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Some fixed point results concerning various contractions in extended *b*- metric space endowed with a graph

Neeraj Kumar [a,1], Seema Mehra [a,1], Dania Santina [b,1], Nabil Mlaiki [b,c,*,1]

[a] *Department of Mathematics, Maharshi Dayanand University, Rohtak 124001, India*
[b] *Department of Mathematics and Sciences, Prince Sultan University, Riyadh 11586, Saudi Arabia*
[c] *Jadara University Research Center, Jadara University, Jordan*

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ABSTRACT

Contraction type mappings are crucial for understanding fixed point theory under specific conditions. We propose generalized (Boyd–Wong) type A **F** and (S - N) rational type contractions in an enlarged b-metric space which are represented by a graphically. Also, we gave a contrast of generalized (Boyd–Wong) type A **F** — contraction in 2D and 3D. We use appropriate illustrations to demonstrate the validity and primacy of our outcomes. Additionally, we use our derived conclusions to solve the Fredholm integral problem.

1. Introduction

Fixed point theory is one of the most celebrated and conventional theories in mathematics has comprehensive applications in different fields. In the 20th century, it was because of Frechet [1] who entered on the notion of metric space, and further because of its validity and practicable execution the notion has been extrapolated in various directions. The most pivotal principle in fixed point theory was given by Banach [2] and kept the astonished status the same. In this principle, the contractive mapping is necessarily continuous while it is not applicable in the case of discontinuity. The major drawback of this principle is how we apply this contractive mapping in case of discontinuity. Kannan [3] previously addressed this issue by demonstrating a fixed point outcome without continuity. In 1972, Chaterjea [4] proved a result independent of the Banach contraction principle and Kannan fixed point theorem. Later on, Fisher [5] presented rational inequality in fixed point theory and established a fixed point result in all metric spaces. Recent research has extended the concept of metric space and the Banach contraction principle to accomplish this result in fixed point theory. Further, researchers have applied fixed point outcomes to ordinary differential and integral equations, ensuring the uniqueness and existence of solutions. There are lots of extensions and generalizations of metric space. As a generalization of metric space, Bakhtin [6] established the postulate of b-metric space which was further upgraded by Czerwik [7], and for more novel information on can see [8–14].

In 2017, Several researchers [15] initiated concept of extended b-metric space. Exploring the metric space into an extended b-metric space allows for a fresh examination of fixed points that satisfied multiple contraction axioms, ensuring their uniqueness and existence see in [16–23]. As we know fixed point theory includes research on contraction mappings and generalized metric

* Corresponding author at: Department of Mathematics and Sciences, Prince Sultan University, Riyadh 11586, Saudi Arabia.
E-mail addresses: neerajrewariya78@gmail.com (N. Kumar), sberwal2007@gmail.com (S. Mehra), dsantina@psu.edu.sa (D. Santina), nmlaiki@psu.edu.sa (N. Mlaiki).

[1] All authors contributed equally and significantly in writing this article.

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spaces, and we generalized the results of Singh et al. [24] and Almari and Ahmed [25]. Further, we provide the generalization (Boyd–Wong) type and (S - N) rational type contraction in a extended b-metric space. We use 2D and 3D graphs to represent our driven outcomes visually. In addition to this, we find the solution of the Fredholm integral equation with the help of our established results.

2. Definitions and preliminaries

Definition 2.1 (*[26]*). "Consider a function $\theta : \Upsilon\times\Upsilon \to [1,\infty)$ with a non-empty set $\Upsilon \neq \phi$.

A function $\varpi_\theta : \Upsilon\times\Upsilon \to [0,\infty)$ is an extended b-metric space if it satisfies the following axioms for all $\xi,\ \beta,\ \gamma \in \Upsilon$.

(ϖ_θ1) $\varpi_\theta(\xi,\ \beta) = 0$ iff $\xi = \beta$

(ϖ_θ2) $\varpi_\theta(\xi,\ \beta)= \varpi_\theta(\beta,\ \xi)$

(ϖ_θ3) $\varpi_\theta(\xi,\ \gamma) \leq \theta(\xi,\ \gamma)[\varpi_\theta(\xi,\ \beta)+\varpi_\theta(\beta,\ \gamma)]$.

So, the pair (Υ, ϖ_θ) is a extended b-metric space".

For more information about convergence, completeness and Cauchy see in [15].

Definition 2.2 (*[27]*). "Let T be self map on Υ and $\alpha : \Upsilon\times\Upsilon \to [0,\infty)$ be function. We say T is α-admissible if $\xi,\ \beta\in \Upsilon$, $\alpha(\xi,\ \beta)\Rightarrow\alpha(\mathrm{T}\xi,\ \mathrm{T}\beta) \geq 1$".

Definition 2.3 (*[28]*). "An α- admissible map T is said to be triangular α-admissible if $\xi,\ \beta,\ \gamma\in \Upsilon$, $\alpha(\xi,\ \gamma)\geq 1$, $\alpha(\gamma,\ \beta)\geq 1 \Rightarrow \alpha(\xi,\ \beta)\geq 1$".

Definition 2.4 (*[29]*). "Let **F**: $\mathbb{R}^+\to\mathbb{R}$ be a mapping satisfies:

(**F1**) **F** is strictly increasing.

(**F2**) For any sequence $\{\beta_n\}$ of positive numbers $\lim_{n\to\infty} (\beta_n) = 0$ iff $\lim_{n\to\infty} \mathbf{F}(\beta_n)$ = - ∞

(**F3**). There exists $\alpha\in(0,\ 1)$, such that $\lim_{\alpha\to 0^+} \beta^\alpha\mathbf{F}(\beta) = 0$".

Definition 2.5 (*[26]*). "Let **F**: $\mathbb{R}^+\to\mathbb{R}$ be a increasing function and $\{\beta_n\}$ be a sequence of positive real numbers. Then the following axioms hold:

(1) $\lim_{n\to\infty} \mathbf{F}(\beta_n) = -\infty$ then $\lim_{n\to\infty} (\beta_n) = 0$.

(2)If inf**F**= - ∞ and $\lim_{n\to\infty} (\beta_n) = 0$, then $\lim_{n\to\infty} \mathbf{F}(\beta_n) = -\infty$".

Secelean [26]" reintegrated the condition (**F2**) by more elementary condition i.e. (**F2′**) (**F2′**) inf**F** = - ∞ or also by (**F2″**), there exists a sequence $\{\beta_n\}$ of positive real numbers such that $\lim_{n\to\infty} \mathbf{F}(\beta_n) = -\infty$".

Currently, Piri et al. [30]" used the following (**F3′**) in place of (**F3**).

(**F3′**), **F** is continuous on (0, ∞)

We denote the set of all functions (**F1**), (**F2′**), (**F3′**) by Λ".

Consider Φ which is the set of functions ϕ:[0, ∞)→[0, ∞) with ϕ is monotonic increasing as well as continuous and $\phi(\alpha)<\alpha$ for any $\alpha> 0$. Let Ψ symbolize the collection of non-increasing functions ψ:(0, ∞)→(0, ∞).

Before discussing how we demonstrate the convergence of a sequence in a graph let us discuss the basic notions about graph theory. Motivated by Jachmski [31], throughout in this note, let Σ be the diagonal product of $\Upsilon\times\Upsilon$. Let $\breve{G}$ stand for a graph and V($\breve{G}$) be the set of vertices that coincide with Υ and E($\breve{G}$) be the set of edges containing all loops. Consider the set

E($\breve{G}^{-1}$)= {(ξ, α)∈$\Upsilon\times\Upsilon$:(ξ, α)∈E($\breve{G}$)} and E($\grave{G}$) = E($\breve{G}$) ⋃ E($\breve{G}^{-1}$). Throughout in this note, graph $\breve{G}$ stands for $\breve{G}$ = (V($\breve{G}$), E($\breve{G}$). For more axioms of graph and fixed point combination, see in [29,32,33].

Definition 2.6. Let (Υ, ϖ_θ) be an extended b-metric space and $\breve{G}$ symbolize a graph in which V($\breve{G}$) = Υ and E($\breve{G}$) = {(ξ, α):(ξ, α)∈$\Upsilon\times\Upsilon$}. Then

(1) A sequence $\{\xi_n\}$ of Υ is converges to a point ξ of Υ if, $(\xi_n,\ \xi)\in$ E($\breve{G}$), $\lim_{n\to\infty}\varpi_\theta(\xi_n,\ \xi)= 0$

(2) A Cauchy sequence is convergent iff $(\xi_n,\ \xi_m)\in$ E($\breve{G}$), $\lim_{n,m\to\infty}\varpi_\theta(\xi_n,\ \xi_m) = 0$

(3) A space (Υ, ϖ_θ) is complete ⇔ any Cauchy sequence is convergent.

3. Main results

Here, we establish generalized (Boyd–Wong) type A **F** — contraction which is defined as:

Definition 3.1. A complete extended b-metric space (Υ, ϖ_θ) on $\Upsilon\neq \phi$ endowed with a graph $\breve{G}$ = (V, E) and Υ = V($\breve{G}$), {(ξ, β):(ξ, β)∈ $\Upsilon \times \Upsilon$} = E($\breve{G}$), where α is triangular α-admissible map along with two self maps S, T on Υ. Then the pair (S, T) satisfies the axiom of (Boyd–Wong) type A **F** — contraction, if $\phi \in \Phi$, **F** ∈Λ, ψ ∈Ψ and for any ξ, $\beta \in \Upsilon$ = V($\breve{G}$), σ, k> 1 with $\varpi_\theta(\mathrm{S}\xi,\ \mathrm{T}\beta) > 0$

$$\varpi_\theta(\xi,\beta)\alpha(\xi,\beta)\mathbf{F}(\sigma^k\varpi_\theta(S\xi,T\beta)) \leq \mathbf{F}(\chi(\xi,\beta)) - \psi(\varpi_\theta(\xi,\beta)) \tag{3.1}$$

$$\chi(\xi,\beta) = max\left\{\phi(\varpi_\theta(\xi,\beta)), \phi(\varpi_\theta(\xi,T\xi)), \phi(\varpi_\theta(\beta),T\beta), \phi\left(\frac{\varpi_\theta(\xi,T\beta)+\varpi_\theta(\beta,S\xi)}{2\sigma}\right)\right\} \tag{3.2}$$

Lemma 3.1. *A complete extended b-metric space $(\Upsilon, \varpi_\theta)$ $(\Upsilon \neq \phi)$ endowed with a graph $\breve{G}$ and two self maps S, T on Υ satisfies the axioms (3.1) and (3.2). Then S or T have a fixed point $\xi \in \Upsilon$ which is unique.*

Proof. First of all, consider a point ξ of $\Upsilon = V(\breve{G})$ such that $S(\xi) = \xi$ also we demonstrate i.e. $T(\xi)= \xi$. For this let $\varpi_\theta(S\xi, T\xi) > 0$. By using (3.1) and (3.2), we write

$$\mathbf{F}(\varpi_\theta(S\xi,T\xi)) \leq \varpi_\theta(\xi,\beta)\alpha(\xi,\xi)\mathbf{F}(\sigma^k\varpi_\theta(S\xi,T\xi)) \leq \mathbf{F}(\chi(\xi,\xi)) - \psi(\varpi_\theta(\xi,\xi)) \tag{3.3}$$

where,

$$\mathbf{F}(\chi(\xi,\xi)) - \psi(\varpi_\theta(\xi,\xi)) =$$

$$\mathbf{F}(max\left\{\phi(\varpi_\theta(\xi,\xi)), \phi(\varpi_\theta(\xi,T\xi)), \phi(\varpi_\theta(\xi),T\xi)), \phi(\left(\frac{\varpi_\theta(\xi,T\xi)+\varpi_\theta(\xi,S\xi)}{2\sigma}\right)\right\}) - \psi(\varpi_\theta(\xi,\xi)) < \mathbf{F}(\varpi_\theta(\xi,S\xi)) \tag{3.4}$$

From (3.3) and (3.4), we arrive at a contradiction. So $\varpi_\theta(S\xi, T\xi) = 0$ i.e. points are equal so ξ is common fixed point for S and T. In addition to this if $\varpi_\theta(\xi, \xi)= 0$ and by replicating the same exercise as discussed above we derive $\mathbf{F}(\varpi_\theta(\xi, \xi))<\mathbf{F}(\varpi_\theta(\xi, \xi))$, which is again contradiction implies $\varpi_\theta(\xi, \xi)= 0$. To prove ξ is unique choose β to be also a fixed point of S and T. Further, let $\varpi_\theta(\xi, \xi)>$ 0 then from (3.2) and (3.3) we write,

$$\mathbf{F}(\varpi_\theta(S\xi,T\beta)) \leq \varpi_\theta(\xi,\beta)\alpha(\xi,\xi)\mathbf{F}(\sigma^k\varpi_\theta(S\xi,T\beta)) \leq \mathbf{F}(\chi(\xi,\beta)) - \psi(\varpi_\theta(\xi,\beta)) < \mathbf{F}\varpi_\theta(\xi,\beta) \tag{3.5}$$

thus $\mathbf{F}(\varpi_\theta(\xi, \beta)) < \mathbf{F}(\varpi_\theta(\xi, \beta))$. Hence we determine that $\varpi_\theta(\xi, \beta) = 0$ implies $\xi = \beta$. So S and T have unique fixed point. □

Theorem 3.1. *A complete extended b-metric space $(\Upsilon, \varpi_\theta)$ $(\Upsilon \neq \phi)$ endowed with a graph $\breve{G} = (V, E)$ and two self maps S, T on Υ which please the axioms i.e. S is α-admissible, there exists ξ in Υ such that $\alpha(\xi, S\xi) \geq 1$ and S, T satisfies generalized (Boyd–Wong) type A* ***F****- contraction. Then S, T have a fixed point $\xi \in \Upsilon$ which is unique.*

Proof. Consider the point ξ_0 of $\Upsilon = V(\breve{G})$ and $\alpha(\xi_0, S\xi_0) \geq 1$. Now, we choose a sequence $\{\xi_n\}$ such that

$$S\xi_n = \xi_{2n+1} \text{ and } T\xi_{2n+1} = \xi_{2n+2}\ \forall\ n \in \mathbb{N} \tag{3.6}$$

As S is α admissible map so $\alpha(\xi_0, \xi_1) = \alpha(\xi_0, S\xi_0) \geq 1$ implies $\alpha(\xi_1, \xi_2)\Rightarrow\alpha(S\xi_0, S\xi_1)\geq 1$. Thus, we can write $\alpha(\xi_n, \xi_{n+1})\geq 1\ \forall$ n $\in\mathbb{N}$. Let $\xi_m = \xi_{m+1}\forall$ m $\in\mathbb{N}$ in addition to this if $\xi_{2n} = \xi_{2n+1}$ and $\varpi_\theta(\xi_{2n}, \xi_{2n+1}) = 0$. Now, from **F1** and using Definition 3.1, we write

$$\begin{aligned}\mathbf{F}(\varpi_\theta(\xi_{2n+1},\xi_{2n+2})) &\leq \mathbf{F}(\sigma^k\varpi_\theta(S\xi_{2n},T\xi_{2n+1}))\\ &\leq \varpi_\theta(\xi_{2n+1},\xi_{2n+2})\alpha(\xi_{2n},\xi_{2n+1})\mathbf{F}(\sigma^k\varpi_\theta(S\xi_{2n},T\xi_{2n+1}))\\ &\leq \mathbf{F}(\chi(\xi_{2n+1},\xi_{2n+2})) - \psi(\varpi_\theta(\xi_{2n},\xi_{2n+1}))\ and\end{aligned} \tag{3.7}$$

Also, we write,

$$\chi(\xi_{2n+1},\xi_{2n+2}) = max\{\phi(\varpi_\theta(\xi_{2n},\xi_{2n+1}))\} \tag{3.8}$$

Using (3.7) in (3.6) and by the postulate of ϕ and ψ we obtain

$\mathbf{F}(\varpi_\theta(\xi_{2n+1}, \xi_{2n+2}) < \mathbf{F}(\varpi_\theta(\xi_{2n+1}, \xi_{2n+2})$, thus we get a contradiction so $\varpi_\theta(\xi_{2n+1}, \xi_{2n+2}) = 0$. This implies $\xi_{2n+1} = \xi_{2n} = \xi_{2n+2} = \xi_{2n+3}=$.... which leads to $S\xi_{2n} = T\xi_{2n} = \xi_{2n}$. Thus, ξ_{2n} is a fixed point of S and T. Now, we assume that $\xi_m \neq \xi_{m+1}\ \forall$ m $\in\mathbb{N}$ and $\varpi_\theta(\xi_{2n+1}, \xi_{2n+2}) \geq 0$. Now, we using condition (3.1) of Definition 3.1 we write,

$$\begin{aligned}\mathbf{F}(\varpi_\theta(\xi_{2n+1},\xi_{2n})) &\leq \mathbf{F}(\sigma^k\varpi_\theta(S\xi_{2n},T\xi_{2n-1}))\\ &\leq \varpi_\theta(\xi_{2n},\xi_{2n-1})\alpha(\xi_{2n},\xi_{2n-1})\mathbf{F}(\sigma^k\varpi_\theta(S\xi_{2n},T\xi_{2n-1}))\\ &\leq \mathbf{F}(\chi(\xi_{2n1},\xi_{2n-1})) - \psi(\varpi_\theta(\xi_{2n},\xi_{2n-1}))\end{aligned} \tag{3.9}$$

Also,

$$\begin{aligned}\chi(\xi_{2n},\xi_{2n-1}) &= max\left\{\phi(\varpi_\theta(\xi_{2n},\xi_{2n-1})), \phi(\varpi_\theta(\xi_{2n},\xi_{2n+1})), \phi\left(\frac{\varpi_\theta(\xi_{2n},\xi_{2n})+\varpi_\theta(\xi_{2n-1},\xi_{2n+1})}{2\sigma}\right)\right\}\\ &= max\left\{\phi(\varpi_\theta(\xi_{2n},\xi_{2n-1})), \phi(\varpi_\theta(\xi_{2n},\xi_{2n+1})), \phi(\varpi_\theta(\xi_{2n-1},\xi_{2n})), \phi\left(\frac{\varpi_\theta(\xi_{2n-1},\xi_{2n})+\varpi_\theta(\xi_{2n},\xi_{2n+1})-\varpi_\theta(\xi_{2n},\xi_{2n}+\varpi_\theta(\xi_{2n},\xi_{2n}))}{2\sigma}\right)\right\}\\ &= max\{\phi(\varpi_\theta(\xi_{2n},\xi_{2n-1})), \phi(\varpi_\theta(\xi_{2n},\xi_{2n+1}))\}.\end{aligned}$$

Now, if

$$\chi(\xi_{2n},\xi_{2n-1})=\phi(\varpi_\theta(\xi_{2n},\xi_{2n+1}))\ \forall\ n\in\mathbb{N}\bigcup 0\ \textit{and consider}\ (3.9)\ \textit{we write}$$

$\mathbf{F}(\varpi_\theta(\xi_{2n+1},\ \xi_{2n})<\mathbf{F}(\varpi_\theta(\xi_{2n+1},\ \xi_{2n})$. Thus we arrive at a contradiction consequently,

$$\chi(\xi_{2n},\xi_{2n-1})=\phi(\varpi_\theta(\xi_{2n},\xi_{2n+1})) \tag{3.10}$$

Using (3.9) and by the property of ϕ and ψ we obtain

$$\mathbf{F}(\varpi_\theta(\xi_{2n+1},\xi_{2n}))\leq\mathbf{F}(\phi(\varpi_\theta(\xi_{2n},\xi_{2n-1}))-\psi(\varpi_\theta(\xi_{2n},\xi_{2n-1}))) \tag{3.11}$$

This leads to

$$\varpi_\theta(\xi_{2n+1},\xi_{2n})<\varpi_\theta(\xi_{2n},\xi_{2n-1})$$

Here, we observe that $\{\varpi_\theta(\xi_{2n+1},\ \xi_{2n}\}$ is a decreasing sequence of positive real numbers and by using the condition of ϕ and (3.10), we get

$$\mathbf{F}(\varpi_\theta(\xi_{2n+1},\xi_{2n}))<\mathbf{F}(\phi(\varpi_\theta(\xi_{2n-1},\xi_{2n-2})))-\psi(\varpi_\theta(\xi_{2n-1},\xi_{2n-2}))-\psi(\varpi_\theta(\xi_{2n},\xi_{2n-1})). \tag{3.12}$$

Since ψ is also non-increasing function, the above inequality

$$\mathbf{F}(\varpi_\theta(\xi_{2n+1},\xi_{2n}))<\mathbf{F}(\phi(\varpi_\theta(\xi_{2n-1},\xi_{2n-2}))-2\psi(\varpi_\theta(\xi_{2n-1},\xi_{2n-2})))$$

Repeat the above procedure, and we get

$$\mathbf{F}(\varpi_\theta(\xi_{2n+1},\xi_{2n}))<\mathbf{F}(\phi(\varpi_\theta(\xi_0,\xi_1)))-2\psi(\varpi_\theta(\xi_0,\xi_1)) \tag{3.13}$$

Also,

$$\mathbf{F}(\varpi_\theta(\xi_{2n+2},\xi_{2n+1}))<\mathbf{F}(\phi(\varpi_\theta(\xi_0,\xi_1))-2(n+1)\psi(\varpi_\theta(\xi_0,\xi_1))) \tag{3.14}$$

Since $\mathbf{F}\in\Lambda$ and n→∞ in (3.13) and (3.14) we obtain
$\lim_{n\to\infty}\varpi_\theta(\xi_n,\ \xi_{n+1})$= -∞. By using (**F2**′) and Lemma3.1 we get

$$\lim_{n\to\infty}\varpi_\theta(\xi_n,\xi_{n+1})=0. \tag{3.15}$$

In addition to this

$$\lim_{n\to\infty}\varpi_\theta(\xi_n,\xi_n)=0. \tag{3.16}$$

Now, we examine that $\{\xi_n\}$ is a Cauchy sequence in Υ, for this, we have to exhibit that $\{\xi_{2n}\}$ is a Cauchy sequence in Υ. On the contrary, there ∃ δ>0 so that for any integer c such that n(c)>m(k)≥ c and

$$\varpi_\theta(\xi_{2m(c)},\xi_{2n(c)})\geq\delta. \tag{3.17}$$

Let m(c) be the least positive integer which exceeds n(c) and satisfies (3.17) and

$$\varpi_\theta(\xi_{2m(c)},\xi_{2n(c)-1})\geq\delta. \tag{3.18}$$

By using inequality of triangle and (3.17) we write

$$\delta\leq\varpi_\theta(\xi_{2m(c)},\xi_{2n(c)})\leq\theta\varpi_\theta(\xi_{2m(c)},\xi_{2n(c)-1})\leq\theta\varpi_\theta(\xi_{2m(c)-1},\xi_{2m(c)})$$

Let c→∞ and using (3.18) we conclude

$$\frac{\delta}{\theta}\leq\lim_{c\to\infty}inf\,\varpi_\theta(\xi_{2m(c)},\xi_{2n(c)-1})\leq\lim_{c\to\infty}sup\varpi_\theta(\xi_{2m(c)},\xi_{2n(c)-1})\leq\delta \tag{3.19}$$

Besides this from (3.18) and (3.19) we get

$$\delta\leq\lim_{c\to\infty}sup\varpi_\theta(\xi_{2m(c)},\xi_{2n(c)-1})\leq\theta\delta. \tag{3.20}$$

Now,

$$\begin{aligned}\varpi_\theta(\xi_{2m(c)+1},\xi_{2n(c)})&\leq\theta\varpi_\theta(\xi_{2m(c)+1},\xi_{2m(c)})+\theta\varpi_\theta(\xi_{2m(c)},\xi_{2n(c)})\\&\leq\theta\varpi_\theta(\xi_{2m(c)+1},\xi_{2m(c)})+\theta^2\delta+\theta^2\varpi_\theta(\xi_{2m(c)-1},\xi_{2n(c)}).\end{aligned}$$

Which gives

$$\delta\leq\lim_{c\to\infty}sup\varpi_\theta(\xi_{2m(c)+1},\xi_{2n(c)})\leq\theta^2\delta.$$

Now we write,

$$\varpi_\theta(\xi_{2m(c)+1},\xi_{2n(c)})\leq\theta\varpi_\theta(\xi_{2m(c)+1},\xi_{2m(c)})+\theta\varpi_\theta(\xi_{2m(c)},\xi_{2n(c)-1})$$

$$\leq\lim_{c\to\infty}sup\varpi_\theta(\xi_{2m(c)+1},\xi_{2n(c)-1})\leq\theta\delta.$$

Now, from above expression and (3.19) we write

$$\lim_{c\to\infty} sup\varpi_\theta(\xi_{2m(c)},\xi_{2n(c)-1}) = 2\lim_{c\to\infty} sup\varpi_\theta(\xi_{2m(c)},\xi_{2n(c)-1}).$$

$$\frac{\delta}{2\theta} \le \lim_{c\to\infty} inf\,\varpi_\theta(\xi_{2m(c)},\xi_{2n(c)-1}) \le \lim_{c\to\infty} sup\varpi_\theta(\xi_{2m(c)},\xi_{2n(c)-1}) \le \frac{\delta}{2}. \quad (3.21)$$

Corresponding, we write

$$\lim_{c\to\infty} inf\,\varpi_\theta(\xi_{2m(c)},\xi_{2n(c)}) \le \frac{\delta}{2}. \quad (3.22)$$

$$\frac{\delta}{2\theta} \le \lim_{c\to\infty} inf\,\varpi_\theta(\xi_{2m(c)+1},\xi_{2n(c)}). \quad (3.23)$$

$$\lim_{c\to\infty} sup\varpi_\theta(\xi_{2m(c)+1},\xi_{2n(c)-1})\frac{\theta\delta}{2}. \quad (3.24)$$

Since

$$\varpi_\theta(\xi_{2m(c)},\xi_{2n(c)-1}) = \varpi_\theta(\xi_{2m(c)+1},\xi_{2n(c)}) > 0.$$

Then by contractive property (3.1) of Definition 3.1 along with axiom of ψ we derive

$$\begin{aligned}\mathbf{F}(\varpi_\theta(\xi_{2m(c)+1},\xi_{2n(c)})) &\le \mathbf{F}(\sigma^k\varpi_\theta(S\xi_{2m(c)},T\xi_{2n(c)-1}))\\ &\le \varpi_\theta(\xi_{2m(c)},\xi_{2n(c)-1})\alpha(\xi_{2m(c)},\xi_{2n(c)-1})\mathbf{F}(\sigma^k\varpi_\theta(S\xi_{2m(c)},T\xi_{2n(c)-1}))\\ &\le \mathbf{F}(\chi(\xi_{2m(c)},\xi_{2n(c)-1})) - \psi(\varpi_\theta(\xi_{2m(c)},\xi_{2n(c)-1}))\\ &\le \mathbf{F}(\chi(\xi_{2m(c)},\xi_{2n(c)-1}))\end{aligned} \quad (3.25)$$

By the property $\chi(\xi,\beta)$ of and using (3.21 −3.24), we get

$$\lim_{c\to\infty} sup\chi(\xi_{2m(c)},\xi_{2n(c)-1})\frac{\delta}{2}. \quad (3.26)$$

On solving $\chi(\xi_{2m(c)},\xi_{2n(c)-1})$ we get

$$\lim_{c\to\infty} sup\chi(\xi_{2m(c)},\xi_{2n(c)-1}) < max\left\{\frac{\delta}{2},0,0,\frac{1}{2\theta}\left[\frac{\theta+\delta}{2}\right]\right\}. \quad (3.27)$$

Moreover, using (3.23), (3.24) and (3.26) we make out

$$\mathbf{F}(\theta\frac{\delta}{2\theta}) \le \mathbf{F}(\lim_{c\to\infty} sup\chi(\xi_{2m(c)},\xi_{2n(c)-1}))$$

so $\mathbf{F}(\frac{\delta}{2})<\mathbf{F}(\frac{\delta}{2})$, this comes out a contradiction so $\{\xi_m\}$ is a Cauchy sequence in (Υ,ϖ_θ). Also (Υ,ϖ_θ) is complete the sequence $(\xi_n\}$ converges to $\xi\in\Upsilon$ = V($\breve{G}$) so

$\lim_{n\to\infty}\varpi_\theta(\xi_n,\xi)$= 0 = $\lim_{n\to\infty}\varpi_\theta(\xi,\xi)$. Now, we shall prove S$\xi$ = Tξ = ξ. As $\mathbf{F}\in$(F3′) $\in\Lambda$ and F is continuous so we have discussed the upcoming two cases,

CaseI: For any n in $\mathbb{N}$ there $\exists$ $\xi_n\in\mathbb{N}$ such that $\varpi_\theta(\xi_{n+1}$, Sξ) = 0 i.e. ξ_{n+1} = Sξ and $\xi_n>\xi_{n-1}$ with ξ_0 i.e.

$$\xi = \lim_{n\to\infty}\xi_{n+1} = \lim_{n\to\infty} S\xi = S\xi. \quad (3.28)$$

Thus ξ is fixed point of S.

CaseII: Let $n_2\in\mathbb{N}$ with $\varpi_\theta(\xi_{n+1}$, Sξ) ≠ 0 ∀ n ≥ n_2, i.e. $\varpi_\theta(\xi_n$, Sξ) > 0. Now, by using inequality (3.1) of Definition 3.1, we deduce that

$$\begin{aligned}\mathbf{F}(\varpi_\theta(S\xi,\xi_{2n+2})) &\le \varpi_\theta(S\xi,\xi_{2n+2})\alpha(\xi,\xi_{2n+1})\mathbf{F}(\sigma^k\varpi_\theta(S\xi,T\xi_{2n+1}))\\ &\le \mathbf{F}(\chi(\xi,\xi_{2n+1})) - \psi(\varpi_\theta(\xi,\xi_{2n+1})).\end{aligned} \quad (3.29)$$

Where

$$\chi(\xi,\xi_{2n+1}) = \varpi_\theta(\xi,S\xi). \quad (3.30)$$

Thus, we derive with the help of (3.29) and (3.30)

$$\mathbf{F}(\varpi_\theta(S\xi,\xi_{2n+2})) \le \mathbf{F}\varpi_\theta(\xi,S\xi)$$

for any n ≥ n_2. As $\mathbf{F}$ is continuous and letting n →∞ in the above inequality, we arrive at,

$$\mathbf{F}(\varpi_\theta(S\xi,\xi_{2n+2})) < \mathbf{F}\varpi_\theta(\xi,S\xi).$$

Which is a contradiction and from Lemma(3.1) ξ is a unique fixed point of S and T. □

Example 3.1. Let $\Upsilon\neq\phi$ be a non empty set where Υ = [0,30] = V($\breve{G}$) and

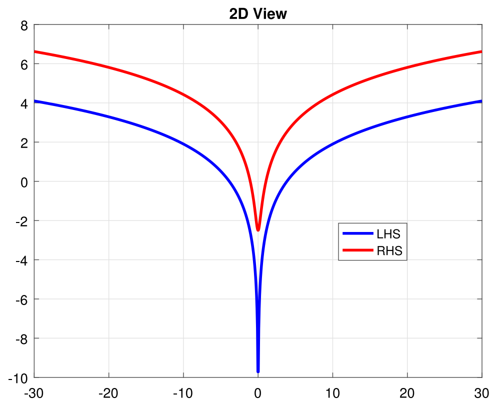

Fig. 1. LHS is depicted by blue surface while RHS is red.

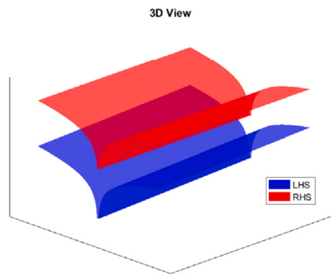

Fig. 2. LHS is depicted by blue surface while RHS is red.

E($\breve{G}$) = {(ξ, β):(ξ, β)∈ $Y \times Y$} then the function on Y defined as ϖ_θ = max{ξ, β} is a complete extended b-metric space where $\theta = 1$, $\sigma = 2$. Consider the maps S, T on [0, 30] which is defined as S(ξ) = $\frac{1}{1024}$log(1+ ξ^3)+ ξ^3 and T(ξ)= $\frac{1}{32}\xi^2$ exp(-ξ). Now, we derive the map $\alpha : Y \times Y \to [0, \infty)$ defined as = $\alpha(\xi, \beta) = 1$, (ξ, β) ∈[0, 30]. Clearly, the map is α-admissible and α(0, S0) = α(0,0) = 1. Consider the function ψ on (0, 30) defined as, $\psi(\xi) = \frac{1}{50(1+\xi)}$,

ϕ on [0, 30] given by, $\phi(\xi) = \frac{10\xi+1}{12}$ and $\mathbf{F}(\xi) = \log\xi$. Clearly, 0 is a unique fixed point of S and T. Now, we discuss the contractive condition (3.1) of Theorem (3.1), for this, we assume that for any ξ, β∈[0, 30] we consider $\xi \geq \beta$. If ξ, β∈[0, 30] and k = 1.1 then from contraction condition (3.1) of Definition 3.1 as shown in Fig. 1 and Fig. 2.(LHS with blue surface while RHS is red surface) we write,

$$L.H.S = \varpi_\theta(\xi,\beta)\alpha(\xi,\beta)\mathbf{F}(\sigma^k\varpi_\theta(S\xi,T\beta)) \leq \mathbf{F}(2^{k-5}max\left\{\frac{1}{1024}[\log(1+\xi^3)+\xi^3], \frac{1}{32}\beta^2\exp^{-\beta}\right\}^2)$$
$$\leq \mathbf{F}(2^{k-5}max\{\xi,\beta\}^2)$$
$$= \log(2^{k-5}\xi^2). \tag{3.31}$$

Also,

$$\chi(\xi,\beta) = \mathbf{F}(\phi(\xi^2)) - \psi(max\{\xi,\beta\}^2)$$
$$= \log\left(\frac{10\xi^2+1}{12} - \frac{1}{50(\xi^2+1)}\right) = R.H.S. \tag{3.32}$$

Proposition 3.1. *An extended b-metric space (Y, ϖ_θ) equipped by graph $\breve{G}$ = (V,E). Let $\theta \geq 1$ and considering two maps, S, T on Y. Derive a $\{\xi_n\}$ by $\{\xi_{2n+1}\} = S\xi_{2n}$ and $\xi_{2n+2} = T\xi_{2n+1}$, ∀ n = 0,1... If there ∃ a function, $\hbar$: $Y \times Y \to [0,1)$ which satisfies, $\hbar(TS\xi, \beta) \leq \hbar(\xi, \beta)$ and $\hbar(\xi, ST\beta,) \leq \hbar(\xi, \beta)$, ∀ ξ, $\beta \in Y$. Then $\hbar(\xi_{2n}, \beta) \leq \hbar(\xi_0, \beta)$ and $\hbar(\alpha, \xi_{2n+1}) \leq \hbar(\alpha, \xi_1)$.*

Proof. Since ξ, β∈Y= V($\breve{G}$) and n = 0,1,2... so we write

$$\hbar(\xi_{2n},\beta) = \hbar(TS\xi_{2n-2}\beta) \leq \hbar(\xi_{2n-2},\beta)$$

$$= \hbar(TS\xi_{2n-4}, \beta) \leq \hbar(\xi_{2n-4}, \beta) \leq \cdots \leq \hbar(\xi_0, \beta).$$

In similar manner, we derive

$$\hbar(\alpha, \xi_{2n+1}) = \hbar(\alpha, ST\xi_{2n-2}) \leq \hbar(\alpha, \xi_{2n-1})$$
$$= \hbar(\alpha, ST\xi_{2n-3}) \leq \hbar(\alpha, \xi_{2n-3}) \leq \cdots \leq \hbar(\alpha, \alpha_1). \quad \square$$

Definition 3.2. If (Υ, ϖ_θ) is a extended b-metric space, $\theta : \Upsilon \times \Upsilon \to [1, \infty)$ and graph $\breve{G}$ = (V, E) which contains loops, where Υ = V($\breve{G}$), E($\breve{G}$)= {(ξ, β):(ξ, β)∈$\Upsilon \times \Upsilon$} then the maps S, T on Υ is called (S - N) rational type contractive mappings if there ∃ control functions

$\hbar$, $\Bbbk$, $\hbar$: $\Upsilon\times\Upsilon\to[0, 1)$ implies

$$\varpi_\theta(S\xi, T\beta) \leq \theta(\xi, \beta)\left(\hbar(\xi, \beta)\varpi_\theta(\beta, S\beta) + \Bbbk(\xi, \beta)[\varpi_\theta(\xi, T\beta) + \varpi_\theta(\beta, S\xi)] + \hbar(\xi, \beta)\frac{\varpi_\theta(\xi, T\xi)\varpi_\theta(\beta, S\beta)}{1 + \varpi_\theta(\xi, S\beta) + \varpi_\theta(\beta, T\xi) + \varpi_\theta(\xi, \beta)} \right) \tag{3.33}$$

Theorem 3.2. *Let (Υ, ϖ_θ) be a extended b-metric space, $\theta : \Upsilon \times \Upsilon \to [1, \infty)$ and graph $\breve{G}$ = (V, E) which contains loops and the maps is (S - N) rational type which satisfies*

(a) $\hbar(TS\xi, \beta) \leq \hbar(\xi, \beta)$ *and* $\hbar(\xi, ST\beta) \leq \hbar(\xi, \beta)$

$\Bbbk(TS\xi, \beta) \leq \Bbbk(\xi, \beta)$ *and* $\Bbbk(\xi, ST\beta) \leq \Bbbk(\xi, \beta)$;

$\hbar(TS\xi, \beta) \leq \hbar(\xi, \beta)$ *and* $\hbar(\xi, ST\beta)$ *and* $\hbar(\xi, ST\beta) \leq \hbar(\xi, \beta)$

(b) $\hbar(\xi, \beta) + 2\theta\Bbbk(\xi, \beta) + \theta\hbar(\xi, \beta) < 1.$

Then S and T have a unique fixed point.

Proof. Let ξ_0 ∈Υ = V($\breve{G}$) and derive a sequence{ξ_n} by {ξ_{2n+1}} = Sξ_{2n} and ξ_{2n+2} = Tξ_{2n+1}, ∀ n = 0,1... and by (3.33) we deduce

$$\varpi_\theta(\xi_{2n+1}, \xi_{2n+2}) = \varpi_\theta(S\xi_{2n}, T\xi_{2n+1})$$
$$\leq \hbar(\xi_{2n}, \xi_{2n+1})\Bbbk(\xi_{2n}, \xi_{2n+1}) + [\varpi_\theta\xi_{2n}, \xi_{2n+1}]$$
$$+ \theta\hbar(\xi_{2n}, \xi_{2n+1})\frac{\varpi_\theta(\xi_{2n}, \xi_{2n+1})\varpi_\theta(\xi_{2n+1}, \xi_{2n+2})}{\theta + \varpi_\theta(\xi_{2n+1}, \xi_{2n+2})}$$
$$\leq \hbar(\xi_{2n}, \xi_{2n+1})\varpi_\theta(\xi_{2n}, \xi_{2n+1}) + \theta\Bbbk(\xi_{2n}, \xi_{2n+1})\varpi_\theta(\xi_{2n+1}, \xi_{2n+2})$$
$$+ \theta\hbar(\xi_{2n}, \xi_{2n+1})\varpi_\theta(\xi_{2n}, \xi_{2n+1}).$$

By using Proposition 3.1 we derive

$$\varpi_\theta(\xi_{2n+1}, \xi_{2n+2}) \leq \hbar(\xi_{2n}, \xi_{2n+1})\varpi_\theta(\xi_{2n}, \xi_{2n+1})\theta\Bbbk(\xi_{2n}, \xi_{2n+1})\varpi_\theta(\xi_{2n}, \xi_{2n+1})$$
$$+ \theta\Bbbk(\xi_{2n}, \xi_{2n+1})\varpi_\theta(\xi_{2n+1}, \xi_{2n+2}) + \theta\hbar(\xi_{2n}, \xi_{2n+1})\varpi_\theta(\xi_{2n}, \xi_{2n+1}).$$

Thus,

$$\varpi_\theta(\xi_{2n+1}, \xi_{2n+2}) \leq \frac{\hbar(\xi_0, \xi_1) + \theta\Bbbk(\xi_0, \xi_1) + \theta\hbar(\xi_0, \xi_1)}{1 - \theta\Bbbk(\xi_0, \xi_1)} \tag{3.34}$$

Also, in same way

$$\varpi_\theta(\xi_{2n+2}, \xi_{2n+3}) \leq \hbar(\xi_{2n+2}, \xi_{2n+1})\varpi_\theta(\xi_{2n+2}, \xi_{2n+1}) + \theta\Bbbk(\xi_{2n+2}, \xi_{2n+1})\varpi_\theta(\xi_{2n+2}, \xi_{2n+3})$$
$$+ \theta\hbar(\xi_{2n+2}, \xi_{2n+1})\varpi_\theta(\xi_{2n+1}, \xi_{2n+2}).$$

From Proposition 3.1 we derive

$$\varpi_\theta(\xi_{2n+2}, \xi_{2n+3}) \leq \hbar(\xi_0, \xi_1)\varpi_\theta(\xi_{2n+2}, \xi_{2n+1}) + \theta\Bbbk(\xi_0, \xi_1)\varpi_\theta(\xi_{2n+2}, \xi_{2n+3})$$
$$+ \theta\hbar(\xi_0, \xi_1)\varpi_\theta(\xi_{2n+1}, \xi_{2n+2}).$$

$$\varpi_\theta(\xi_{2n+2}, \xi_{2n+3}) \leq \frac{\hbar(\xi_0, \xi_1) + \theta\Bbbk(\xi_0, \xi_1) + \theta\hbar(\xi_0, \xi_1)}{1 - \theta\Bbbk(\xi_0, \xi_1)}\varpi_\theta(\xi_{2n+1}, \xi_{2n+2}). \tag{3.35}$$

$$Let, \quad \frac{\hbar(\xi_0, \xi_1) + \theta\Bbbk(\xi_0, \xi_1) + \theta\hbar(\xi_0, \xi_1)}{1 - \theta\Bbbk(\xi_0, \xi_1)} < \kappa. \tag{3.36}$$

So by using (3.35) and (3.36), we write

$$\varpi_\theta(\xi_n, \xi_{n+1}) \leq \kappa\varpi_\theta(\xi_{n-1}, \xi_n).$$

Now, we set up a sequence $\{\xi_n\}$ i.e

$$\varpi_\theta(\xi_n,\xi_{n+1}) \le \kappa\varpi_\theta(\xi_{n-1},\xi_n) \le \cdots .\kappa^n\varpi_\theta(\xi_0,\xi_1). \tag{3.37}$$

For $m>$ n, we deduce

$$\begin{aligned}\varpi_\theta(\xi_n,\xi_m) &\le \theta[\varpi_\theta(\xi_n,\xi_{n+1})+\varpi_\theta(\xi_{n+},\xi_m)]\\ &\le \theta\varpi_\theta(\xi_n,\xi_{n+1})\theta^2\varpi_\theta(\xi_{n+1},\xi_{n+2}) \le \cdots .+\theta^{m-1}\varpi_\theta(\xi_{m-1},\xi_m).\end{aligned} \tag{3.38}$$

Using (3.37) we get

$$\begin{aligned}\varpi_\theta(\xi_n,\xi_m) &\le \varpi_\theta(\xi_0,\xi_1)+\theta^2\kappa^{n+1}\varpi_\theta(\xi_0,\xi_1)+\cdots+\theta^{m-1}\kappa^{m-1}\varpi_\theta(\xi_0,\xi_1)\\ &\le \theta\kappa^n\left[1+(\theta\kappa)^1+(\theta\kappa)^2+\cdots+(\theta\kappa)^{m-n-1}\right]\varpi_\theta(\xi_0,\xi_1)\\ &\le \frac{\theta\kappa^n}{1-\theta\kappa}\varpi_\theta(\xi_0,\xi_1).\end{aligned} \tag{3.39}$$

As n→∞,

$$\varpi_\theta(\xi_n,\xi_m) \to 0.$$

Thus, the sequence $\{\xi_n\}$ is a Cauchy and Υ is a complete so there ∃ $\xi\in\Upsilon=$ V($\breve{G}$) such that $\{\xi_n\}\to\xi$ or $\lim_{n\to\infty}\xi_n=\xi$. Thus $\lim_{n\to\infty}\xi_{2n+1}=\xi$ and $\lim_{n\to\infty}\xi_{2n+2}=\xi$. Here, now we show that ξ is the fixed point of S and T, using (3.33) we get

$$\begin{aligned}\varpi_\theta(\xi,S\xi) &\le \theta[\varpi_\theta(\xi,T\xi_{2n+1})+\varpi_\theta(T\xi_{2n+1},S\xi)]\\ &\le \Bigg(\varpi_\theta(\xi,\xi_{2n+2})+\hbar(\xi,\xi_1)\varpi_\theta(\xi,\xi_{2n+2})+\theta\Bbbk(\xi,\xi_1)[\varpi_\theta(\xi,\xi_{2n+2})+\varpi_\theta(\xi_{2n+1},S\xi)]\\ &\quad+\theta\Bbbk(\xi_{2n},\xi_{2n+1})\varpi_\theta(\xi_{2n+1},\xi_{2n+2})+\theta\hbar(\xi,\xi_1)\frac{\varpi_\theta(\xi,S\xi)+\varpi_\theta(\xi_{2n+1},\xi_{2n+2})}{1+\varpi_\theta(\xi,\xi_{2n+2})+\varpi_\theta(\xi_{2n+1},S\xi)+\varpi_\theta(\xi,\xi_{2n+1})}\Bigg).\end{aligned}$$

As n→∞ in above expression, we deduce

$$\begin{aligned}\varpi_\theta(\xi,S\xi) &\le \theta\Bbbk(\xi,\xi_1)\varpi_\theta(\xi,S\xi)\\ &\le (\hbar(\xi,\xi_1))+2\theta(\Bbbk(\xi,\xi_1)+\theta\hbar(\xi,\xi_1)\varpi_\theta(\xi,S\xi))\\ &< \varpi_\theta(\xi,S\xi)\end{aligned}$$

which comes out a contradiction so S$\xi=\xi$ and in the same way we also demonstrate S$\xi=\xi$. Now, we prove that S and T have a unique fixed point. Let β be any other fixed point of S and T where $\xi\neq\beta$ and using (3.33)

$$\begin{aligned}\varpi_\theta(\xi,\beta)=\varpi_\theta(S\xi,T\beta) &\le \theta(\xi,\beta)\Bigg(\hbar(\xi,\beta)\varpi_\theta(\beta,S\beta)+\Bbbk(\xi,\beta)[\varpi_\theta(\xi,T\beta)+\varpi_\theta(\beta,S\xi)]\\ &\quad+\hbar(\xi,\beta)\frac{\varpi_\theta(\xi,T\xi)\varpi_\theta(\beta,S\beta)}{1+\varpi_\theta(\xi,S\beta)+\varpi_\theta(\beta,T\xi)+\varpi_\theta(\xi,\beta)}\Bigg)\\ &\le \hbar(\xi,\beta)\varpi_\theta(\xi,\beta)+2\theta\Bbbk(\xi,\beta)+\theta\hbar(\xi,\beta)\varpi_\theta(\xi,\beta)<1.\end{aligned}$$

Thus,

$$\varpi_\theta(\xi,\beta)=0 \Rightarrow \xi=\beta. \quad \square$$

Corollary 3.1. *Let $\Upsilon\neq\phi$ and (Υ, ϖ_θ) be a extended b-metric space, $\theta:\Upsilon\times\Upsilon\to[1,\infty)$ and graph $\breve{G}$ = (V, E). Let S, T be the self maps on Υ and if there ∃ control functions,*
$\hbar$, $\Bbbk$: $\Upsilon\times\Upsilon\to$[0, 1) which satisfies

$$(a)\hbar\ (TS\xi,\beta)\le\hbar(\xi,\beta)\ \ \text{and}\ \ \ \hbar\ (\xi,ST\beta)\le\hbar(\xi,\beta)$$

$$\Bbbk(TS\xi,\beta)\le\Bbbk(\xi,\beta) \text{and} \Bbbk(\xi,ST\beta)\le\Bbbk(\xi,\beta);$$

$$(b)\ \hbar(\xi,\beta)+2\theta\Bbbk(\xi,\beta)<1,$$

$$(c)\ \varpi_\theta(S\xi,T\beta)\le\theta(\xi,\beta)\Bigg(\hbar(\xi,\beta)\varpi_\theta(\beta,S\beta)+\Bbbk(\xi,\beta)[\varpi_\theta(\xi,T\beta)+\varpi_\theta(\beta,S\xi)]\Bigg). \tag{3.40}$$

So, S and T have a fixed point.

Proof. If we choose $\hbar(\xi,\beta)=0$ in Theorem 3.2 we get the required result. □

Corollary 3.2. *Let $\Upsilon \neq \phi$ $(\Upsilon, \varpi_\theta)$ be an extended b-metric space, $\theta : \Upsilon \times \Upsilon \to [1,\infty)$ and graph $\breve{G} = (V, E)$. Let S, T be the self maps on Υ and if there ∃ control functions,*
ℏ, ℏ: $\Upsilon\times \Upsilon \to [0, 1)$ which satisfies:

$$(a)\quad \hbar(TS\xi,\beta) \le \hbar(\xi,\beta) \text{ and } \hbar(\xi,ST\beta) \le \hbar(\xi,\beta)$$

$$\hbar(TS\xi,\beta) \le \hbar(\xi,\beta) \text{ and } \hbar(\xi,ST\beta) \text{ and } \hbar(\xi,ST\beta) \le \hbar(\xi,\beta);$$

$$(b)\quad \hbar(\xi,\beta) + \theta\hbar(\xi,\beta) < 1.$$

$$(c)\ \varpi_\theta(S\xi,T\beta) \le \theta(\xi,\beta)\left(\hbar(\xi,\beta)\varpi_\theta(\beta,S\beta) + \hbar(\xi,\beta)\frac{\varpi_\theta(\xi,T\xi)\varpi_\theta(\beta,S\beta)}{1+\varpi_\theta(\xi,S\beta)+\varpi_\theta(\beta,T\xi)+\varpi_\theta(\xi,\beta)}\right) \tag{3.41}$$

Then S and T have a fixed point.

Proof. Let $\Bbbk(\xi,\ \beta) = 0$ in Theorem 3.2 we deduce the required proof. □

Corollary 3.3. *Let $\Upsilon \neq \phi$ and $(\Upsilon, \varpi_\theta)$ be an extended b-metric space, $\theta : \Upsilon \times \Upsilon \to [1,\infty)$ and graph $\breve{G} = V((\breve{G}), E(\breve{G}))$ which contains loops. Let S, T be the self maps on Υ and if there ∃ control functions, ℏ: $\Upsilon\times\Upsilon\to [0, 1)$ which satisfies:*

$$(a)\ \hbar(TS\xi,\beta) \le \hbar(\xi,\beta) \text{ and } \hbar(\xi,ST\beta) \le \hbar(\xi,\beta)$$

$$(b)\ \hbar(\xi,\beta) < 1.$$

$$(c)\ \varpi_\theta(S\xi,T\beta) \le \theta(\xi,\beta)\hbar(\xi,\beta)\varpi_\theta(\beta,S\beta), \tag{3.42}$$

Then S and T have a fixed point.

Proof. Let $\Bbbk(\xi,\ \beta) = \hbar(\xi,\ \beta) = 0$ in Theorem 3.2 we deduce required proof. □

Corollary 3.4. *Let $\Upsilon\neq \phi$ and $(\Upsilon, \varpi_\theta)$ be an extended b-metric space, $\theta : \Upsilon \times \Upsilon \to [1,\infty)$ and graph $\breve{G} = (V, E)$. Let S be the map on Υ and if there ∃ control functions, ℏ: $\Upsilon\times\Upsilon\to[0,1)$ which satisfies:*

$$(a)\ \hbar(S\xi,\beta) \le \hbar(\xi,\beta) \text{ and } \hbar(\xi,S\beta) \le \hbar(\xi,\beta)$$

$$(b)\ \hbar(\xi,\beta) < 1.$$

$$(c)\ \varpi_\theta(\xi,T\beta) \le \theta(\xi,\beta)\hbar(\xi,\beta)\varpi_\theta(\beta,\beta), \tag{3.43}$$

Implies S has a fixed point.

Proof. Let $\Bbbk(\xi,\ \beta) = \hbar(\xi,\ \beta) = 0$ and S = I (identity map) in Theorem 3.2 we deduce required proof. □

Example 3.2. Let $\Upsilon = [0,\ 1)$ and $\varpi_\theta : \Upsilon \times \Upsilon \to \mathbb{C}$ defined as:

$$\varpi_\theta(\xi,\beta) = |\xi-\beta|^2$$

$\forall\ \xi,\ \beta \in \Upsilon$ = V(Υ), and E($\breve{G}$)) = {(ξ, β):(ξ, β)∈$\Upsilon\times\Upsilon$}. Define $\theta(\xi,\ \beta) = 2+ \max\{\xi,\ \beta\}$ then $(\Upsilon,\ \varpi_\theta)$ is complete extended b-metric space. Derive the maps S, T on [0, 1) as S(ξ) = $\frac{\xi}{3}$ and T(ξ) = $\frac{\xi}{4}$. Now, we define control functions, ℏ, 𝕜, ℏ: $\Upsilon\times\Upsilon\to$[0,1) as $\hbar = \frac{\xi}{17}+\frac{\beta}{20}$, $\Bbbk = \frac{\xi}{16}+\frac{\beta}{21}$ and
$\hbar = \frac{\xi\beta}{39}$. Now, we prove conditions (a) and (b),

$$\hbar(TS\xi,\beta) = \frac{\xi}{204}+\frac{\beta}{20} \le \frac{\xi}{17}+\frac{\beta}{20} = \hbar(\xi,\beta) \text{ and } \hbar(\xi,ST\beta) = \frac{\xi}{17}+\frac{\beta}{240} \le \frac{\xi}{17}+\frac{\beta}{20} = \hbar(\xi,\beta)$$

$$\Bbbk(TS\xi,\beta) = \frac{\xi}{192}+\frac{\beta}{21} \le \frac{\xi}{16}+\frac{\beta}{21} = \Bbbk(\xi,\beta) \text{ and } \Bbbk(\xi,ST\beta) = \frac{\xi}{16}+\frac{\beta}{252} \le \frac{\xi}{16}+\frac{\beta}{21} = \Bbbk(\xi,\beta);$$

$$\hbar(ST\beta,\xi) = \frac{\xi\beta}{468} \le \frac{\xi\beta}{39} = \hbar(\xi,\beta) \text{ and } \hbar(\xi,ST\beta) \text{ and } \hbar(\xi,ST\beta) = \frac{\xi\beta}{468} \le \frac{\xi\beta}{39}. \le \hbar(\xi,ST\beta).$$

Since, the condition (b) is also true i.e.

$$\hbar(\xi,\beta) + 2\theta\Bbbk(\xi,\beta) + \theta\hbar(\xi,\beta) < 1.$$

Now,

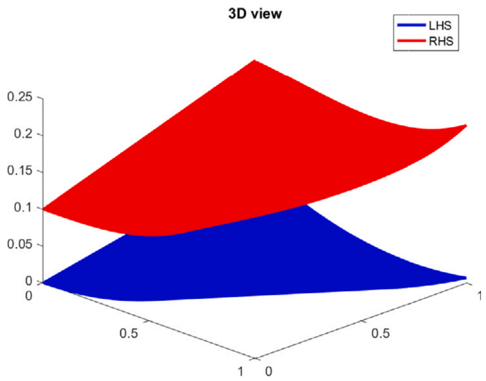

Fig. 3. LHS is depicted by blue surface while RHS is red.

$$\begin{aligned}\varpi_\theta(S\xi, T\beta) &= \left|\frac{\xi}{3} - \frac{\beta}{4}\right|^2 \\ &\le \left(\frac{\xi}{17} + \frac{\beta}{20}\right)|\xi-\beta|^2 + \left(\frac{\xi}{17} + \frac{\beta}{20}\right)\left(\left|\xi - \frac{\beta}{4}\right|^2 + \left|\xi - \frac{\beta}{3}\right|^2\right) \\ &+ \frac{\xi\beta}{39}\frac{\left|\xi - \frac{\beta}{4}\right|^2\left|\xi - \frac{\beta}{3}\right|^2}{1 + \left|\xi - \frac{\beta}{4}\right|^2 + \left|\xi - \frac{\beta}{3}\right|^2 + |\beta-\beta|^2} \\ &= \left(\hbar(\xi,\beta)\varpi_\theta(\beta, S\beta) + \Bbbk(\xi,\beta)[\varpi_\theta(\xi, T\beta) + \varpi_\theta(\beta, S\xi)] + \hbar(\xi,\beta)\frac{\varpi_\theta(\xi, T\xi)\varpi_\theta(\beta, S\beta)}{1 + \varpi_\theta(\xi, S\beta) + \varpi_\theta(\beta, T\xi) + \varpi_\theta(\xi,\beta)}\right).\end{aligned}$$

Hence, all the postulates of Theorem 3.2 are fulfilled and 0 is a unique fixed point of S and T and we depict here the comparison of LHS and RHS of (S - N) contraction as shown in Fig. 3.

4. Application

Here, we confer the possibility of a solution to the Fredholm integral equation

$$\tau(\xi) = \int_0^1 \top(\xi, \theta, \tau(\xi))\, d\theta \tag{4.1}$$

where $\top : [0,1]\times[0,1]\times\mathbb{R} \to \mathbb{R}^+$ is continuous function. Set Υ = C[0, 1] which denotes the set of real continuous functions on [0, 1] also,

$$\varpi_\theta(\tau(\xi), \tau(\beta)) = max_{\xi\in[0,1]}(\|\tau(\xi), \tau(\beta)\|)^m$$

$\forall$ ξ, $\beta \in \Upsilon$ and m $\ge$1 and $\theta : \Upsilon\times\Upsilon \to [1,\infty) = 2+ \xi$. If $\breve{G}$ = (V, E) is a graph with Υ = V($\breve{G}$),

E($\breve{G}$) = {$\tau(\xi)$, $\tau(\beta)$: $\tau(\xi)\le\tau(\beta)$, $\forall$ ξ, $\beta\in$[0, 1]} then (Υ, ϖ_θ) is an complete extended b-metric space.

Theorem 4.1. *Choose an Eq. (4.1), we propose that*

$$\top : [0,1]\times[0,1]\times\mathbb{R} \to \mathbb{R}^+ \; is\; continuous function. \tag{4.2}$$

$$\Pi : [0,1]\times[0,1]\times\mathbb{R} \to \mathbb{R}^+ \; is\; continuous\; function\; suchthat \int_0^1 \Pi(\xi,\theta,)\,d\theta \le 1 \tag{4.3}$$

$$there\; \exists\; control\; functions\; , \exists\; \hbar : \Upsilon\times\Upsilon \to [0,1)\; which\; satisfies : \hbar(S\xi,\beta) \le \hbar(\xi,\beta)\; and\; \hbar(\xi, S\beta) \le \hbar(\xi,\beta)$$

$$\forall \xi,\theta \in [0,1]^2 and \xi,\beta \in [0,1]\Upsilon,$$

$$|\top(\xi,\theta,\tau(\theta)) - \top(\xi,\theta,\beta(\theta))| \le \hbar(\xi,\beta)^{\frac{1}{m}}\,\Pi(\xi,\theta)|\tau(\theta) - \beta(\theta)|. \tag{4.4}$$

This implies (4.1) has a unique solutionτ in Υ.

Proof. Set the map S i.e.

$$S\tau(\xi) = \int_0^1 \mathsf{T}(\xi, \theta, \tau(\xi))\, d\theta \tag{4.5}$$

Now, $\tau(\xi)$, $\beta(\xi)\in E(\breve{G})$ and $\xi\in[0,1]$ we deduce,

$$\begin{aligned}
\varpi_\theta(\tau(\xi), \beta(\xi)) &= \Big(|\tau(\xi) - \beta(\xi)|\Big)^m \\
&\le \left(\int_0^1 |\mathsf{T}(\xi, \theta, \tau(\xi)) - \mathsf{T}(\xi, \theta, \beta(\theta))| d\theta\right)^m \\
&\le \left(\int_0^1 \hbar(\mathsf{T}, \beta)^{\frac{1}{m}}\, \Pi(\xi, \theta)(|\tau(\theta) - \beta(\theta)|^m)^{\frac{1}{m}}\, d\theta\right)^m \\
&\le \left(\int_0^1 \hbar(\xi, \beta)^{\frac{1}{m}}\, \Pi(\xi, \theta)\varpi_\theta(\tau(\xi), \tau(\beta))^{\frac{1}{m}}\, d\theta\right)^m \\
&\le \hbar(\xi, \beta)\varpi_\theta(\tau(\xi), \tau(\beta)) \int_0^1 \Big(\Pi(\xi, \theta)\Big)^m \\
&\le \hbar(\xi, \beta)\varpi_\theta(\tau(\xi), \tau(\beta)).
\end{aligned}$$

Thus we write,

$$\varpi_\theta(\tau(\xi), \beta(\xi)) \le \hbar(\xi, \beta)\varpi_\theta(\xi, \beta) \tag{4.6}$$

Consequently, all postulates of corollary (3.4) are satisfied, and S has a unique fixed point in Υ which is a solution of (4.1). □

5. Conclusion

This note, inspired by Singh et al. [24] and Almari and Ahmed [25] introduces the concept of generalized (Boyd–Wong) type A **F** and (S - N) contractions in a extended b - metric space visualized by graphs. Furthermore, we present numerical representations to support our findings. In addition, we introduce graphs in both 2D and 3D to contrast the (Boyd–Wong) type A **F** — contraction. Fixed point theory relies heavily on metric space generalizations and contractive mappings. Here, we outline some of the future goals of our findings.

(I) Apply Theorem 3.2 to both controlled metric type and doubled controlled metric type spaces

(II) To establish a non-trivial innovative use of corollary (3.4).

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Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Data availability

Data will be made available on request.

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