_1\sqrt{t}$.

Proof.

$$\mathcal{F}(X, Y, Y', Y'', \dots, Y^{(n)}) = 0$$

$$\mathcal{F}(x_0 + x_1 J, y_0 + y_1 J, y'_0 + y'_1 J, y''_0 + y''_1 J, \dots, y_0{}^{(n)} + y_1{}^{(n)} J) = 0$$

By using the function φ, we get

$$\varphi^{-1}(\mathcal{F}_1(x_0 - x_1\sqrt{t}, y_0 - y_1\sqrt{t}, y'_0 - y'_1\sqrt{t}, y''_0 - y''_1\sqrt{t}, \dots, y_0{}^{(n)} - y_1{}^{(n)}\sqrt{t}),\ \mathcal{F}_2(x_0 + x_1\sqrt{t}, y_0 + y_1\sqrt{t}, y'_0 + y'_1\sqrt{t}, y''_0 + y''_1\sqrt{t}, \dots, y_0{}^{(n)} + y_1{}^{(n)}\sqrt{t})) = 0$$

$$\Rightarrow \begin{cases} \mathcal{F}_1(x_0 - x_1\sqrt{t}, y_0 - y_1\sqrt{t}, y'_0 - y'_1\sqrt{t}, y''_1 - y''_2\sqrt{t}, \dots, y_1{}^{(n)} - y_2{}^{(n)}\sqrt{t}) = 0 & (2-1) \\ \mathcal{F}_2(x_0 + x_1\sqrt{t}, y_0 + y_1\sqrt{t}, y'_0 + y'_1\sqrt{t}, y''_1 + y''_2\sqrt{t}, \dots, y_1{}^{(n)} + y_2{}^{(n)}\sqrt{t}) = 0 & (2-2) \end{cases} \qquad \textbf{(2)}$$

where $\mathcal{F} = \varphi^{-1}(\mathcal{F}_1, \mathcal{F}_2) = \frac{1}{2}(\mathcal{F}_1 + \mathcal{F}_2) + \frac{1}{2\sqrt{t}} J(\mathcal{F}_2 - \mathcal{F}_1)$.

Definition 14 The functions $y_0 - y_1\sqrt{t} = f_1(x_0 - x_1\sqrt{t})$ and $y_0 + y_1\sqrt{t} = f_2(x_0 + x_1\sqrt{t})$ are called *general solutions* to (2-1) and (2-2), respectively, on intervals where $f_1\, and\, f_2$ are defined (domain of solutions) on $\boldsymbol{I}_1 \subseteq \boldsymbol{R}$ and $\boldsymbol{I}_2 \subseteq \boldsymbol{R}$ if each of them is an n-times differentiable in the related interval with respect to their own variables $x_0 - x_1\sqrt{t}$, $x_0 + x_1\sqrt{t}$, respectively.

Remark 5. Each general solution of (2-1) and (2-2) contains in general n real arbitrary constants of integrations respectively. Definition 15. A function

$$Y = f(X) = \varphi^{-1}(f_1(x_0 - x_1\sqrt{t}), f_2(x_0 + x_1\sqrt{t})) = \frac{1}{2}(f_1 + f_2) + \frac{1}{2\sqrt{t}} J(f_2 - f_1)$$

is called *the general solution* of (1) on $I = \varphi^{-1}(I_1 \times I_2) \subseteq F_J$ if $f_1$and f_2 are the general solutions for (2-1) and (2-2), respectively. Such that (1) is satisfied identically when $f(X)$ and its derivatives are substituted into $\mathcal{F}$:

$$\mathcal{F}(X, f(X), f'(X), f''(X), \dots, f^{(n)}(X)) = 0 \qquad \textbf{(3)}$$

where

$$\mathcal{F}(X, Y, Y', Y'', \dots, Y^{(n)}) = \mathcal{F}(X, f(X), f'(X), f''(X), \dots, f^{(n)}(X))$$
$$= \varphi^{-1}\varphi(\mathcal{F}(X, f(X), f'(X), f''(X), \dots, f^{(n)}(X)))$$

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$= \varphi^{-1}\mathcal{F}(\varphi(X, f(X), f'(X), f''(X), \dots, f^{(n)}(X)))$

$= \varphi^{-1}\mathcal{F}((x_0 - x_1\sqrt{t}, x_0 + x_1\sqrt{t}),$

$(f_1(x_0 - x_1\sqrt{t}), f_2(x_0 + x_1\sqrt{t})),$

$(f'_1(x_0 - x_1\sqrt{t}), f'_2(x_0 + x_1\sqrt{t})),$

$(f''_1(x_0 - x_1\sqrt{t}), f''_2(x_0 + x_1\sqrt{t})), \dots,$

$(f_1^{(n)}(x_0 - x_1\sqrt{t}), f_2^{(n)}(x_0 + x_1\sqrt{t})))$

$= \varphi^{-1}(\mathcal{F}_1[(x_0 - x_1\sqrt{t}), f_1(x_0 - x_1\sqrt{t}), f'_1(x_0 - x_1\sqrt{t}), f''_1(x_0 - x_1\sqrt{t}), \dots, f_1^{(n)}(x_0 - x_1\sqrt{t})],$

$\mathcal{F}_2[(x_0 + x_1\sqrt{t}), f_2(x_0 + x_1\sqrt{t}), f'_2(x_0 + x_1\sqrt{t}), f''_2(x_0 + x_1\sqrt{t}), \dots, f_2^{(n)}(x_0 + x_1\sqrt{t})]).$

So that, (3) is equivalent to:

$$\begin{cases} \mathcal{F}_1[(x_0 - x_1\sqrt{t}), f_1(x_0 - x_1\sqrt{t}), f'_1(x_0 - x_1\sqrt{t}), f''_1(x_0 - x_1\sqrt{t}), \dots, f_1^{(n)}(x_0 - x_1\sqrt{t})] = 0 \ (4-1) \\ \mathcal{F}_2[(x_0 + x_1\sqrt{t}), f_2(x_0 + x_1\sqrt{t}), f'_2(x_0 + x_1\sqrt{t}), f''_2(x_0 + x_1\sqrt{t}), \dots, f_2^{(n)}(x_0 + x_1\sqrt{t})] = 0 \ (4-2) \end{cases} \quad \textbf{(4)}$$

where: $\mathcal{F} = \varphi^{-1}(\mathcal{F}_1, \mathcal{F}_2)$, $x_0 - x_1\sqrt{t} \in \boldsymbol{I}_1, x_0 + x_1\sqrt{t} \in \boldsymbol{I}_2$

- In other words, $Y = f(X) = \varphi^{-1}(f_1(x_0 - x_1\sqrt{t}), f_2(x_0 + x_1\sqrt{t}))$ is *the general solution* of (1) on $\boldsymbol{I} \subseteq \boldsymbol{F}_J$ if Y is n-times differentiable WFC function on $\boldsymbol{I}$ and satisfies the equation on I.

4. First-order and First-degree ODEs in F_J

Definition 16. *The weak fuzzy complex differential equation of the first order* is written as follows:

$$\mathcal{F}(X, Y, Y') = 0 \quad \textbf{(5)}$$

$$\Leftrightarrow \begin{cases} \mathcal{F}_1(x_0 - x_1\sqrt{t}, y_0 - y_1\sqrt{t}, y'_0 - y'_1\sqrt{t}) = 0 & (6-1) \\ \mathcal{F}_2(x_0 + x_1\sqrt{t}, y_0 + y_1\sqrt{t}, y'_0 + y'_1\sqrt{t}) = 0 & (6-2) \end{cases} \quad \textbf{(6)}$$

where $y_0 - y_1\sqrt{t} = f_1(x_0 - x_1\sqrt{t})$, $y_0 + y_1\sqrt{t} = f_2(x_0 + x_1\sqrt{t})$

$Y' = \frac{dY}{dX} = f'(X) = \varphi^{-1}(f'_1(x_0 - x_1\sqrt{t}), f'_2(x_0 + x_1\sqrt{t}))$

$$\frac{d}{dX} = \varphi^{-1}\left(\frac{d}{d(x_0 - x_1\sqrt{t})}, \frac{d}{d(x_0 + x_1\sqrt{t})}\right)$$

and $\mathcal{F} = \varphi^{-1}(\mathcal{F}_1, \mathcal{F}_2)$

Definition 17. Let $y_0 - y_1\sqrt{t} = f_1(x_0 - x_1\sqrt{t})$ and $y_0 + y_1\sqrt{t} = f_2(x_0 + x_1\sqrt{t})$ are the general solutions to (6-1) on $\boldsymbol{I}_1 \subseteq \boldsymbol{R}$ and (6-2) on $\boldsymbol{I}_2 \subseteq \boldsymbol{R}$ with real arbitrary constants C_0, C_1, respectively, then

$Y = f(X) = \varphi^{-1}(f_1(x_0 - x_1\sqrt{t}), f_2(x_0 + x_1\sqrt{t})),$

is the general solution of (5) on $\boldsymbol{I} = \varphi^{-1}(\boldsymbol{I}_1 \times \boldsymbol{I}_2) \subseteq \boldsymbol{F}_J$ with a WFC constant $C = \varphi^{-1}(C_0, C_1)$ if

$\mathcal{F}(X, f(X), f'(X)) \equiv 0$ holds for all X,

where $= \varphi^{-1}(x_0 - x_1\sqrt{t}, x_0 + x_1\sqrt{t})$, $x_0 - x_1\sqrt{t} \in \boldsymbol{I}_1$, $x_0 + x_1\sqrt{t} \in \boldsymbol{I}_2$ and

$$\begin{cases} \mathcal{F}_1[(x_0 - x_1\sqrt{t}), f_1(x_0 - x_1\sqrt{t}), f'_1(x_0 - x_1\sqrt{t})] = 0 \\ \mathcal{F}_2[(x_0 + x_1\sqrt{t}), f_2(x_0 + x_1\sqrt{t}), f'_2(x_0 + x_1\sqrt{t})] = 0. \end{cases}$$

Therefore, the integrals of (6-1) and (6-2) with respect to $x_0 - x_1\sqrt{t}$, $x_0 + x_1\sqrt{t}$ respectively, are defined by the following forms:

$$\begin{cases} \int \mathcal{F}_1(x_0 - x_1\sqrt{t}, y_0 - y_1\sqrt{t}, y'_0 - y'_1\sqrt{t})d(x_0 - x_1\sqrt{t}) = C_0 \\ \int \mathcal{F}_2(x_0 + x_1\sqrt{t}, y_0 + y_1\sqrt{t}, y'_0 + y'_1\sqrt{t})d(x_0 + x_1\sqrt{t}) = C_1 \end{cases}.$$

Then, the integral can be written as follows:

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$$\int \mathcal{F}(X,Y,Y')dX = \varphi^{-1}\left(\int \mathcal{F}_1 d\left(x_0 - x_1\sqrt{t}\right), \int \mathcal{F}_2 d\left(x_0 + x_1\sqrt{t}\right)\right)$$

$$= \frac{1}{2}\left(\int \mathcal{F}_1 d\left(x_0 - x_1\sqrt{t}\right) + \int \mathcal{F}_2 d\left(x_0 + x_1\sqrt{t}\right)\right) + \frac{1}{2\sqrt{t}}J\left(\int \mathcal{F}_2 d\left(x_0 + x_1\sqrt{t}\right) - \int \mathcal{F}_1 d\left(x_0 - x_1\sqrt{t}\right)\right)$$

Which means that the solution of (5) equivalents to

$$\varphi^{-1}\left(\int \mathcal{F}_1 d\left(x_0 - x_1\sqrt{t}\right), \int \mathcal{F}_2 d\left(x_0 + x_1\sqrt{t}\right)\right) = \varphi^{-1}(C_0, C_1).$$

- ❖ In other words, $Y = f(X) = \varphi^{-1}\left(f_1(x_0 - x_1\sqrt{t}), f_2(x_0 + x_1\sqrt{t})\right)$ is the general solution of (4) on $\boldsymbol{I} \subseteq \boldsymbol{F}_J$ if Y is a differentiable WFC function on $\boldsymbol{I}$.

Definition 18. We consider *the explicit* WFC first order differential equation

$$Y' = \mathcal{F}(X,Y) \qquad \textbf{(7)}$$

where $Y = y_0 + y_1 J = \varphi^{-1}(y_0 - y_1\sqrt{t}, y_0 + y_1\sqrt{t}) = \varphi^{-1}(f_1(x_0 - x_1\sqrt{t}), f_2(x_0 + x_1\sqrt{t}))$,

$\boldsymbol{I} = \varphi^{-1}(\boldsymbol{I}_1 \times \boldsymbol{I}_2) \subseteq \boldsymbol{F}_J, = \varphi^{-1}(x_0 - x_1\sqrt{t}, x_0 + x_1\sqrt{t}), x_0 - x_1\sqrt{t} \in \boldsymbol{I}_1 \subseteq \boldsymbol{R}, x_0 + x_1\sqrt{t} \in \boldsymbol{I}_2 \subseteq \boldsymbol{R}$ ؛

Definition 19. A function $(X): \boldsymbol{I} \to \boldsymbol{F}_J$, is called *the general solution* to the differential equation (7) on $\boldsymbol{I} \subseteq \boldsymbol{F}_J$ if $f_1 and\ f_2$ are differentiable in $\boldsymbol{I}_1 and\ \boldsymbol{I}_2$ with respect to their own variables $x_0 - x_1\sqrt{t}$, $x_0 + x_1\sqrt{t}$ respectively, i.e., if

$$\begin{cases}(y_0 - y_1\sqrt{t})' = \mathcal{F}_1(x_0 - x_1\sqrt{t}, y_0 - y_1\sqrt{t}) \text{ for all } x_0 - x_1\sqrt{t} \in \boldsymbol{I}_1 \\ (y_0 + y_1\sqrt{t})' = \mathcal{F}_2(x_0 + x_1\sqrt{t}, y_0 + y_1\sqrt{t}) \text{ for all } x_0 + x_1\sqrt{t} \in \boldsymbol{I}_2\end{cases}$$

$$\Rightarrow \begin{cases} y_0 - y_1\sqrt{t} = \int \mathcal{F}_1 d\left(x_0 - x_1\sqrt{t}\right) \\ y_0 + y_1\sqrt{t} = \int \mathcal{F}_2 d\left(x_0 + x_1\sqrt{t}\right)\end{cases}$$

$$\Rightarrow Y = \varphi^{-1}(y_0 - y_1\sqrt{t}, y_0 + y_1\sqrt{t}) = \varphi^{-1}\left(\int \mathcal{F}_1 d\left(x_0 - x_1\sqrt{t}\right), \int \mathcal{F}_2 d\left(x_0 + x_1\sqrt{t}\right)\right) \in \boldsymbol{I}$$

$$Y = \frac{1}{2}\left(\int \mathcal{F}_1 d\left(x_0 - x_1\sqrt{t}\right) + \int \mathcal{F}_2 d\left(x_0 + x_1\sqrt{t}\right)\right) + \frac{1}{2\sqrt{t}}J\left(\int \mathcal{F}_2 d\left(x_0 + x_1\sqrt{t}\right) - \int \mathcal{F}_1 d\left(x_0 - x_1\sqrt{t}\right)\right)$$

$\boldsymbol{I} = \varphi^{-1}(\boldsymbol{I}_1 \times \boldsymbol{I}_2) \subseteq \boldsymbol{F}_J$, $X = \varphi^{-1}(x_0 - x_1\sqrt{t}, x_0 + x_1\sqrt{t}), x_0 - x_1\sqrt{t} \in \boldsymbol{I}_1 \subseteq \boldsymbol{R}, x_0 + x_1\sqrt{t} \in \boldsymbol{I}_2 \subseteq \boldsymbol{R}$.

General Algorithm to solve 1st order 1st degree WFC-ODEs:

Step1: Use φ to write the WFC-DE as two Real DEs

Step2: Solve each one of Real DEs with respect to their own variables

Step3: Write the WFC general solution as a structure of the two real solutions

Famous Types of First order and First degree WFC-ODEs:

4.1. Separable type

4.1.1. $Y' = h(X)g(Y)$ **(I)**

Step1:

This ODE can be written as:

$$\begin{cases}\left(y_0 - y_1\sqrt{t}\right)' = h_1\left(x_0 - x_1\sqrt{t}\right)g_1\left(y_0 - y_1\sqrt{t}\right) & (8-1) \\ \left(y_0 + y_1\sqrt{t}\right)' = h_2\left(x_0 + x_1\sqrt{t}\right)g_2\left(y_0 + y_1\sqrt{t}\right) & (8-2)\end{cases} \qquad \textbf{(8)}$$

Dividing (8 −1) and (8-2) by g_1 and g_2, respectively, where $g_1(y_0 - y_1\sqrt{t}) \neq 0$ and $g_2(y_0 + y_1\sqrt{t}) \neq 0$

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$$\Rightarrow \begin{cases} \frac{1}{g_1(y_0 - y_1\sqrt{t})} d(y_0 - y_1\sqrt{t}\,) = h_1(x_0 - x_1\sqrt{t})d(x_0 - x_1\sqrt{t}) \quad (9-1) \\ \frac{1}{g_2(y_0 + y_1\sqrt{t})} d(y_0 + y_1\sqrt{t}\,) = h_2(x_0 + x_1\sqrt{t})d(x_0 + x_1\sqrt{t}) \quad (9-2) \end{cases} \quad \textbf{(9)}$$

It means $\frac{1}{g(Y)} dY = h(X)dX$ (where g is invertible).

$(9-1), (9-2)$ are called *separable* ODEs where h_1, h_2, g_1 and g_2 are real functions.

Step2:

Integrating both sides with respect to $x_0 - x_1\sqrt{t}$ and $x_0 + x_1\sqrt{t}$ respectively, we get

$$\Rightarrow \begin{cases} \int \frac{1}{g_1(y_0 - y_1\sqrt{t})} \frac{d(y_0 - y_1\sqrt{t}\,)}{d(x_0 - x_1\sqrt{t})} d(x_0 - x_1\sqrt{t}) = \int h_1(x_0 - x_1\sqrt{t})d(x_0 - x_1\sqrt{t}) \\ \int \frac{1}{g_2(y_0 + y_1\sqrt{t})} \frac{d(y_0 + y_1\sqrt{t}\,)}{d(x_0 + x_1\sqrt{t})} d(x_0 + x_1\sqrt{t}) = \int h_2(x_0 + x_1\sqrt{t})d(x_0 + x_1\sqrt{t}) \end{cases}$$

$$\Leftrightarrow \int \frac{1}{g(Y)} \frac{dY}{dX} dX = \int h(X)dX.$$

Assuming that these integrals maybe evaluated, we get the general solutions to $(10-1), (10-2)$ respectively:

$$\begin{cases} G_1(y_0 - y_1\sqrt{t}\,) = H_1(x_0 - x_1\sqrt{t}) + C_0 \quad (11-1) \\ G_2(y_0 + y_1\sqrt{t}\,) = H_2(x_0 + x_1\sqrt{t}) + C_1 \quad (11-2) \end{cases} \quad \textbf{(11)}$$

where $C_0, C_1 \in \boldsymbol{R}$ are arbitrary constants.

Step3:

$\varphi^{-1}(G_1, G_2) = \varphi^{-1}(H_1, H_2) + \varphi^{-1}(C_0, C_1); C = \varphi^{-1}(C_0, C_1)$

$\Rightarrow G(Y) = H(X) + C.$

Example 1.

$Y' = e^Y sin(X) \quad \Leftrightarrow dY = e^Y sin(X)dX$

$\varphi^{-1}\left(d(y_0 - y_1\sqrt{t}), d(y_0 + y_1\sqrt{t})\right)$

$\quad = \varphi^{-1}(e^{y_0 - y_1\sqrt{t}} sin(x_0 - x_1\sqrt{t})d(x_0 - x_1\sqrt{t}), e^{y_0 + y\sqrt{t}} sin(x_0 + x_1\sqrt{t})d(x_0 + x_1\sqrt{t}))$

$$\Rightarrow \begin{cases} \frac{1}{e^{y_0 - y_1\sqrt{t}}} d(y_0 - y_1\sqrt{t}\,) = sin(x_0 - x_1\sqrt{t})d(x_0 - x_1\sqrt{t}) \\ \frac{1}{e^{y_0 + y\sqrt{t}}} d(y_0 + y_1\sqrt{t}\,) = sin(x_0 + x_1\sqrt{t})d(x_0 + x_1\sqrt{t}) \end{cases}$$

by integrating both sides of the equation with respect to $x_0 - x_1\sqrt{t}$ and $x_0 + x_1\sqrt{t}$, respectively:

$$\begin{cases} \int \frac{1}{e^{y_0 - y_1\sqrt{t}}} d(y_0 - y_1\sqrt{t}) = \int sin(x_0 - x_1\sqrt{t})d(x_0 - x_1\sqrt{t}) \\ \int \frac{1}{e^{y_0 + y\sqrt{t}}} d(y_0 + y_1\sqrt{t}) = \int sin(x_0 + x_1\sqrt{t})\, d(x_0 + x_1\sqrt{t}) \end{cases}$$

$$\Rightarrow \begin{cases} -e^{-(y_0 - y_1\sqrt{t})} = -cos(x_0 - x_1\sqrt{t}) - C_0 \\ -e^{-(y_0 + y\sqrt{t})} = -cos(x_0 + x_1\sqrt{t}) - C_1 \end{cases}$$

$$\Rightarrow \begin{cases} y_0 - y_1\sqrt{t} = -ln[cos(x_0 - x_1\sqrt{t}) + C_0] \text{ (where } cos(x_0 - x_1\sqrt{t}) + C_0 > 0) \\ y_0 + y\sqrt{t} = -ln[cos(x_0 + x_1\sqrt{t}) + C_1\,] \text{ (where } cos(x_0 + x_1\sqrt{t}) + C_1 > 0) \end{cases}$$

$$\overset{\varphi^{-1}}{\Longrightarrow} \quad Y = \frac{1}{2}(y_0 - y_1\sqrt{t} + y_0 + y_1\sqrt{t}) + \frac{1}{2\sqrt{t}} J(y_0 + y_1\sqrt{t} - y_0 + y_1\sqrt{t}))$$

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$$= \frac{1}{2}(-ln[cos(x_0 - x_1\sqrt{t}) + C_0] - ln[cos(x_0 + x_1\sqrt{t}) + C_1]) + \frac{1}{2\sqrt{t}}J(-ln[cos(x_0 + x_1\sqrt{t}) + C_1] + ln[cos(x_0 - x_1\sqrt{t}) + C_0]).$$

$$= \varphi^{-1}(-ln[cos(x_0 - x_1\sqrt{t}) + C_0], -ln[cos(x_0 + x_1\sqrt{t}) + C_1])$$

$$= -ln(\varphi^{-1}[cos(x_0 - x_1\sqrt{t}) + C_0, cos(x_0 + x_1\sqrt{t}) + C_1])$$

$$= -ln(\varphi^{-1}(cos(x_0 - x_1\sqrt{t}), cos(x_0 + x_1\sqrt{t})) + \varphi^{-1}(C_0, C_1))$$

Thus, the general solution to the given WFC-ODE is

$Y = -ln[cos(X) + C]$ where $cos(X) + C > 0$.

4.1.2. $\boldsymbol{Y' = \mathcal{F}(X)}$ **(II)**

$$\Rightarrow dY = \mathcal{F}(X)dX$$

Step1:

$$\text{(II)} \Leftrightarrow \begin{cases} d(y_0 - y_1\sqrt{t}\,) = \mathcal{F}_1(x_0 - x_1\sqrt{t})d(x_0 - x_1\sqrt{t}) & (12-1) \\ d(y_0 + y_1\sqrt{t}\,) = \mathcal{F}_2(x_0 + x_1\sqrt{t})d(x_0 + x_1\sqrt{t}) & (12-2) \end{cases} \quad \textbf{(12)}$$

Step2:

Solving the WFC-DE (II) is equivalent to solve two Real separable DEs $(12-1), (12-2)$:

Suppose the function $\mathcal{F}_1, \mathcal{F}_2$ is continuous functions in intervals $\boldsymbol{I}_1, \boldsymbol{I}_2$, respectively. Integrating both sides with respect to $x_0 - x_1\sqrt{t}$ and $x_0 + x_1\sqrt{t}$ respectively, we get

$$\begin{cases} \int d(y_0 - y_1\sqrt{t}\,) = \int \mathcal{F}_1(x_0 - x_1\sqrt{t})d(x_0 - x_1\sqrt{t}) & (13-1) \\ \int d(y_0 + y_1\sqrt{t}\,) = \int \mathcal{F}_2(x_0 + x_1\sqrt{t})d(x_0 + x_1\sqrt{t}) & (13-2) \end{cases} \quad \textbf{(13)}$$

Assuming that these integrals maybe evaluated, we get the general solutions to $(13-1), (13-2)$ respectively:

$$\Rightarrow \begin{cases} y_0 - y_1\sqrt{t} = r_1(x_0 - x_1\sqrt{t}) + C_0 \\ y_0 + y_1\sqrt{t} = r_2(x_0 + x_1\sqrt{t}) + C_1 \end{cases}$$

where $C_0, C_1 \in R$ are arbitrary constants.

Also,

$\int \mathcal{F}_1(x_0 - x_1\sqrt{t})d(x_0 - x_1\sqrt{t}) = r_1(x_0 - x_1\sqrt{t}) + C_0$ and $\int \mathcal{F}_2(x_0 + x_1\sqrt{t})d(x_0 + x_1\sqrt{t}) = r_2(x_0 + x_1\sqrt{t}) + C_1$

Step3:

Thus, the general solution to the DE (II) is: $\varphi^{-1}(y_0 - y_1\sqrt{t}, y_0 + y_1\sqrt{t}) = \varphi^{-1}(r_1(x_0 - x_1\sqrt{t}) + C_0, r_2(x_0 + x_1\sqrt{t}) + C_1)$

$$\Rightarrow Y = r(X) + C$$

where $r(X) = \varphi^{-1}(r_1(x_0 - x_1\sqrt{t}), r_2(x_0 + x_1\sqrt{t}))$ *and* $C = \varphi^{-1}(C_0, C_1)$

Example 2.

$Y' = e^X \Leftrightarrow dY = e^X dX$

$$\varphi^{-1}(d(y_0 - y_1\sqrt{t}), d(y_0 + y_1\sqrt{t})) = \varphi^{-1}(e^{x_0 - x_1\sqrt{t}}d(x_0 - x_1\sqrt{t}), e^{x_0 + x_1\sqrt{t}}d(x_0 + x_1\sqrt{t}))$$

by integrating both sides of the equation with respect to $x_0 - x_1\sqrt{t}$ and $x_0 + x_1\sqrt{t}$, respectively

$$\varphi^{-1}\left(\int d(y_0 - y_1\sqrt{t}), \int d(y_0 + y_1\sqrt{t})\right) = \varphi^{-1}\left(\int e^{x_0 - x_1\sqrt{t}}d(x_0 - x_1\sqrt{t}), \int e^{x_0 + x_1\sqrt{t}}d(x_0 + x_1\sqrt{t})\right),$$

$$\Rightarrow \varphi^{-1}(y_0 - y_1\sqrt{t}, y_0 + y_1\sqrt{t}) = \varphi^{-1}(e^{x_0 - x_1\sqrt{t}} + C_0, e^{x_0 + x_1\sqrt{t}} + C_1)$$

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$$\Rightarrow \frac{1}{2}(y_0 - y_1\sqrt{t} + y_0 + y_1\sqrt{t}) + \frac{1}{2\sqrt{t}}J(y_0 + y_1\sqrt{t} - y_0 + y_1\sqrt{t})$$

$$= \frac{1}{2}(e^{x_0 - x_1\sqrt{t}} + C_0 + e^{x_0 + x_1\sqrt{t}} + C_1) + \frac{1}{2\sqrt{t}}J(e^{x_0 + x_1\sqrt{t}} + C_1 - e^{x_0 - x_1\sqrt{t}} - C_0)$$

$$= \frac{1}{2}(e^{x_0 - x_1\sqrt{t}} + e^{x_0 + x_1\sqrt{t}}) + \frac{1}{2\sqrt{t}}J(e^{x_0 + x_1\sqrt{t}} - e^{x_0 - x_1\sqrt{t}}) + \frac{1}{2}(C_0 + C_1) + \frac{1}{2\sqrt{t}}J(C_1 - C_0)$$

$$\Rightarrow y_0 + y_1 J = e^{x_0 + x_1 J} + C_0 + C_1 J.$$

The general solution is $Y = e^X + C$.

Example 3.

$$(\sin(X) + \cos(X))dY + (\cos(X) - \sin(X))dX = 0$$

$$\begin{aligned}\varphi^{-1}(&\sin(x_0 - x_1\sqrt{t}) + \cos(x_0 - x_1\sqrt{t}), \sin(x_0 + x_1\sqrt{t}) + \cos(x_0 + x_1\sqrt{t}))(d(y_0 - y_1\sqrt{t}), d(y_0 + y_1\sqrt{t})) \\ &+ (\sin(x_0 - x_1\sqrt{t}) - \cos(x_0 - x_1\sqrt{t}), \sin(x_0 + x_1\sqrt{t}) \\ &- \cos(x_0 + x_1\sqrt{t}))(d(x_0 - x_1\sqrt{t}), d(x_0 + x_1\sqrt{t})) = 0\end{aligned}$$

$$\Rightarrow$$

$$\begin{cases}[\sin(x_0 - x_1\sqrt{t}) + \cos(x_0 - x_1\sqrt{t})]d(y_0 - y_1\sqrt{t}) = [\sin(x_0 - x_1\sqrt{t}) - \cos(x_0 - x_1\sqrt{t})]d(x_0 - x_1\sqrt{t}) \ (14-1) \\ [\sin(x_0 + x_1\sqrt{t}) + \cos(x_0 + x_1\sqrt{t})]d(y_0 + y_1\sqrt{t}) = [\sin(x_0 + x_1\sqrt{t}) - \cos(x_0 + x_1\sqrt{t})]d(x_0 + x_1\sqrt{t}) \ (14-2)\end{cases} \quad (\mathbf{14})$$

Dividing $(14-1)$ by $\sin(x_0 - x_1\sqrt{t}) + \cos(x_0 - x_1\sqrt{t}) \neq 0$ and integrating both sides with respect to $x_0 - x_1\sqrt{t}$. And dividing $(14-2)$ by $\sin(x_0 + x_1\sqrt{t}) + \cos(x_0 + x_1\sqrt{t}) \neq 0$ and integrating both sides with respect to $x_0 + x_1\sqrt{t}$, we get

$$\begin{cases}\int d(y_0 - y_1\sqrt{t}) = \int \frac{\cos(x_0 - x_1\sqrt{t}) - \sin(x_0 - x_1\sqrt{t})}{\sin(x_0 - x_1\sqrt{t}) + \cos(x_0 - x_1\sqrt{t})} d(x_0 - x_1\sqrt{t}) \\ \int d(y_0 + y_1\sqrt{t}) = \int \frac{\cos(x_0 + x_1\sqrt{t}) - \sin(x_0 + x_1\sqrt{t})}{\sin(x_0 + x_1\sqrt{t}) + \cos(x_0 + x_1\sqrt{t})} d(x_0 + x_1\sqrt{t}),\end{cases}$$

$$\Leftrightarrow \int dY = \int \frac{\cos(X) - \sin(X)}{\sin(X) + \cos(X)} dX,$$

$$\begin{cases}y_0 - y_1\sqrt{t} = ln|\sin(x_0 - x_1\sqrt{t}) + \cos(x_0 - x_1\sqrt{t})| + C_0 \\ y_0 + y_1\sqrt{t} = ln|\sin(x_0 + x_1\sqrt{t}) + \cos(x_0 + x_1\sqrt{t})| + C_1\end{cases},$$

$$\Rightarrow \frac{1}{2}(y_0 - y_1\sqrt{t} + y_0 + y_1\sqrt{t}) + \frac{1}{2\sqrt{t}}J(y_0 + y_1\sqrt{t} - y_0 + y_1\sqrt{t})$$

$$\begin{aligned}= \frac{1}{2}(&ln|\sin(x_0 - x_1\sqrt{t}) + \cos(x_0 - x_1\sqrt{t})| + C_0 + ln|\sin(x_0 + x_1\sqrt{t}) + \cos(x_0 + x_1\sqrt{t})| + C_1) \\ &+ \frac{1}{2\sqrt{t}}J(ln|\sin(x_0 + x_1\sqrt{t}) + \cos(x_0 + x_1\sqrt{t})| + C_1 - ln|\sin(x_0 - x_1\sqrt{t}) + \cos(x_0 - x_1\sqrt{t})| - C_0)\end{aligned}$$

$$\Rightarrow y_0 + y_1 J = ln|\sin(x_0 + x_1 J) + \cos(x_0 + x_1 J)| + C_0 + C_1 J.$$

The general solution is

$$Y = -ln|\sin(X) + \cos(X)| + C; \ C = C_0 + C_1 J$$

$$Y = ln\frac{1}{|\sin(X) + \cos(X)|} + C; \ \sin(X) + \cos(X) \neq 0.$$

Example 4.

$$Y' = X^3 + cos(X)$$

we write it in the following form:

$$dY = (X^3 + cos(X))dX$$

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$$\varphi^{-1}\left(\int d\left(y_0-y_1\sqrt{t}\right),\int d\left(y_0+y_1\sqrt{t}\right)\right)$$

$$=\varphi^{-1}(\int((x_0-x_1\sqrt{t})^3+cos(x_0-x_1\sqrt{t}))d\left(x_0-x_1\sqrt{t}\right),\int((x_0+x_1\sqrt{t})^3+cos(x_0+x_1\sqrt{t}))d\left(x_0+x_1\sqrt{t}\right))$$

$$\varphi^{-1}\left(y_0-y_1\sqrt{t},y_0+y_1\sqrt{t}\right)$$

$$=\varphi^{-1}(\frac{(x_0-x_1\sqrt{t})^4}{4}+sin(x_0-x_1\sqrt{t})+C_0,\frac{(x_0+x_1\sqrt{t})^4}{4}+sin(x_0+x_1\sqrt{t})+C_1)$$

$$\Rightarrow\frac{1}{2}(y_0-y_1\sqrt{t}+y_0+y_1\sqrt{t})+\frac{1}{2\sqrt{t}}J(y_0+y_1\sqrt{t}-y_0+y_1\sqrt{t}))$$

$$=\frac{1}{2}(\frac{(x_0-x_1\sqrt{t})^4}{4}+sin(x_0-x_1\sqrt{t})+\frac{(x_0+x_1\sqrt{t})^4}{4}+sin(x_0+x_1\sqrt{t})+C_0+C_1)$$

$$+\frac{1}{2\sqrt{t}}J(\frac{(x_0-x_1\sqrt{t})^4}{4}+sin(x_0+x_1\sqrt{t})-\frac{(x_0-x_1\sqrt{t})^4}{4}-sin(x_0-x_1\sqrt{t})+C_1-C_0)$$

$$\Rightarrow y_0+y_1J$$

$$=\frac{1}{2}\left(\frac{\left(x_0-x_1\sqrt{t}\right)^4}{4}+\frac{\left(x_0+x_1\sqrt{t}\right)^4}{4}\right)+\frac{1}{2\sqrt{t}}J\left(\frac{\left(x_0+x_1\sqrt{t}\right)^4}{4}-\frac{\left(x_0-x_1\sqrt{t}\right)^4}{4}\right)$$

$$+\frac{1}{2}\left(sin\left(x_0-x_1\sqrt{t}\right)+sin\left(x_0+x_1\sqrt{t}\right)\right)+\frac{1}{2\sqrt{t}}J\left(sin\left(x_0+x_1\sqrt{t}\right)-sin\left(x_0-x_1\sqrt{t}\right)\right)$$

$$+\frac{1}{2}(C_0+C_1)+\frac{1}{2\sqrt{t}}J(C_1-C_0)$$

$$\Rightarrow y_0+y_1J=\frac{(x_0+x_1J)^4}{4}+sin(x_0+x_1J)+C_0+C_1J$$

The general solution is $Y=\frac{X^4}{4}+sin(X)+C$.

4.1.3. $\boldsymbol{Y'=\mathcal{F}(Y)}$ **(III)**

$$\Rightarrow dY=\mathcal{F}(Y)dX$$

Here is similar to the previous ODE(II) but with exchange roles of X and Y.

Step1:

$$\text{(III)}\iff\begin{cases}d(y_0-y_1\sqrt{t}\,)=\mathcal{F}_1(y_0-y_1\sqrt{t})d(x_0-x_1\sqrt{t}) & (15-1)\\ d(y_0+y_1\sqrt{t}\,)=\mathcal{F}_2(y_0+y_1\sqrt{t})d(x_0+x_1\sqrt{t}) & (15-2)\end{cases}\tag{15}$$

Step2:

Solving the WFC-DE (II) is equivalent to solve two Real separable DEs $(15-1),(15-2)$.

Suppose the function $\mathcal{F}_1,\mathcal{F}_2$ is continuous functions in intervals $\boldsymbol{I}_1,\boldsymbol{I}_2$, respectively. Integrating both sides with respect to $x_0-x_1\sqrt{t}$ and $x_0+x_1\sqrt{t}$ respectively, we get

$$\begin{cases}\int\frac{1}{\mathcal{F}_1(y_0-y_1\sqrt{t})}d(y_0-y_1\sqrt{t}\,)=\int d(x_0-x_1\sqrt{t}) & (16-1)\\ \int\frac{1}{\mathcal{F}_2\left(y_0+y_1\sqrt{t}\right)}d\left(y_0+y_1\sqrt{t}\,\right)=\int d\left(x_0+x_1\sqrt{t}\right) & (16-2),\end{cases}\tag{16}$$

where $\mathcal{F}_1(y_0-y_1\sqrt{t})\neq0$ and $\mathcal{F}_2(y_0+y_1\sqrt{t})\neq0$ (where $\mathcal{F}$ is invertible).

We get the general solutions to $(16-1),(16-2)$, respectively:

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$$\begin{cases} S_1(y_0 - y_1\sqrt{t}) = x_0 - x_1\sqrt{t} + C_0 & \Rightarrow x_0 - x_1\sqrt{t} = S_1(y_0 - y_1\sqrt{t}) - C_0 \;\; (17-1) \\ S_2(y_0 + y_1\sqrt{t}) = x_0 + x_1\sqrt{t} + C_1 & \Rightarrow x_0 + x_1\sqrt{t} = S_2(y_0 + y_1\sqrt{t}) - C_1 \;\; (17-2) \end{cases} \quad \textbf{(17)}$$

where $C_0, C_1 \in \boldsymbol{R}$ are arbitrary constants.

Note that in $(17-1)$ we exchange roles of $x_0 - x_1\sqrt{t}$ and $y_0 - y_1\sqrt{t}$ and in $(17-2)$ we exchange roles of $x_0 + x_1\sqrt{t}$ and $y_0 + y_1\sqrt{t}$.

Step3:

$\varphi^{-1}\big(x_0 - x_1\sqrt{t}, x_0 + x_1\sqrt{t}\big) = \varphi^{-1}(S_1(y_0 - y_1\sqrt{t}) + C_0,\ S_2(y_0 + y_1\sqrt{t}) + C_1),$

$\Longrightarrow X = S(Y) + C$

where $S(Y) = \varphi^{-1}(S_1(y_0 - y_1\sqrt{t}), S_2(y_0 + y_1\sqrt{t}))\ and\ \ C = \varphi^{-1}(C_0, C_1)$.

Hence, the general solution $Y(\mathrm{X})$ to the WFC-ODE (III) is the inverse of the function $X = X(Y)$.

Example 5.

$$Y' = -2Y \quad \textbf{(18)}$$

$$\Leftrightarrow \begin{cases} d(y_0 - y_1\sqrt{t}\,) = -2(y_0 - y_1\sqrt{t})d(x_0 - x_1\sqrt{t}) \\ d(y_0 + y_1\sqrt{t}\,) = -2(y_0 + y_1\sqrt{t})d(x_0 + x_1\sqrt{t}) \end{cases}$$

$$\begin{cases} \int \dfrac{1}{(y_0 - y_1\sqrt{t})} d(y_0 - y_1\sqrt{t}\,) = -2\int d(x_0 - x_1\sqrt{t}) \\ \int \dfrac{1}{(y_0 + y_1\sqrt{t})} d(y_0 + y_1\sqrt{t}\,) = -2\int d(x_0 + x_1\sqrt{t}) \end{cases}$$

where $y_0 - y_1\sqrt{t} \neq 0$ and $y_0 + y_1\sqrt{t} \neq 0$.

$$\Leftrightarrow \begin{cases} ln|y_0 - y_1\sqrt{t}| = -2(x_0 - x_1\sqrt{t}) + C_0 & \Leftrightarrow |y_0 - y_1\sqrt{t}| = e^{C_0 - 2(x_0 - x_1\sqrt{t})} \\ ln|y_0 + y_1\sqrt{t}| = -2(x_0 + x_1\sqrt{t}\,) + C_1 & \Leftrightarrow |y_0 + y_1\sqrt{t}| = e^{C_1 - 2(x_0 + x_1\sqrt{t})} \end{cases}$$

where $C_0, C_1 \in \boldsymbol{R}$ are arbitrary constants.

The general solutions with $\pm e^{C_0}, \pm e^{C_1}$ replaced with C_0, C_1, respectively:

$$\Leftrightarrow \begin{cases} y_0 - y_1\sqrt{t} = C_0 e^{-2(x_0 - x_1\sqrt{t})} \\ y_0 + y_1\sqrt{t} = C_1 e^{-2(x_0 + x_1\sqrt{t})} \end{cases}$$

The general solution to (18) is written as:

$\varphi^{-1}(y_0 - y_1\sqrt{t}, y_0 + y_1\sqrt{t}) = \varphi^{-1}(C_0 e^{-2(x_0 - x_1\sqrt{t})}, C_1 e^{-2(x_0 + x_1\sqrt{t})}) \Longleftrightarrow Y = C.e^{-2X}$.

4.1.4. $\boldsymbol{Y' = \mathcal{F}(AX + BY + C)}$ **(IV)**

This structure of the differential equation can be reduced by simple transformation to one of the previous types already discussed. Assuming the function $\mathcal{F}$ is continuous in an interval $\boldsymbol{I} \subseteq \boldsymbol{F_J}$.

$$\varphi^{-1}((y_0 - y_1\sqrt{t}\,)', (y_0 + y_1\sqrt{t})')$$
$$= \varphi^{-1}(\mathcal{F}_1[(A_0 - A_1\sqrt{t})(x_0 - x_1\sqrt{t}) + (B_0 - B_1\sqrt{t})(y_0 - y_1\sqrt{t}) + C_0 - C_1\sqrt{t}], \mathcal{F}_2[(A_0 + A_1\sqrt{t})(x_0 + x_1\sqrt{t}) + (B_0 + B_1\sqrt{t})(y_0 + y_1\sqrt{t}) + C_0 + C_1\sqrt{t}])$$

$\Longrightarrow$

$$\begin{cases} (y_0 - y_1\sqrt{t}\,)' = \mathcal{F}_1\big[(A_0 - A_1\sqrt{t})(x_0 - x_1\sqrt{t}) + (B_0 - B_1\sqrt{t})(y_0 - y_1\sqrt{t}) + C_0 - C_1\sqrt{t}\big] \;\; (19-1) \\ (y_0 + y_1\sqrt{t})' = \mathcal{F}_2\big[(A_0 + A_1\sqrt{t})(x_0 + x_1\sqrt{t}) + (B_0 + B_1\sqrt{t})(y_0 + y_1\sqrt{t}) + C_0 + C_1\sqrt{t}\big] \;\; (19-2) \end{cases} \quad \textbf{(19)}$$

where $A = A_0 + A_1 J,\ B = B_0 + B_1 J \in \boldsymbol{F_J},\ \ C = C_0 + C_1 J \in \boldsymbol{F_J}$.

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We can say that solving (IV) is equivalent to solve the two ODEs $(19-1\ \&\ 19-2)$ in real field with respect to variables $x_0 - x_1\sqrt{t}$ and $x_0 + x_1\sqrt{t}$, respectively. For that, we suppose:

$$\begin{cases} U_1(x_0 - x_1\sqrt{t}) = (A_0 - A_1\sqrt{t})(x_0 - x_1\sqrt{t}) + (B_0 - B_1\sqrt{t})(y_0 - y_1\sqrt{t}) + C_0 - C_1\sqrt{t} \\ U_2(x_0 + x_1\sqrt{t}) = (A_0 + A_1\sqrt{t})(x_0 + x_1\sqrt{t}) + (B_0 + B_1\sqrt{t})(y_0 + y_1\sqrt{t}) + C_0 + C_1\sqrt{t} \end{cases}$$

$$\Rightarrow \begin{cases} U_1\left(x_0 - x_1\sqrt{t}\right) = \left(A_0 - A_1\sqrt{t}\right)\left(x_0 - x_1\sqrt{t}\right) + \left(B_0 - B_1\sqrt{t}\right)Y_1\left(x_0 - x_1\sqrt{t}\right) + C_0 - C_1\sqrt{t} & (20-1) \\ U_2\left(x_0 + x_1\sqrt{t}\right) = \left(A_0 + A_1\sqrt{t}\right)\left(x_0 + x_1\sqrt{t}\right) + \left(B_0 + B_1\sqrt{t}\right)Y_2\left(x_0 + x_1\sqrt{t}\right) + C_0 + C_1\sqrt{t} & (20-2) \end{cases} \quad \mathbf{(20)}$$

$$\text{(IV)} \Rightarrow \begin{cases} (y_0 - y_1\sqrt{t}\,)' = \mathcal{F}_1[U_1(x_0 - x_1\sqrt{t})] \\ (y_0 + y_1\sqrt{t}\,)' = \mathcal{F}_2[U_2(x_0 + x_1\sqrt{t})] \end{cases}$$

(We are interested in the case where $B_0 + B_1\sqrt{t} \neq 0$, $B_0 - B_1\sqrt{t} \neq 0$)

$$Y(X) = \varphi^{-1}(Y_1(x_0 - x_1\sqrt{t}), Y_2(x_0 + x_1\sqrt{t})) \in \boldsymbol{F_J}(X)$$
$$= \varphi^{-1}(y_0 - y_1\sqrt{t}, y_0 + y_1\sqrt{t})$$
$$U(X) = \varphi^{-1}(U_1(x_0 - x_1\sqrt{t}), U_2(x_0 + x_1\sqrt{t})) \in \boldsymbol{F_J}(X)$$

So that, we can write

$$U(X) = AX + BY(X) + C \quad \mathbf{(21)}$$

(If Y_1 is a solution to $(20-1)$ and Y_2 is a solution to $(20-2)$, then U_1 and U_2 satisfie

$$\begin{cases} U'_1(x_0 - x_1\sqrt{t}) = (A_0 - A_1\sqrt{t}) + (B_0 - B_1\sqrt{t})(y_0 - y_1\sqrt{t})' \\ U'_2(x_0 + x_1\sqrt{t}) = (A_0 + A_1\sqrt{t}) + (B_0 + B_1\sqrt{t})(y_0 + y_1\sqrt{t})' \end{cases}$$

Thus,

$$U'(X) = A + BY'(X)$$
$$U' = A + B\mathcal{F}(U) \quad \mathbf{(22)}$$

which is a solvable equation of the type (III).

Conversely, from a solution $U(X)$ to (22) we can get a solution $Y(X)$ to (21).

Example 6.

$$Y' = (X + Y)^2 \quad \mathbf{(23)}$$

$$\Leftrightarrow \varphi^{-1}((y_0 - y_1\sqrt{t}\,)', (y_0 + y_1\sqrt{t})') = \varphi^{-1}((x_0 - x_1\sqrt{t} + y_0 - y_1\sqrt{t})^2, (x_0 + x_1\sqrt{t} + y_0 + y_1\sqrt{t})^2).$$

To solve it, we use the following ansatz:

$$\begin{cases} U_1(x_0 - x_1\sqrt{t}) = x_0 - x_1\sqrt{t} + y_0 - y_1\sqrt{t} \\ U_2(x_0 + x_1\sqrt{t}) = x_0 + x_1\sqrt{t} + y_0 + y_1\sqrt{t} \end{cases},$$

$$\frac{1}{2}(U_1 + U_2) + \frac{1}{2\sqrt{t}}J(U_2 - U_1) = \frac{1}{2}(x_0 + y_0 + x_0 + y_0) + \frac{1}{2\sqrt{t}}J\left(2x_1\sqrt{t} - 2y_1\sqrt{t}\right),$$

$$U(X) = \varphi^{-1}(U_1, U_2) = (x_0 + y_0) + (x_1 + y_1)J,$$
$$\Rightarrow U(X) = X + Y(X).$$

Then, (23) becomes like

$$U'(X) = U^2 + 1$$

$$\frac{dU}{dX} = U^2 + 1 \Longrightarrow \frac{dU}{U^2+1} - dX = 0 \quad \text{(Separable ODE)}$$

By integrating:

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$arc\tan(U) - X = C \Rightarrow arc\tan(U) = C + X \Rightarrow U = tan(C + X)$

Going back to the old variables, we get the general solution:

$Y = tan(C + X) - X.$

4.1.5. The Homogeneous ODE

$$Y' = \mathcal{F}\left(\frac{Y}{X}\right);\ X \in F_J \backslash \{(\tfrac{1}{2} \mp \tfrac{1}{2\sqrt{t}}J)k, 0; k \in R\} \qquad \textbf{(V)}$$

$$\Leftrightarrow \varphi^{-1}((y_0 - y_1\sqrt{t}\,)', (y_0 + y_1\sqrt{t})') = \varphi^{-1}(\mathcal{F}_1\left(\frac{y_0 - y_1\sqrt{t}}{x_0 - x_1\sqrt{t}}\right), \mathcal{F}_2\left(\frac{y_0 + y_1\sqrt{t}}{x_0 + x_1\sqrt{t}}\right))$$

Supposing

$U_1(x_0 - x_1\sqrt{t}) = \frac{y_0 - y_1\sqrt{t}}{x_0 - x_1\sqrt{t}}$ and $U_2(x_0 + x_1\sqrt{t}) = \frac{y_0 + y_1\sqrt{t}}{x_0 + x_1\sqrt{t}}$.

Then, supposing $U(X) = \frac{Y(X)}{X}$ and calculating the derivative

$$\begin{cases} y_0 - y_1\sqrt{t} = (x_0 - x_1\sqrt{t})U_1(x_0 - x_1\sqrt{t}) \Rightarrow (y_0 - y_1\sqrt{t}\,)' = U_1 + (x_0 - x_1\sqrt{t})U'_1 & (24-1) \\ y_0 + y_1\sqrt{t} = (x_0 + x_1\sqrt{t})U_2(x_0 + x_1\sqrt{t}) \Rightarrow (y_0 + y_1\sqrt{t}\,)' = U_2 + (x_0 + x_1\sqrt{t})U'_2 & (24-2) \end{cases} \quad (24)$$

But we have $\begin{cases} (y_0 - y_1\sqrt{t}\,)' = \mathcal{F}_1(U_1) \\ (y_0 + y_1\sqrt{t}\,)' = \mathcal{F}_2(U_2) \end{cases}$

It means $Y(X) = XU(X) \Longrightarrow Y'(X) = U(X) + XU'(X) = \mathcal{F}(U)$.

Then,

$$\begin{cases} U'_1(x_0 - x_1\sqrt{t}) = \dfrac{\mathcal{F}_1(U_1) - U_1(x_0 - x_1\sqrt{t})}{x_0 - x_1\sqrt{t}} \\ U'_2(x_0 + x_1\sqrt{t}) = \dfrac{\mathcal{F}_2(U_2) - U_2(x_0 + x_1\sqrt{t})}{x_0 + x_1\sqrt{t}} \end{cases}$$

Hence, we get ODEs for the new function with separated variables,

$$U'(X) = \frac{\mathcal{F}(U) - U(X)}{X}$$

Every solution $U_1(X)$, $U_2(X)$ to last ODEs leads to a solution $Y_1(x_0 - x_1\sqrt{t}) = (x_0 - x_1\sqrt{t})U_1(x_0 - x_1\sqrt{t})$, $Y_2(x_0 + x_1\sqrt{t}) = (x_0 + x_1\sqrt{t})U_2(x_0 + x_1\sqrt{t})$ of the homogeneous ODE $(24-1)$ and $(24-2)$, respectively, so the general solution is $Y = \varphi^{-1}(Y_1, Y_2)$.

Example 7.

Let the ODE:

$$Y' = \frac{Y}{X} - \frac{X^2}{Y^2} \quad ; X \in F_J \backslash \{(\tfrac{1}{2} \mp \tfrac{1}{2\sqrt{t}}J)k, 0; k \in R\} \qquad \textbf{(25)}$$

$$\Rightarrow \varphi^{-1}((y_0 - y_1\sqrt{t}\,)', (y_0 + y_1\sqrt{t})') = \varphi^{-1}(\frac{y_0 - y_1\sqrt{t}}{x_0 - x_1\sqrt{t}} - \left(\frac{x_0 - x_1\sqrt{t}}{y_0 - y_1\sqrt{t}}\right)^2, \frac{y_0 + y_1\sqrt{t}}{x_0 + x_1\sqrt{t}} - \left(\frac{x_0 + x_1\sqrt{t}}{y_0 + y_1\sqrt{t}}\right)^2)$$

$$\Leftrightarrow \begin{cases} (y_0 - y_1\sqrt{t}\,)' = U_1 - \dfrac{1}{U_1{}^2} = \mathcal{F}_1(U_1) \\ (y_0 + y_1\sqrt{t}\,)' = U_2 - \dfrac{1}{U_2{}^2} = \mathcal{F}_2(U_2) \end{cases}$$

Where

$U_1(x_0 - x_1\sqrt{t}) = \frac{y_0 - y_1\sqrt{t}}{x_0 - x_1\sqrt{t}}$ and $U_2(x_0 + x_1\sqrt{t}) = \frac{y_0 + y_1\sqrt{t}}{x_0 + x_1\sqrt{t}}$ $(U(X) = \frac{Y(X)}{X})$,

$$Y(X) = XU(X), \qquad \textbf{(26)}$$

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$$\Rightarrow Y'(X) = U(X) + XU'(X) \Leftrightarrow \begin{cases} \left(y_0 - y_1\sqrt{t}\,\right)' = U_1 + \left(x_0 - x_1\sqrt{t}\right)U'_1 \\ \left(y_0 + y_1\sqrt{t}\,\right)' = U_2 + \left(x_0 + x_1\sqrt{t}\right)U'_2 \end{cases}.$$

Then, the ODE (25) transforms into separable ODEs:

$$\begin{cases} U'_1\left(x_0 - x_1\sqrt{t}\right) = \dfrac{\mathcal{F}_1(U_1) - U_1}{x_0 - x_1\sqrt{t}} = -\dfrac{1}{\left(x_0 - x_1\sqrt{t}\right){U_1}^2} \\ U'_2\left(x_0 + x_1\sqrt{t}\right) = \dfrac{\mathcal{F}_2(U_2) - U_2}{x_0 + x_1\sqrt{t}} = -\dfrac{1}{\left(x_0 + x_1\sqrt{t}\right){U_2}^2} \end{cases} \qquad \left(U' = -\frac{1}{XU^2}\right)$$

$$\Rightarrow \begin{cases} {U_1}^2 dU_1 = -\dfrac{d(x_0 - x_1\sqrt{t})}{x_0 - x_1\sqrt{t}} \\ {U_2}^2 dU_2 = -\dfrac{d(x_0 + x_1\sqrt{t})}{x_0 + x_1\sqrt{t}} \end{cases} \qquad \left(U^2 dU = -\frac{dX}{X}\right)$$

By integrating:

$$\begin{cases} \int {U_1}^2 dU_1 + \int \dfrac{d(x_0 - x_1\sqrt{t})}{x_0 - x_1\sqrt{t}} = 0 \\ \int {U_2}^2 dU_2 + \int \dfrac{d(x_0 + x_1\sqrt{t})}{x_0 + x_1\sqrt{t}} = 0 \end{cases} \qquad \left(\int U^2 dU + \int \frac{dX}{X} = 0\right)$$

$$\Rightarrow \begin{cases} \dfrac{{U_1}^3}{3} + log(x_0 - x_1\sqrt{t}) = C_0 \\ \dfrac{{U_2}^3}{3} + log(x_0 + x_1\sqrt{t}) = C_1 \end{cases} \qquad \left(\frac{U^3}{3} + log(X) = C\right)$$

$$\Rightarrow \begin{cases} {U_1}^3 = -\,3log(x_0 - x_1\sqrt{t}) + 3C_0 \\ {U_2}^3 = -\,3log(x_0 + x_1\sqrt{t}) + 3C_1 \end{cases}$$

$$\xRightarrow{from(26)} \begin{cases} {Y_1}^3 = ((x_0 - x_1\sqrt{t})U_1)^3 = (x_0 - x_1\sqrt{t})^3[-\,3log(x_0 - x_1\sqrt{t}) + 3C_0] \\ {Y_2}^3 = ((x_0 + x_1\sqrt{t})U_2)^3 = (x_0 + x_1\sqrt{t})^3[-\,3log(x_0 + x_1\sqrt{t}) + 3C_1] \end{cases}$$

$$\Rightarrow \begin{cases} Y_1 = \sqrt[3]{(x_0 - x_1\sqrt{t})^3\,(-3\,log(x_0 - x_1\sqrt{t}) + 3C_0)} \\ Y_2 = \sqrt[3]{(x_0 + x_1\sqrt{t})^3\,(-3\,log(x_0 + x_1\sqrt{t}) + 3C_1)} \end{cases}$$

The general solution is $Y = \varphi^{-1}(Y_1, Y_2) = \sqrt[3]{X^3\,(-3\,log(X) + 3C)}$.

4.2. Exact type

Definition 19. Let $X \in F_J$ and $Y(X) \in F_J(X)$, the following DE:

$M(X,Y) + N(X,Y)Y' = 0$

Or

$$M(X,Y)dX + N(X,Y)dY = 0 \qquad (27)$$

is called an *Exact* (or *total*) *WFC-DE* when there exists a function $U(X,Y)$ of which it is *the Exact differential,* so that $\frac{\partial U}{\partial X} = M(X,Y)$ and $\frac{\partial U}{\partial Y} = N(X,Y)$, i.e. $dU = M(X,Y)dX + N(X,Y)dY = \frac{\partial U}{\partial X}dX + \frac{\partial U}{\partial Y}dY$, where $U(X,Y) = C$ is the general solution.

Remark 6. The previous definition is equivalent to say that (27) is an exact WFC-DE when its consequent equations by using φ for $(28-1)$, $(28-2)$ are exact DEs:

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$$(27) \Leftrightarrow \begin{cases} M_1(x_0 - x_1\sqrt{t}, y_0 - y_1\sqrt{t}\,)d(x_0 - x_1\sqrt{t}) + N_1(x_0 - x_1\sqrt{t}, y_0 - y_1\sqrt{t}\,)d(y_0 - y_1\sqrt{t}\,) = 0 \ (28-1) \\ M_2(x_0 + x_1\sqrt{t}, y_0 + y_1\sqrt{t}\,)d(x_0 + x_1\sqrt{t}) + N_2(x_0 + x_1\sqrt{t}, y_0 + y_1\sqrt{t}\,)d(y_0 + y_1\sqrt{t}\,) = 0 \ (28-2) \end{cases}$$

(28)

where $M = \varphi^{-1}(M_1, M_2), N = \varphi^{-1}(N_1, N_2)$ and $U = \varphi^{-1}(U_1, U_2)$. Since,

$$dU_1 = M_1(x_0 - x_1\sqrt{t}, y_0 - y_1\sqrt{t}\,)d(x_0 - x_1\sqrt{t}) + N_1(x_0 - x_1\sqrt{t}, y_0 - y_1\sqrt{t}\,)d(y_0 - y_1\sqrt{t}\,)$$

$$= \frac{\partial U_1}{\partial(x_0 - x_1\sqrt{t})}d(x_0 - x_1\sqrt{t}) + \frac{\partial U_1}{\partial(y_0 - y_1\sqrt{t})}d(y_0 - y_1\sqrt{t}\,)$$

$$dU_2 = M_2(x_0 + x_1\sqrt{t}, y_0 + y_1\sqrt{t}\,)d(x_0 + x_1\sqrt{t}) + N_2(x_0 + x_1\sqrt{t}, y_0 + y_1\sqrt{t}\,)d(y_0 + y_1\sqrt{t}\,)$$

$$= \frac{\partial U_2}{\partial(x_0 + x_1\sqrt{t})}d(x_0 + x_1\sqrt{t}) + \frac{\partial U_2}{\partial(y_0 + y_1\sqrt{t})}d(y_0 + y_1\sqrt{t}\,)$$

Theorem. If $M(X,Y)$ and $N(X,Y)$ are continuously differentiable WFC functions in a connected domain, then (27) is an exact differential WFC if and only if $\frac{\partial N}{\partial X} = \frac{\partial M}{\partial Y}$.

Proof.

We know that WFC functions $M(X,Y)$ and $N(X,Y)$ are continuous if and only if M_1, M_2, N_1, N_2 are continuous real functions. Also, (28-1) and (28-2) are exact DEs if and only if

$\frac{\partial N_1}{\partial(x_0-x_1\sqrt{t})} = \frac{\partial M_1}{\partial(y_0-y_1\sqrt{t})}, \frac{\partial N_2}{\partial(x_0+x_1\sqrt{t})} = \frac{\partial M_2}{\partial(y_0+y_1\sqrt{t})}$, respectively.

So that,

$$\frac{\partial N}{\partial X} = \varphi^{-1}\left(\frac{\partial N_1}{\partial(x_0 - x_1\sqrt{t})}, \frac{\partial N_2}{\partial(x_0 + x_1\sqrt{t})}\right)$$

$$= \frac{1}{2}\left(\frac{\partial N_1}{\partial(x_0 - x_1\sqrt{t})} + \frac{\partial N_2}{\partial(x_0 + x_1\sqrt{t})}\right) + \frac{1}{2\sqrt{t}}J\left(\frac{\partial N_2}{\partial(x_0 + x_1\sqrt{t})} - \frac{\partial N_1}{\partial(x_0 - x_1\sqrt{t})}\right)$$

$$= \frac{1}{2}\left(\frac{\partial M_1}{\partial(x_0 - x_1\sqrt{t})} + \frac{\partial M_2}{\partial(x_0 + x_1\sqrt{t})}\right) + \frac{1}{2\sqrt{t}}J\left(\frac{\partial M_2}{\partial(x_0 + x_1\sqrt{t})} - \frac{\partial M_1}{\partial(x_0 - x_1\sqrt{t})}\right)$$

$$= \varphi^{-1}\left(\frac{\partial M_1}{\partial(x_0 - x_1\sqrt{t})}, \frac{\partial M_2}{\partial(x_0 + x_1\sqrt{t})}\right) = \frac{\partial M}{\partial Y}.$$

Example 8.

$$\frac{Y}{\sqrt{X}}dX + 2\sqrt{X}\,dY = 0$$

It is an exact WFC-DE since $\frac{\partial N}{\partial X} = \frac{\partial M}{\partial Y}$, because of $\frac{\partial N_1}{\partial(x_0-x_1\sqrt{t})} = \frac{\partial M_1}{\partial(y_0-y_1\sqrt{t})}, \frac{\partial N_2}{\partial(x_0+x_1\sqrt{t})} = \frac{\partial M_2}{\partial(y_0+y_1\sqrt{t})}$,

where

$$M = \frac{Y}{\sqrt{X}} \Rightarrow \frac{\partial M}{\partial Y} = \frac{1}{\sqrt{X}} \Rightarrow \varphi^{-1}\left(\frac{\partial M_1}{\partial(x_0 - x_1\sqrt{t})}, \frac{\partial M_2}{\partial(x_0 + x_1\sqrt{t})}\right) = \left(\frac{1}{\sqrt{x_0 - x_1\sqrt{t}}}, \frac{1}{\sqrt{x_0 + x_1\sqrt{t}}}\right),$$

$$N = 2\sqrt{X} \Rightarrow \frac{\partial N}{\partial X} = \frac{1}{\sqrt{X}} \Rightarrow \varphi^{-1}\left(\frac{\partial N_1}{\partial(x_0 - x_1\sqrt{t})}, \frac{\partial N_2}{\partial(x_0 + x_1\sqrt{t})}\right) = \left(\frac{1}{\sqrt{x_0 - x_1\sqrt{t}}}, \frac{1}{\sqrt{x_0 + x_1\sqrt{t}}}\right).$$

To find the general solution, suppose

$$dU_1 = M_1(x_0 - x_1\sqrt{t}, y_0 - y_1\sqrt{t}\,)d(x_0 - x_1\sqrt{t}) + N_1(x_0 - x_1\sqrt{t}, y_0 - y_1\sqrt{t}\,)d(y_0 - y_1\sqrt{t}\,)$$

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$$= \frac{y_0 - y_1\sqrt{t}}{\sqrt{x_0 - x_1\sqrt{t}}} d(x_0 - x_1\sqrt{t}) + 2\sqrt{x_0 - x_1\sqrt{t}} d(y_0 - y_1\sqrt{t})$$

$$dU_2 = M_2(x_0 + x_1\sqrt{t}, y_0 + y_1\sqrt{t})d(x_0 + x_1\sqrt{t}) + N_2(x_0 + x_1\sqrt{t}, y_0 + y_1\sqrt{t})d(y_0 + y_1\sqrt{t})$$

$$= \frac{y_0 + y_1\sqrt{t}}{\sqrt{x_0 + x_1\sqrt{t}}} d(x_0 + x_1\sqrt{t}) + 2\sqrt{x_0 + x_1\sqrt{t}} d(y_0 + y_1\sqrt{t})$$

$$dU = M(X,Y)dX + N(X,Y)dY = \frac{Y}{\sqrt{X}}dX + 2\sqrt{X}\,dY = \frac{\partial U}{\partial X}dX + \frac{\partial U}{\partial Y}dY$$

where $M = \varphi^{-1}(M_1, M_2), N = \varphi^{-1}(N_1, N_2)$ and $U = \varphi^{-1}(U_1, U_2)$

$\Rightarrow U(X,Y) = \int \frac{Y}{\sqrt{X}}dX + \int 2\sqrt{X}\,dY = 2Y\sqrt{X} \;\; ; X > 0.$

4.3. Linear type

Definition 20. Let Y is the WFC dependent variable, the following DE is called the standard form of *a linear first-order WFC-DE* in the dependent WFC variable Y:

$$Y' + P(X)Y = Q(X) \qquad \textbf{(29)}$$

assuming that the two given functions $P(X)$ and $Q(X)$ are both continuous on the interval $I = \varphi^{-1}(I_1, I_2)$.

$$(29) \Leftrightarrow \begin{cases} (y_0 - y_1\sqrt{t})' + (y_0 - y_1\sqrt{t})P_1(x_0 - x_1\sqrt{t}) = Q_1(x_0 - x_1\sqrt{t}) & (30-1) \\ (y_0 + y_1\sqrt{t})' + (y_0 + y_1\sqrt{t})P_2(x_0 + x_1\sqrt{t}) = Q_2(x_0 + x_1\sqrt{t}) & (30-2) \end{cases} \qquad \textbf{(30)}$$

where $Q = \varphi^{-1}(Q_1, Q_2)$, $P = \varphi^{-1}(P_1, P_2)$, Q_1, P_1 are continuous functions on I_1 and Q_2, P_2 are continuous functions on I_2.

Remark 7. When $Q(X) = 0$, the linear equation (29) is said to be homogeneous; otherwise, it is nonhomogeneous, i.e., $Q_1(x_0 - x_1\sqrt{t}) = 0$ and $Q_2(x_0 + x_1\sqrt{t}) = 0$. Such that, we have two homogeneous DEs:

$$Y' + P(X)Y = 0 \;\; \Leftrightarrow \begin{cases} (y_0 - y_1\sqrt{t})' + (y_0 - y_1\sqrt{t})P_1(x_0 - x_1\sqrt{t}) = 0 & (31-1) \\ (y_0 + y_1\sqrt{t})' + (y_0 + y_1\sqrt{t})P_2(x_0 + x_1\sqrt{t}) = 0 & (31-2) \end{cases} \qquad \textbf{(31)}$$

Definition 21. Finding Y the solution to the WFC linear ODE (29) is equivalently to find the solutions $y_0 - y_1\sqrt{t}$, $y_0 + y_1\sqrt{t}$ to the linear ODEs $(30-1), (30-2)$, respectively, where

$Y = \varphi^{-1}(y_0 - y_1\sqrt{t}, y_0 + y_1\sqrt{t})$.

Find the General Solution to the Linear First-order WFC-ODE:

FIRST STEP: Solve the homogeneous equation $\;Y' + P(X)Y = 0$ **(32)**

To do that we have to solve the homogeneous equations $(31-1)$ and $(31-2)$ which are equivalent to the following separable DEs:

$$\begin{cases} \frac{d(y_0 - y_1\sqrt{t})}{y_0 - y_1\sqrt{t}} + P_1 d\left(x_0 - x_1\sqrt{t}\right) = 0 & (33-1) \\ \frac{d(y_0 + y_1\sqrt{t})}{y_0 + y_1\sqrt{t}} + P_2 d\left(x_0 + x_1\sqrt{t}\right) = 0 & (33-2) \end{cases} \qquad \textbf{(33)}$$

by integrating and solving $(33-1)$ and $(33-2)$ for $y_0 - y_1\sqrt{t}$ and $y_0 + y_1\sqrt{t}$, respectively:

$$\begin{cases} Y_{h1} = C_1 e^{-\int P_1(x_0 - x_1\sqrt{t})d(x_0 - x_1\sqrt{t})} \\ Y_{h2} = C_2 e^{-\int P_2(x_0 + x_1\sqrt{t})d(x_0 + x_1\sqrt{t})} \end{cases}$$

$$\Rightarrow Y_h = \varphi^{-1}(Y_{h1}, Y_{h2}) = \varphi^{-1}(C_1, C_2)\varphi^{-1}\left(e^{-\int P_1 d(x_0 - x_1\sqrt{t})}, e^{-\int P_2 d(x_0 + x_1\sqrt{t})}\right)$$

After calculating:

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$$C = \frac{1}{2}[C_1 + C_2] + \frac{1}{2\sqrt{t}}J[C_2 - C_1]$$

$$e^{-\int P(X)d(X)} = \frac{1}{2}[e^{-\int P_1 d(x_0 - x_1\sqrt{t})} + e^{-\int P_2 d(x_0 + x_1\sqrt{t})}] + \frac{1}{2\sqrt{t}}J[e^{-\int P_2 d(x_0 + x_1\sqrt{t})} - e^{-\int P_1 d(x_0 - x_1\sqrt{t})}]$$

We find that the solution to homogeneous WFC-DE (32) is as follows:

$$Y_h = Ce^{-\int P(X)dX}$$

SECOND STEP: Find the particular solutions to the nonhomogeneous equations $(30-1)$ and $(30-2)$, respectively are:

$$\begin{cases} Y_{r1} = e^{-\int P_1(x_0 - x_1\sqrt{t})d(x_0 - x_1\sqrt{t})} \int Q_1(x_0 - x_1\sqrt{t}) e^{\int P_1(x_0 - x_1\sqrt{t})d(x_0 - x_1\sqrt{t})} d(x_0 - x_1\sqrt{t}) \\ Y_{r2} = e^{-\int P_2(x_0 + x_1\sqrt{t})d(x_0 + x_1\sqrt{t})} \int Q_2(x_0 + x_1\sqrt{t}) e^{\int P_2(x_0 + \sqrt{t})d(x_0 + x_1\sqrt{t})} d(x_0 + x_1\sqrt{t}) \end{cases}$$

$$\Rightarrow Y_r = \varphi^{-1}(Y_{r1}, Y_{r2}) = e^{-\int P(X)dX} \int Q(X) e^{\int P(X)dX} dX$$

THIRD STEP: We know that the general solutions to each $(30-1)$ and $(30-2)$is the sum of two solutions:

$$\begin{cases} y_0 - y_1\sqrt{t} = Y_{h1} + Y_{r1} \\ y_0 + y_1\sqrt{t} = Y_{h2} + Y_{r2} \end{cases}$$

Then, the general solution to (29) can be written as the sum of two solutions:

$$Y = Y_h + Y_r$$

where $= \varphi^{-1}(y_0 - y_1\sqrt{t}, y_0 + y_1\sqrt{t})$,

$Y_h = \varphi^{-1}(Y_{h1}, Y_{h2})$ is the solution of the associated homogeneous equation (32) and

$Y_r = \varphi^{-1}(Y_{r1}, Y_{r2})$) is a particular solution to the nonhomogeneous (29).

Example 9

Let the WFC-DE is $XY' - 2Y - X^3e^X = 0$

We can write it as follows:

$$Y' - \frac{2}{X}Y = X^2e^X.$$

It is a Linear 1st order WFC-DE, we can solve it using the previous steps, where

$$P(X) = -\frac{2}{X} = \varphi^{-1}(-\frac{2}{x_0 - x_1\sqrt{t}}, -\frac{2}{x_0 + x_1\sqrt{t}})$$

$$Q(X) = X^2e^X = \varphi^{-1}((x_0 - x_1\sqrt{t})^2 e^{x_0 - x_1\sqrt{t}}, (x_0 + x_1\sqrt{t})^2 e^{x_0 + x_1\sqrt{t}})$$

The solution to homogeneous associated DE is $Y_h(X) = \varphi^{-1}(Y_{h1}, Y_{h2}) = CX^2$

$$\text{where} \begin{cases} Y_{h1} = C_1 e^{\int \frac{2}{x_0 - x_1\sqrt{t}} d(x_0 - x_1\sqrt{t})} = C_1(x_0 - x_1\sqrt{t})^2 \\ Y_{h2} = C_2 e^{\int \frac{2}{x_0 + x_1\sqrt{t}} d(x_0 + x_1\sqrt{t})} = C_2(x_0 + x_1\sqrt{t})^2 \end{cases} ; C = \varphi^{-1}(C_1, C_2)$$

The particular solution to nonhomogeneous associated DE is

$$Y_r(X) = e^{\int X^2 dX} \int X^2 e^X e^{-\int X^2 dX} dX = X^2 \int e^X dX = X^2 e^X$$

where,

$$\begin{cases} Y_{r1} = e^{\int \frac{2}{x_0 - x_1\sqrt{t}} d(x_0 - x_1\sqrt{t})} \int (x_0 - x_1\sqrt{t})^2 e^{x_0 - x_1\sqrt{t}} e^{-\int \frac{2}{x_0 - x_1\sqrt{t}} d(x_0 - x_1\sqrt{t})} d(x_0 - x_1\sqrt{t}) = (x_0 - x_1\sqrt{t})^2 e^{x_0 - x_1\sqrt{t}} \\ Y_{r2} = e^{\int \frac{2}{x_0 + x_1\sqrt{t}} d(x_0 + x_1\sqrt{t})} \int (x_0 + x_1\sqrt{t})^2 e^{x_0 + x_1\sqrt{t}} e^{-\int \frac{2}{x_0 + x_1\sqrt{t}} d(x_0 + x_1\sqrt{t})} d(x_0 + x_1\sqrt{t}) = (x_0 + x_1\sqrt{t})^2 e^{x_0 + x_1\sqrt{t}} \end{cases}$$

The general solution to the given DE is

$$Y = Y_h + Y_r$$
$$= CX^2 + X^2e^X.$$

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5. Conclusion

In this paper, we have defined the Weak Fuzzy Complex Differential Equation depending on a special isomorphic transformation function which helps us to deal easily with two ODEs in R then come back to F_J to get the general solution to WFC-ODEs. Therefore, we introduce some types of WFC-ODEs of first-order first-degree (Separable - Exact - Linear) with their general solutions and some explicit examples. In the future, we aim to solve WFC Initial Value Problems.

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