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Research article

Functional differential equations in the non-canonical case: New conditions for oscillation

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Abstract: In this paper, we study the oscillation of a class of second-order nonlinear differential equations with mixed neutral terms in the non-canonical case. New criteria are derived that ensure the oscillation of the studied equation. The results obtained here greatly improve and extend some of the results reported in previous studies. To illustrate this, we present some examples.

Keywords: functional differential equations; mixed neutral terms; second-order; oscillation
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1. Introduction

This paper is concerned with the oscillatory behavior of solutions to a nonlinear second-order neutral differential equation

$$\left(\ell(\zeta)\left(\psi'(\zeta)\right)^{\epsilon}\right)' + q(\zeta)\varkappa^{\beta}(\varepsilon(\zeta)) = 0,\ \zeta \geq \zeta_0 > 0, \tag{1.1}$$

where $\psi(\zeta) = \varkappa(\zeta) + p_1(\zeta)\varkappa(\tau(\zeta)) + p_2(\zeta)\varkappa(\mu(\zeta))$, ϵ,β are the ratios of odd natural numbers. The following assumptions are satisfied:

(G1) $\ell \in C([\zeta_0,\infty),(0,\infty))$ satisfies the condition (i.e., the non-canonical case)

$$\int_{\zeta_0}^{\infty} \frac{1}{\ell^{1/\epsilon}(\xi)}\mathrm{d}\xi < \infty; \tag{1.2}$$

(G2) $\varepsilon \in C([\zeta_0,\infty),(0,\infty))$, $\varepsilon(\zeta) \leq \zeta$, $\varepsilon'(\zeta) > 0$, and $\lim_{\zeta\to\infty}\varepsilon(\zeta) = \infty$;

(G3) $p_1, p_2 \in C([\zeta_0, \infty), [0,1))$, $q \in C([\zeta_0, \infty), [0, \infty))$ and $q(\zeta)$ is not identically zero in any interval of $[\zeta_0, \infty)$;

(G4) $\tau, \mu \in C([\zeta_0, \infty), (0, \infty))$, $\tau(\zeta) \le \zeta$, $\mu(\zeta) \ge \zeta$ and $\lim_{\zeta \to \infty} \tau(\zeta) = \infty$.

By a solution of (1.1), we mean a function $\varkappa \in C([\zeta_\varkappa, \infty), \mathbb{R})$ $\zeta_\varkappa \ge \zeta_0$, which has the property $\ell(\zeta)(\psi'(\zeta))^\epsilon \in C^1([\zeta_\varkappa, \infty), \mathbb{R})$ and satisfies (1.1) for all $\zeta \ge \zeta_\varkappa$. We consider only those solutions $\varkappa(\zeta)$ of (1.1) satisfying $\sup\{|\varkappa(\zeta)| : \zeta \ge \zeta_a\} > 0$ for all $\zeta_a \ge \zeta_\varkappa$, and we assume that (1.1) possesses such solutions.

A solution of (1.1) is called oscillatory if it has arbitrarily many zeros on $[\zeta_\varkappa, \infty)$; otherwise, it is termed non-oscillatory. If every solution to Eq (1.1) is oscillatory, then the equation is considered oscillatory.

Oscillation theory has grown significantly since this phenomenon appears in various real-world models; for example, the papers [1–3] that discuss biological mechanisms (for models from mathematical biology where the oscillation and deviation scenarios may be formulated by means of external sources and/or nonlinear diffusion, perturbing the natural evolution of related systems). Neutral functional differential equations have also drawn a lot of interest since they are used in a wide range of disciplines, including economics, physics, biodynamics, mechanical engineering, control theory, and communication (see [4–8] and the references therein). The oscillation area for several classes of second-order difference equations; see [9–12], second-order differential equations; see [13–17] and second-order dynamic equations; see [18, 19] was the focus of researchers due to the aforementioned observations.

The oscillation and asymptotic behavior of different forms of second-order differential equations have been discussed. Some of them are given below.

Dzurina et al. [20] investigated the oscillation of the second-order differential equation

$$\left(\ell(\zeta)\left((\varkappa(\zeta) + p_1(\zeta)\varkappa^\kappa(\tau(\zeta)))'\right)^\epsilon\right)' + q(\zeta)\varkappa^\beta(\varepsilon(\zeta)) = 0,\ \zeta \ge \zeta_0, \tag{1.3}$$

under the condition (1.2), where $0 < \kappa < 1$ is a ratio of odd natural numbers. They found sufficient conditions to ensure that all solutions to (1.3) oscillate.

Li and Rogovchenko [4] and Shi and Han [21] studied the oscillation of the half-linear neutral differential equation of second-order

$$(\ell(\zeta)((\varkappa(\zeta) + p_1(\zeta)\varkappa(\tau(\zeta)))')^\epsilon)' + q(\zeta)\varkappa^\epsilon(\varepsilon(\zeta)) = 0,\ \zeta \ge \zeta_0, \tag{1.4}$$

under the condition (i.e., the canonical case)

$$\int_{\zeta_0}^{\zeta} \frac{1}{\ell^{1/\epsilon}(\xi)} \mathrm{d}\xi = \infty \text{ as } \zeta \to \infty, \tag{1.5}$$

where $\varepsilon(\zeta) \ge \zeta$ and $\tau(\zeta) \le \zeta$. They obtained sufficient conditions for the oscillation of the studied equations by the inequality principle and the Riccati transformation.

Grace et al. [22] considered the oscillatory behavior of all solutions of second-order nonlinear differential equations with positive and negative neutral terms

$$\left(\ell(\zeta)\left((\varkappa(\zeta) + p_1(\zeta)\varkappa^{\kappa_1}(\tau(\zeta)) - p_2(\zeta)\varkappa^{\kappa_2}(\tau(\zeta)))'\right)^\epsilon\right)' + q(\zeta)\varkappa^\beta(\varepsilon(\zeta)) = 0,\ \zeta \ge \zeta_0, \tag{1.6}$$

under the condition (1.5), where κ_1 and κ_2 are the ratios of positive odd integers. They introduced new oscillation criteria, by which they proved that equation (1.6) is oscillatory.

Moaaz et al. [23] focused on studying the differential equation

$$\left(\ell(\zeta)\left(\left(\varkappa(\zeta)+p_1(\zeta)\varkappa(\tau(\zeta))+p_2(\zeta)\varkappa(\mu(\zeta))\right)'\right)^{\epsilon}\right)'+q(\zeta)\varkappa^{\epsilon}(\varepsilon(\zeta))+q_2(\zeta)\varkappa^{\epsilon}(\varepsilon_2(\zeta))=0,\ \zeta\geq\zeta_0, \tag{1.7}$$

in the non-canonical case, where $\varepsilon_2(\zeta)\geq\zeta$, $q_2(\zeta)\in C([\zeta_0,\infty),[0,\infty))$, and $q_2(\zeta)$ is not identically zero for large ζ. They created criteria with one condition through which they guaranteed the oscillation of the differential Eq (1.7). To illustrate the importance of the results they obtained, they presented the following differential equation as an example:

$$\left(\zeta^2\left(\varkappa(\zeta)+p_1^*\varkappa\left(\frac{\zeta}{\gamma}\right)+p_2^*\varkappa(\gamma\zeta)\right)'\right)'+q_1^*\varkappa\left(\frac{\zeta}{\varrho}\right)+q_2^*\varkappa(\varrho\zeta)=0,\ \zeta\geq\zeta_0, \tag{1.8}$$

where $\gamma,\varrho\geq 1$. They proved using Theorem 2 in [23] that (1.8) is oscillatory if

$$q_1^*+q_2^*>\frac{\varrho}{1-\gamma p_1^*-p_2^*}. \tag{1.9}$$

Wu et al. [24] studied the oscillatory properties of a second-order delay differential equation with a sublinear neutral term

$$\left(\ell(\zeta)\left(\left(\varkappa(\zeta)+p_1(\zeta)\varkappa^{\kappa}(\tau(\zeta))\right)'\right)^{\epsilon}\right)'+q(\zeta)\varkappa^{\beta}(\varepsilon(\zeta))=0,\ \zeta\geq\zeta_0, \tag{1.10}$$

where $0<\kappa\leq 1$ is a ratio of odd natural numbers. They introduced oscillation criteria that extend and improve some of the well-known results in the literature. Here we mention one of their results for clarification.

Theorem 1.1. *[24, Corollary 2.1] Let $\kappa=1$, $\epsilon=\beta$, and*

$$\max\left\{p_1(\varepsilon(\zeta)),\,p_1(\varepsilon(\zeta))\frac{\int_{\tau(\varepsilon(\zeta))}^{\infty}\ell^{-1/\epsilon}(\xi)\,\mathrm{d}\xi}{\int_{\varepsilon(\zeta)}^{\infty}\ell^{-1/\epsilon}(\xi)\,\mathrm{d}\xi}\right\}<1. \tag{1.11}$$

If

$$\int^{\infty}\left(1-p_1(\varepsilon(\zeta))\right)^{\beta}q(\zeta)\,\mathrm{d}\zeta=\infty, \tag{1.12}$$

and

$$\int^{\infty}\left(\left(1-p_1(\varepsilon(\zeta))\frac{\int_{\tau(\varepsilon(\zeta))}^{\infty}\ell^{-1/\epsilon}(\xi)\,\mathrm{d}\xi}{\int_{\varepsilon(\zeta)}^{\infty}\ell^{-1/\epsilon}(\xi)\,\mathrm{d}\xi}\right)^{\beta}q(\zeta)\left(\int_{\zeta}^{\infty}\ell^{-1/\epsilon}(\xi)\,\mathrm{d}\xi\right)^{\beta}-\frac{\ell^{-1/\beta}(\zeta)\left(\frac{\beta}{\beta+1}\right)^{\beta+1}}{\int_{\zeta}^{\infty}\ell^{-1/\epsilon}(\xi)\,\mathrm{d}\xi}\right)\mathrm{d}\zeta=\infty \tag{1.13}$$

hold, then (1.10) is oscillatory.

In light of these considerations, our goal is to study the oscillatory behavior of Eq (1.1), and find new oscillation criteria, where we obtain these criteria by deducing some monotonic properties and some new inequalities between the solution and the corresponding function. By verifying these criteria, we can ensure that Eq (1.1) is oscillatory. To see the effectiveness and importance of the criteria we have obtained, we present some examples and compare them with some previous studies.

2. Main results

Let us introduce the following notation:

$$\hbar(\zeta)=\int_{\zeta}^{\infty}\frac{1}{\ell^{1/\epsilon}(\xi)}\mathrm{d}\xi,\ \Re(\zeta)=\int_{\zeta_0}^{\zeta}\frac{1}{\ell^{1/\epsilon}(\xi)}\mathrm{d}\xi,$$

and

$$\Phi(\zeta)=p_1(\varepsilon(\zeta))\frac{\hbar(\tau(\varepsilon(\zeta)))}{\hbar(\varepsilon(\zeta))}+p_2(\varepsilon(\zeta))\frac{\Re(\mu(\varepsilon(\zeta)))}{\Re(\varepsilon(\zeta))}.$$

Theorem 2.1. *If*

$$\int_{\zeta_1}^{\infty}\left(\frac{1}{\ell^{1/\epsilon}(\theta)}\left(\int_{\zeta_1}^{\theta}q(u)\hbar^{\beta}(\varepsilon(u))(1-\Phi(u))^{\beta}\mathrm{d}u\right)^{1/\epsilon}\right)\mathrm{d}\theta=\infty, \tag{2.1}$$

where $\Phi(u)<1$, then (1.1) is oscillatory.

Proof. Assume that (1.1) is not oscillatory. In this case, it has solutions that eventually do not change sign. Without loss of generality, we can suppose that $\varkappa(\zeta)$ is a positive solution of (1.1). Then, we see that $\varkappa(\tau(\zeta))>0$ and $\varkappa(\varepsilon(\zeta))>0$ for all $\zeta\geq\zeta_1$. From (1.1), we obtain

$$\left(\ell(\zeta)(\psi'(\zeta))^{\epsilon}\right)'=-q(\zeta)\varkappa^{\beta}(\varepsilon(\zeta))\leq 0, \tag{2.2}$$

thus, the function $\ell(\zeta)(\psi'(\zeta))^{\epsilon}$ is nonincreasing on $[\zeta_0,\infty)$ and of one sign, i.e., $\psi'(\zeta)<0$ or $\psi'(\zeta)>0$. First, assume that $\psi'(\zeta)<0$. From the monotonicity of $\ell(\zeta)(\psi'(\zeta))^{\epsilon}$, we obtain

$$\ell(\zeta)(\psi'(\zeta))^{\epsilon}\leq\ell(\zeta_1)(\psi'(\zeta_1))^{\epsilon}:=-K<0,\ \zeta\geq\zeta_1. \tag{2.3}$$

We know that $\psi(\zeta)=\varkappa(\zeta)+p_1(\zeta)\varkappa(\tau(\zeta))+p_2(\zeta)\varkappa(\mu(\zeta))$, therefore, we obtain

$$\varkappa(\zeta)=\psi(\zeta)-p_1(\zeta)\varkappa(\tau(\zeta))-p_2(\zeta)\varkappa(\mu(\zeta))\geq\psi(\zeta)-p_1(\zeta)\psi(\tau(\zeta))-p_2(\zeta)\psi(\mu(\zeta)). \tag{2.4}$$

Since $\psi'(\zeta)<0$ and $(\ell(\zeta)(\psi'(\zeta))^{\epsilon})'\leq 0$, we see that

$$\begin{aligned}\psi(\zeta)&\geq-\int_{\zeta}^{\infty}\frac{1}{\ell^{1/\epsilon}(\varsigma)}\left(\ell^{1/\epsilon}(\varsigma)\psi'(\varsigma)\right)\mathrm{d}\varsigma\\&\geq-\ell^{1/\epsilon}(\zeta)\psi'(\zeta)\hbar(\zeta),\end{aligned} \tag{2.5}$$

using (2.5) and (2.3), we have

$$\psi(\zeta)\geq K^{1/\epsilon}\hbar(\zeta). \tag{2.6}$$

From (2.5), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}\zeta}\left(\frac{\psi(\zeta)}{\hbar(\zeta)}\right)=\frac{\hbar(\zeta)\ell^{1/\epsilon}(\zeta)\psi'(\zeta)+\psi(\zeta)}{\hbar^{2}(\zeta)\ell^{1/\epsilon}(\zeta)}\geq 0, \tag{2.7}$$

using (2.7), (2.4), and (G4), we have

$$\varkappa(\zeta)\geq\psi(\zeta)-p_1(\zeta)\frac{\psi(\zeta)\hbar(\tau(\zeta))}{\hbar(\zeta)}-p_2(\zeta)\psi(\zeta)=\psi(\zeta)\left(1-p_1(\zeta)\frac{\hbar(\tau(\zeta))}{\hbar(\zeta)}-p_2(\zeta)\right),$$

and so,

$$\varkappa\left(\varepsilon\left(\zeta\right)\right) \geq \psi\left(\varepsilon\left(\zeta\right)\right)\left(1 - p_1\left(\varepsilon\left(\zeta\right)\right)\frac{\hbar\left(\tau\left(\varepsilon\left(\zeta\right)\right)\right)}{\hbar\left(\varepsilon\left(\zeta\right)\right)} - p_2\left(\varepsilon\left(\zeta\right)\right)\right),$$

from (2.2), we obtain

$$\left(\ell\left(\zeta\right)\left(\psi'\left(\zeta\right)\right)^{\epsilon}\right)' \leq -q\left(\zeta\right)\psi^{\beta}\left(\varepsilon\left(\zeta\right)\right)\left(1 - p_1\left(\varepsilon\left(\zeta\right)\right)\frac{\hbar\left(\tau\left(\varepsilon\left(\zeta\right)\right)\right)}{\hbar\left(\varepsilon\left(\zeta\right)\right)} - p_2\left(\varepsilon\left(\zeta\right)\right)\right)^{\beta}, \tag{2.8}$$

from (2.6), we obtain

$$\left(\ell\left(\zeta\right)\left(\psi'\left(\zeta\right)\right)^{\epsilon}\right)' \leq -q\left(\zeta\right)K^{\beta/\epsilon}\hbar^{\beta}\left(\varepsilon\left(\zeta\right)\right)\left(1 - p_1\left(\varepsilon\left(\zeta\right)\right)\frac{\hbar\left(\tau\left(\varepsilon\left(\zeta\right)\right)\right)}{\hbar\left(\varepsilon\left(\zeta\right)\right)} - p_2\left(\varepsilon\left(\zeta\right)\right)\right)^{\beta}. \tag{2.9}$$

Since $\mathfrak{R}'\left(\zeta\right) > 0$, we obtain

$$\mathfrak{R}\left(\mu\left(\varepsilon\left(\zeta\right)\right)\right) \geq \mathfrak{R}\left(\varepsilon\left(\zeta\right)\right). \tag{2.10}$$

By integrating (2.9) from ζ_1 to ζ, and using (2.10), we find

$$\ell\left(\zeta\right)\left(\psi'\left(\zeta\right)\right)^{\epsilon} \leq -K^{\beta/\epsilon}\int_{\zeta_1}^{\zeta} q\left(u\right)\hbar^{\beta}\left(\varepsilon\left(u\right)\right)\left(1 - p_1\left(\varepsilon\left(u\right)\right)\frac{\hbar\left(\tau\left(\varepsilon\left(u\right)\right)\right)}{\hbar\left(\varepsilon\left(u\right)\right)} - p_2\left(\varepsilon\left(u\right)\right)\frac{\mathfrak{R}\left(\mu\left(\varepsilon\left(u\right)\right)\right)}{\mathfrak{R}\left(\varepsilon\left(u\right)\right)}\right)^{\beta}\mathrm{d}u. \tag{2.11}$$

Integrating (2.11) from ζ_1 to ζ, we obtain

$$\psi\left(\zeta\right) \leq \psi\left(\zeta_1\right) - \left(K^{\beta/\epsilon}\right)^{1/\epsilon}\int_{\zeta_1}^{\zeta}\left(\frac{1}{\ell^{1/\epsilon}\left(\theta\right)}\left(\int_{\zeta_1}^{\theta} q\left(u\right)\hbar^{\beta}\left(\varepsilon\left(u\right)\right)\left(1 - \Phi\left(u\right)\right)^{\beta}\mathrm{d}u\right)^{1/\epsilon}\right)\mathrm{d}\theta, \tag{2.12}$$

combining (2.1) and (2.12), we see that $\psi\left(\zeta\right) \to -\infty$ as $\zeta \to \infty$, a contradiction.
Next, assume that $\psi'\left(\zeta\right) > 0$. Hence,

$$\begin{aligned}\psi\left(\zeta\right) &= \psi\left(\zeta_1\right) + \int_{\zeta_1}^{\zeta}\frac{1}{\ell^{1/\epsilon}\left(\varsigma\right)}\left(\ell^{1/\epsilon}\left(\varsigma\right)\psi'\left(\varsigma\right)\right)\mathrm{d}\varsigma \geq \left(\ell^{1/\epsilon}\left(\zeta\right)\psi'\left(\zeta\right)\right)\int_{\zeta_1}^{\zeta}\frac{1}{\ell^{1/\epsilon}\left(\varsigma\right)}\mathrm{d}\varsigma \\ &\geq \ell^{1/\epsilon}\left(\zeta\right)\psi'\left(\zeta\right)\mathfrak{R}\left(\zeta\right),\end{aligned}$$

and so,

$$\frac{\mathrm{d}}{\mathrm{d}\zeta}\left(\frac{\psi\left(\zeta\right)}{\mathfrak{R}\left(\zeta\right)}\right) = \frac{\mathfrak{R}\left(\zeta\right)\ell^{1/\epsilon}\left(\zeta\right)\psi'\left(\zeta\right) - \psi\left(\zeta\right)}{\mathfrak{R}^2\left(\zeta\right)\ell^{1/\epsilon}\left(\zeta\right)} \leq 0, \tag{2.13}$$

combining (2.4) and (2.13), we see that

$$\varkappa\left(\zeta\right) \geq \psi\left(\zeta\right) - p_1\left(\zeta\right)\psi\left(\zeta\right) - p_2\left(\zeta\right)\frac{\psi\left(\zeta\right)\mathfrak{R}\left(\mu\left(\zeta\right)\right)}{\mathfrak{R}\left(\zeta\right)} = \psi\left(\zeta\right)\left(1 - p_1\left(\zeta\right) - p_2\left(\zeta\right)\frac{\mathfrak{R}\left(\mu\left(\zeta\right)\right)}{\mathfrak{R}\left(\zeta\right)}\right),$$

and so,

$$\varkappa\left(\varepsilon\left(\zeta\right)\right) \geq \psi\left(\varepsilon\left(\zeta\right)\right)\left(1 - p_1\left(\varepsilon\left(\zeta\right)\right) - p_2\left(\varepsilon\left(\zeta\right)\right)\frac{\mathfrak{R}\left(\mu\left(\varepsilon\left(\zeta\right)\right)\right)}{\mathfrak{R}\left(\varepsilon\left(\zeta\right)\right)}\right).$$

From (2.2), we obtain

$$\left(\ell\left(\zeta\right)\left(\psi'\left(\zeta\right)\right)^{\epsilon}\right)' \leq -q\left(\zeta\right)\psi^{\beta}\left(\varepsilon\left(\zeta\right)\right)\left(1 - p_1\left(\varepsilon\left(\zeta\right)\right) - p_2\left(\varepsilon\left(\zeta\right)\right)\frac{\mathfrak{R}\left(\mu\left(\varepsilon\left(\zeta\right)\right)\right)}{\mathfrak{R}\left(\varepsilon\left(\zeta\right)\right)}\right)^{\beta}. \tag{2.14}$$

Since $\hbar'(\zeta) < 0$, we obtain

$$\hbar(\tau(\varepsilon(\zeta))) \geq \hbar(\varepsilon(\zeta)), \tag{2.15}$$

integrating (2.14) from ζ_1 to ζ and using (2.15), we find

$$\begin{aligned}
\ell(\zeta)(\psi'(\zeta))^{\epsilon} &\leq -\int_{\zeta_1}^{\zeta} q(u)\psi^{\beta}(\varepsilon(u))\left(1 - p_1(\varepsilon(u)) - p_2(\varepsilon(u))\frac{\Re(\mu(\varepsilon(u)))}{\Re(\varepsilon(u))}\right)^{\beta} \mathrm{d}u \\
&\quad + \ell(\zeta_1)(\psi'(\zeta_1))^{\epsilon} \\
&\leq -\psi^{\beta}(\varepsilon(\zeta_1))\int_{\zeta_1}^{\zeta} q(u)\left(1 - p_1(\varepsilon(u)) - p_2(\varepsilon(u))\frac{\Re(\mu(\varepsilon(u)))}{\Re(\varepsilon(u))}\right)^{\beta} \mathrm{d}u \\
&\quad + \ell(\zeta_1)(\psi'(\zeta_1))^{\epsilon} \\
&\leq -\psi^{\beta}(\varepsilon(\zeta_1))\int_{\zeta_1}^{\zeta} q(u)\left(1 - p_1(\varepsilon(u))\frac{\hbar(\tau(\varepsilon(u)))}{\hbar(\varepsilon(u))} - p_2(\varepsilon(u))\frac{\Re(\mu(\varepsilon(u)))}{\Re(\varepsilon(u))}\right)^{\beta} \mathrm{d}u \\
&\quad + \ell(\zeta_1)(\psi'(\zeta_1))^{\epsilon}.
\end{aligned} \tag{2.16}$$

Since $\hbar'(\zeta) < 0$, we obtain

$$\int_{\zeta_1}^{\zeta} \hbar^{\beta}(\varepsilon(u))\, q(u)(1 - \Phi(u))^{\beta} \mathrm{d}u \leq \hbar^{\beta}(\varepsilon(\zeta_1))\int_{\zeta_1}^{\zeta} q(u)(1 - \Phi(u))^{\beta} \mathrm{d}u. \tag{2.17}$$

It follows from (2.1) and (G1) that $\int_{\zeta_1}^{\zeta} q(u)\hbar^{\beta}(\varepsilon(u))(1 - \Phi(u))^{\beta} \mathrm{d}u$ must be unbounded. Hence, from (2.17), we find

$$\int_{\zeta_1}^{\zeta} q(u)(1 - \Phi(u))^{\beta} \mathrm{d}u \to \infty \text{ as } \zeta \to \infty. \tag{2.18}$$

Thus, from (2.16), we see that $\psi'(\zeta) \to -\infty$ as $\zeta \to \infty$, a contradiction. Then, the proof is completed. □

Theorem 2.2. *Assume that*

$$\psi'(\zeta) + \frac{1}{\ell^{1/\epsilon}(\zeta)}\left(\int_{\zeta_1}^{\zeta} q(u)(1 - \Phi(u))^{\beta} \mathrm{d}u\right)^{1/\epsilon} \psi^{\beta/\epsilon}(\varepsilon(\zeta)) = 0 \tag{2.19}$$

is oscillatory and

$$\int_{\zeta_0}^{\infty} q(u)(1 - \Phi(u))^{\beta} \mathrm{d}u = \infty, \tag{2.20}$$

where $\Phi(\zeta) < 1$. Then, (1.1) is oscillatory.

Proof. We can proceed exactly as in the proof of Theorem 2.1. Then, $\ell(\zeta)(\psi'(\zeta))^{\epsilon}$ is of one sign eventually. Now, suppose that $\psi'(\zeta) < 0$. Then, we find that (2.8) and (2.10) are satisfied. Integrating (2.8) from ζ_1 to ζ and using (2.10), we have

$$\ell(\zeta)(\psi'(\zeta))^{\epsilon} \leq \ell(\zeta_1)(\psi'(\zeta_1))^{\epsilon} - \int_{\zeta_1}^{\zeta} q(u)\psi^{\beta}(\varepsilon(u))(1 - \Phi(u))^{\beta} \mathrm{d}u,$$

and so,

$$\psi'(\zeta) \le -\frac{\psi^{\beta/\epsilon}(\varepsilon(\zeta))}{\ell^{1/\epsilon}(\zeta)} \left(\int_{\zeta_1}^{\zeta} q(u)(1-\Phi(u))^{\beta} \mathrm{d}u \right)^{1/\epsilon}.$$

Hence, we find that

$$\psi'(\zeta) + \frac{1}{\ell^{1/\epsilon}(\zeta)} \left(\int_{\zeta_1}^{\zeta} q(u)(1-\Phi(u))^{\beta} \mathrm{d}u \right)^{1/\epsilon} \psi^{\beta/\epsilon}(\varepsilon(\zeta)) \le 0 \tag{2.21}$$

has a positive solution, hence, we conclude that (2.19) has a positive solution, a contradiction; see [25, Lemma 1].

Next, let $\psi'(\zeta) > 0$. Hence, we find that (2.20) leads to (2.18), and in light of this, the rest of the proof of this theorem is similar to the proof of Theorem 2.1. □

We can now obtain other oscillation criteria for (1.1) by using the results given in [25–27].

Corollary 2.1. *Assume that $\epsilon = \beta$. If*

$$\liminf_{\zeta\to\infty} \int_{\varepsilon(\zeta)}^{\zeta} \frac{1}{\ell^{1/\epsilon}(\theta)} \left(\int_{\zeta_1}^{\theta} q(u)(1-\Phi(u))^{\beta} \mathrm{d}u \right)^{1/\epsilon} \mathrm{d}\theta > \frac{1}{\mathrm{e}}, \tag{2.22}$$

and (2.20) hold, where $\Phi(\zeta) < 1$, then (1.1) is oscillatory.

Corollary 2.2. *Assume that $\epsilon > \beta$. If*

$$\limsup_{\zeta\to\infty} \int_{\zeta_0}^{\zeta} \frac{1}{\ell^{1/\epsilon}(\theta)} \left(\int_{\zeta_1}^{\theta} q(u)(1-\Phi(u))^{\beta} \mathrm{d}u \right)^{1/\epsilon} \mathrm{d}\theta = \infty \tag{2.23}$$

holds, where $\Phi(\zeta) < 1$, then (1.1) is oscillatory.

Corollary 2.3. *Assume that $\beta > \epsilon$ and (2.20) holds. Let*

$$\limsup_{\zeta\to\infty} \frac{\beta\varpi'(\varepsilon(\zeta))\varepsilon'(\zeta)}{\epsilon\varpi'(\zeta)} < 1,$$

and

$$\liminf_{\zeta\to\infty} \frac{\mathrm{e}^{-\varpi(\zeta)}}{\varpi'(\zeta)} \frac{1}{\ell^{1/\epsilon}(\zeta)} \left(\int_{\zeta_1}^{\zeta} q(u)(1-\Phi(u))^{\beta} \mathrm{d}u \right)^{1/\epsilon} > 0. \tag{2.24}$$

Then, (1.1) is oscillatory, where $\Phi(\zeta) < 1$, $\varpi(\zeta) \in C^1([\zeta_0,\infty),\mathbb{R})$ such that $\varpi'(\zeta) > 0$ and $\lim_{\zeta\to\infty} \varpi(\zeta) = \infty$.

Example 2.1. *Consider the neutral differential equation*

$$\left(\zeta^{2/5} \left([\varkappa(\zeta) + A\varkappa(\tau_0\zeta) + B\varkappa(\mu_0\zeta)]' \right)^{1/5} \right)' + q_0 \varkappa^{1/7}(\varepsilon_0\zeta) = 0, \tag{2.25}$$

where $\zeta \geq 1$, $\epsilon = 1/5$, $\beta = 1/7$, $\ell(\zeta) = \zeta^{2/5}$, $p_1(\zeta) = A$, $p_2(\zeta) = B$, $A, B \in [0, 1)$, $\tau(\zeta) = \tau_0\zeta$, $\mu(\zeta) = \mu_0\zeta$, $\varepsilon(\zeta) = \varepsilon_0\zeta$, $\tau_0, \varepsilon_0 \in (0, 1)$, $\mu_0 \geq 1$, and $q(\zeta) = q_0$. Now, we see that

$$\hbar(\zeta) = \int_\zeta^\infty \frac{1}{\left(\xi^{2/5}\right)^{1/(1/5)}} \mathrm{d}\xi = \frac{1}{\zeta}, \ \Re(\zeta) = \int_1^\zeta \frac{1}{\left(\xi^{2/5}\right)^{1/(1/5)}} \mathrm{d}\xi = -\frac{1}{\zeta} + 1, \text{ where } \zeta_0 = 1,$$

$$\hbar(\varepsilon(\zeta)) = \frac{1}{\varepsilon_0\zeta}, \hbar(\tau(\varepsilon(\zeta))) = \frac{1}{\tau_0\varepsilon_0\zeta}, \ \Re(\varepsilon(\zeta)) = -\frac{1}{\varepsilon_0\zeta} + 1, \Re(\mu(\varepsilon(\zeta))) = -\frac{1}{\mu_0\varepsilon_0\zeta} + 1,$$

and

$$\frac{\Re(\mu(\varepsilon(\zeta)))}{\Re(\varepsilon(\zeta))} = \frac{\left(-\frac{1}{\mu_0\varepsilon_0\zeta} + 1\right)}{\left(-\frac{1}{\varepsilon_0\zeta} + 1\right)} = \frac{\varepsilon_0\zeta - \frac{1}{\mu_0}}{\varepsilon_0\zeta - 1}.$$

Set $\Omega(\zeta) = \left(\varepsilon_0\zeta - \frac{1}{\mu_0}\right) / (\varepsilon_0\zeta - 1)$, since $lim_{\zeta\to\infty}\Omega(\zeta) = 1$, there exists $\zeta_\epsilon > \zeta_0$ such that $\Omega(\zeta) < 1 + \epsilon$ for all $\epsilon > 0$ and every $\zeta \geq \zeta_\epsilon$. By choosing $\epsilon = \mu_0 - 1$, we obtain

$$\Omega(\zeta) = \left(\frac{\varepsilon_0\zeta - \frac{1}{\mu_0}}{\varepsilon_0\zeta - 1}\right) < \mu_0 \text{ for all } \zeta \geq \zeta_*.$$

Therefore, the condition (2.1) becomes

$$\begin{aligned}
&\int_{\zeta_1}^\infty \left(\frac{1}{\ell^{1/\epsilon}(\theta)} \left(\int_{\zeta_1}^\theta q(u)\hbar^\beta(\varepsilon(u))(1 - \Phi(u))^\beta \mathrm{d}u\right)^{1/\epsilon}\right)\mathrm{d}\theta \\
= \ &\int_{\zeta_1}^\infty \left(\frac{1}{\left(\theta^{2/5}\right)^{1/(1/5)}} \left(\int_{\zeta_1}^\theta q_0 \left(\frac{1}{\varepsilon_0 u}\right)^{1/7} \left(1 - A\left(\frac{1}{\tau_0}\right) - B\mu_0\right)^{1/7} \mathrm{d}u\right)^{1/(1/5)}\right)\mathrm{d}\theta \\
= \ &q_0^5 \left(1 - A\left(\frac{1}{\tau_0}\right) - B\mu_0\right)^{5/7} \left(\frac{1}{\varepsilon_0}\right)^{5/7} \int_{\zeta_1}^\infty \left(\frac{1}{\theta^2}\left(\int_{\zeta_1}^\theta \left(\frac{1}{u}\right)^{1/7} \mathrm{d}u\right)^5\right)\mathrm{d}\theta = \infty,
\end{aligned}$$

where

$$A\left(\frac{1}{\tau_0}\right) + B\mu_0 < 1.$$

Thus, by using Theorem 2.1, we see that (2.25) is oscillatory.

Example 2.2. *Consider the neutral differential equation*

$$\left(\zeta^2\left[\varkappa(\zeta) + \frac{1}{16}\varkappa\left(\frac{\zeta}{3}\right) + \frac{1}{32}\varkappa(3\zeta)\right]'\right)' + q_0\varkappa\left(\frac{\zeta}{4}\right) = 0, \tag{2.26}$$

where $\zeta \geq 1$, $\epsilon = \beta = 1$, $\ell(\zeta) = \zeta^2$, $p_1(\zeta) = 1/16$, $p_2(\zeta) = 1/32$, $\tau(\zeta) = \zeta/3$, $\mu(\zeta) = 3\zeta$, $\varepsilon(\zeta) = \zeta/4$, and $q(\zeta) = q_0$. Now, we see that

$$\tau(\varepsilon(\zeta)) = \frac{\zeta}{12}, \mu(\varepsilon(\zeta)) = \frac{3\zeta}{4}, \ \hbar(\varepsilon(\zeta)) = \frac{4}{\zeta}, \hbar(\tau(\varepsilon(\zeta))) = \frac{12}{\zeta},$$

$$\Re(\varepsilon(\zeta)) = -\frac{4}{\zeta} + 1, \ \Re(\mu(\varepsilon(\zeta))) = -\frac{4}{3\zeta} + 1,$$

and

$$\frac{\Re(\mu(\varepsilon(\zeta)))}{\Re(\varepsilon(\zeta))} = \frac{\left(-\frac{4}{3\zeta}+1\right)}{\left(-\frac{4}{\zeta}+1\right)} = \frac{\left(\frac{1}{4}\right)\zeta - \frac{1}{3}}{\left(\frac{1}{4}\right)\zeta - 1}.$$

Set $\Theta(\zeta) = \left(\left(\frac{1}{4}\right)\zeta - \frac{1}{3}\right) / \left(\left(\frac{1}{4}\right)\zeta - 1\right)$, since $lim_{\zeta\to\infty}\Theta(\zeta) = 1$, there exists $\zeta_{\epsilon_1} > \zeta_0$ such that $\Theta(\zeta) < 1 + \epsilon_1$ for all $\epsilon_1 > 0$ and every $\zeta \geq \zeta_{\epsilon_1}$. By choosing $\epsilon_1 = 3 - 1$, we obtain

$$\Theta(\zeta) = \left(\frac{\left(\frac{1}{4}\right)\zeta - \frac{1}{3}}{\left(\frac{1}{4}\right)\zeta - 1}\right) < 3 \, for\ all\ \zeta \geq \zeta_{*_1}.$$

Therefore, the condition (2.20) is satisfied, where

$$\begin{aligned} &\int_{\zeta_0}^{\infty} q(u)\left(1 - p_1(\varepsilon(u))\frac{\hbar(\tau(\varepsilon(u)))}{\hbar(\varepsilon(u))} - p_2(\varepsilon(u))\frac{\Re(\mu(\varepsilon(u)))}{\Re(\varepsilon(u))}\right)^{\beta} \mathrm{d}u \\ = \;& q_0\left(1 - \frac{1}{16}(3) - \frac{1}{32}(3)\right)\int_{\zeta_0}^{\infty} \mathrm{d}u = \infty, \end{aligned}$$

and the condition (2.22) becomes

$$\begin{aligned} &\liminf_{\zeta\to\infty} \int_{\varepsilon(\zeta)}^{\zeta} \frac{1}{\ell^{1/\epsilon}(\theta)}\left(\int_{\zeta_1}^{\theta} q(u)\left(1 - p_1(\varepsilon(u))\frac{\hbar(\tau(\varepsilon(u)))}{\hbar(\varepsilon(u))} - p_2(\varepsilon(u))\frac{\Re(\mu(\varepsilon(u)))}{\Re(\varepsilon(u))}\right)^{\beta} \mathrm{d}u\right)^{1/\epsilon} \mathrm{d}\theta \\ = \;& q_0\left(1 - \frac{1}{16}(3) - \frac{1}{32}(3)\right)\ln 4 > \frac{1}{\mathrm{e}}, \end{aligned}$$

thus, by using Corollary 2.1, we see that (2.26) is oscillatory if $q_0 > 0.36921$.

Now, by comparing (1.8) with (2.26), we notice that $p_1^* = 1/16$, $p_2^* = 1/32$, $\gamma = 3$, $\varrho = 4$, $q_1^* = q_0$ and $q_2^* = 0$. By using (1.9), we see that

$$q_0 > \frac{4}{1 - (3)\frac{1}{16} - \frac{1}{32}},$$

therefore, we find that (2.26) is oscillatory if $q_0 > 5.12$.

From the above, we notice that our results improve the results of [23].

Example 2.3. *Let us assume the special case*

$$\left(\zeta^6\left(\left[\varkappa(\zeta) + \frac{1}{4}\varkappa\left(\frac{\zeta}{2}\right)\right]'\right)^3\right)' + 2\zeta^2\varkappa^3\left(\frac{\zeta}{3}\right) = 0,\ \zeta \geq 1, \tag{2.27}$$

for Eq (1.1), where $\epsilon = \beta = 3$, $\ell(\zeta) = \zeta^6$, $p_1(\zeta) = 1/4$, $p_2(\zeta) = 0$, $\tau(\zeta) = \zeta/2$, $\varepsilon(\zeta) = \zeta/3$, and $q(\zeta) = 2\zeta^2$. Now, we see that

$$\tau(\varepsilon(\zeta)) = \frac{\zeta}{6},\ \hbar(\varepsilon(\zeta)) = \frac{3}{\zeta},\ \hbar(\tau(\varepsilon(\zeta))) = \frac{6}{\zeta} \text{ and } \Phi(\zeta) = \frac{1}{2}.$$

Therefore, the condition (2.20) is satisfied, where

$$\begin{aligned}&\int_{\zeta_0}^{\infty} q(u)\left(1-p_1(\varepsilon(u))\frac{\hbar(\tau(\varepsilon(u)))}{\hbar(\varepsilon(u))}-p_2(\varepsilon(u))\frac{\Re(\mu(\varepsilon(u)))}{\Re(\varepsilon(u))}\right)^{\beta}\mathrm{d}u\\=\;&\frac{1}{2^2}\int_{\zeta_0}^{\infty}u^2\mathrm{d}u=\infty,\end{aligned}$$

and the condition (2.22) is satisfied, where

$$\begin{aligned}&\liminf_{\zeta\to\infty}\int_{\varepsilon(\zeta)}^{\zeta}\frac{1}{\ell^{1/\epsilon}(\theta)}\left(\int_{\zeta_1}^{\theta}q(u)\left(1-p_1(\varepsilon(u))\frac{\hbar(\tau(\varepsilon(u)))}{\hbar(\varepsilon(u))}-p_2(\varepsilon(u))\frac{\Re(\mu(\varepsilon(u)))}{\Re(\varepsilon(u))}\right)^{\beta}\mathrm{d}u\right)^{1/\epsilon}\mathrm{d}\theta\\=\;&\left(1-\frac{1}{2}\right)\left(\frac{2}{3}\right)^{1/3}\ln 3>\frac{1}{\mathrm{e}},\end{aligned}$$

thus, by using Corollary 2.1, we see that (2.27) is oscillatory.

Now, using Theorem 1.1, we find that condition (1.13) is not satisfied, where

$$\begin{aligned}&\int^{\infty}\left(\left(1-p_1(\varepsilon(\zeta))\frac{\int_{\tau(\varepsilon(\zeta))}^{\infty}\ell^{-1/\epsilon}(\xi)\,\mathrm{d}\xi}{\int_{\varepsilon(\zeta)}^{\infty}\ell^{-1/\epsilon}(\xi)\,\mathrm{d}\xi}\right)^{\beta}q(\zeta)\left(\int_{\zeta}^{\infty}\ell^{-1/\epsilon}(\xi)\,\mathrm{d}\xi\right)^{\beta}-\frac{\ell^{-1/\beta}(\zeta)\left(\frac{\beta}{\beta+1}\right)^{\beta+1}}{\int_{\zeta}^{\infty}\ell^{-1/\epsilon}(\xi)\,\mathrm{d}\xi}\right)\mathrm{d}\zeta\\=&\left(\left(\frac{1}{2}\right)^{3}(2)-\left(\frac{3}{4}\right)^{4}\right)\int^{\infty}\frac{1}{\zeta}\mathrm{d}\zeta\neq\infty.\end{aligned}$$

From the above, we notice that our results improve the results of [24].

Remark 2.1. *From the previous examples, the following can be concluded:*

1) From Example 2.2, we note that using the results we obtained, we proved that differential Eq (2.26) is oscillatory if $q_0 > 0.36921$, while using the results of [23], they proved that differential Eq (2.26) is oscillatory if $q_0 > 5.12$.

2) From Example 2.3, we note that using the results we obtained, we proved that differential Eq (2.27) is oscillatory, while the results of [24] fail to study the oscillation of differential Eq (2.27) due to the failure to meet condition (1.13).

Thus, we find that our results improve the results of [23] and [24].

3. Conclusions

In this paper, the oscillatory and asymptotic behavior of second-order neutral differential equations is studied. New conditions are introduced to ensure that all solutions of (1.1) are oscillatory. Furthermore, we provide examples that demonstrate the theoretical significance and practical application of our criteria. These examples demonstrate how well our method improves and extends some previous theorems in this field and provides new directions for future studies. We recommend that future research investigate when our techniques can be applied to higher-order differential equations

$$\left(\ell(\zeta)\left((\varkappa(\zeta)+p_1(\zeta)\varkappa(\tau(\zeta))+p_2(\zeta)\varkappa(\mu(\zeta)))^{(n-1)}\right)^{\epsilon}\right)'+q(\zeta)\varkappa^{\beta}(\varepsilon(\zeta))=0,\ \zeta\geq\zeta_0,$$

where $n \geq 4$ is even.

Author contributions

Abdulaziz Khalid Alsharidi: Conceptualization, methodology, formal analysis, investigation, writing-review and editing; Ali Muhib: Conceptualization, methodology, formal analysis, investigation, writing-original draft preparation, writing-review and editing. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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