Numerical Proceduers for Computing the Exact Solutions to Systems of Ordinary Differential Equations

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Abstract

This paper presents a modified homotopy perturbation method (HPM), which aimed at solving systems of ordinary differential equations (ODEs). The MHPM, which combines the HPM, Laplace transform, and Padé approximants, offers an alternative approach to address the challenges associated with solving such problems. By employing this method, it becomes feasible to overcome these challenges and obtain a dependable approximation for the exact solution. The effectiveness and applicability of the proposed scheme are demonstrated through preliminary results derived from illustrative examples, all of which correspond to exact solutions.

Keywords: Numerical Approximation; HPM; MHPM; Laplace transformation; Padé approximants

1 Introduction

Solving systems of ODEs is a critical task in various fields of applied science and engineering. Whether one is modelling physical phenomena or optimizing industrial processes, accurately solving these equations is essential for understanding the behaviour of dynamic systems. However, finding analytical solutions can be challenging and even impractical, especially for complex systems with nonlinearities. In such cases, numerical methods offer a powerful range of tools for approximating the solution. There are different numerical schemes in fields of numerical analysis that are used in finding approximate solutions for various types of equations [1-20, 28-30]. The solutions resulting from these schemes are excellent and agree with the exact solutions in all cases. However, the difference between them lies only in the structure of the method, which reflects the ease of calculations or requires more effort and time.

In this study, we present a novel enhancement to the utilization of the HPM for resolving systems of ODEs. Our approach can be applied to any given problem, providing exact solutions, opposite the HPM solution that converge towards the exact solution as the number of approximation terms increases. It is noteworthy to acknowledge that the accuracy of our approach does not depend about the approximation employed, which may require additional computational resources and time, especially when dealing with nonlinear problems. Consequently, researchers are continuously striving to develop or adapt numerical techniques to achieve higher accuracy or exact solutions. The primary objective of this paper is to improve the accuracy of the HPM through the application of an alternative methodology. This methodology entails modifying the series solution of the HPM by utilizing the Laplace

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transform on the truncated HPM solution. Subsequently, the transformed series is converted into a meromorphic function using Padé approximants. Lastly, the inverse Laplace transform is employed to derive the desired solution for the given problem. This approach is simple, requiring minimal effort and is highly efficient in obtaining accurate results.

This work organized as follows: Section 2 introduces the fundamental concept of the HPM, along with a brief explanation of the Pade approximants. In Section 3, numerical examples are presented to demonstrate the effectiveness of the discussed procedure in obtaining the exact solution for systems of ODEs. The results highlight that accurate solutions can be obtained with only a few terms. The final section summarizes the conclusions of this work.

2. Fundamental Idea of HPM Procedure

To demonstrate the fundamental concept of the HPM procedure [21-25], let's examine the following equation.

$$A(u) - f(r) = 0, \quad r \in \Omega, \tag{1}$$

where A is the integral operato consiste of the linear and nonlinear operator L and N,respictively, while B is a boundary operator, $f(r)$is a known function, and Γ is the boundary of the domain Ω. Eq. (1) can be rewritten as

$$L(u) - N(u) - f(r) = 0. \tag{2}$$

A homotopy equation $v : \Omega[0,1] \rightarrow$R which satisfies

$$H(v;\, p) = L(v) - L(v_0) + pL(v_0) + p[N(v) - f(r)] = 0, \tag{3}$$

or

$$H(v;\, p) = (1-p)[L(v) - L(v_0)] + p[A(v_0) - F(r)] = 0, \tag{4}$$

is constructed, here, $r \,\epsilon\, \Omega, p \,\epsilon\, [0,1]$ is the homotopy parameter, and $v_0(x)$ is an initial approximation of Eq. (1). It is noted that

$$(v;\, 0) = L(u) - L(v_0) = 0, \quad H(v;\, 1) = A(v) - F(r) = 0. \tag{5}$$

The changing process of p from 0 to 1, include the changing of $H(v;\, p)$ from $L(u)$– $L(v_0)$ to $A(v) - F(r)$ that is called deformation, Moreovere, $L(u) - L(v_0)$and $A(v)$– $F(r)$ are called homotopic. Note that, $0 \le p \le 1$ it is considered as a small parameter, the solution of Eqs. (3) or (4) can be expressed as a series in p, as follows:

$$v = v_0 + pv_1 + p^2 v_2 + p^3 v_3 + \dots \tag{6}$$

when $p \rightarrow 1$, the approximate solution of Eq. (1). i.e.,

$$u(x) = \lim_{p \to 1} v(x) = v_1 + v_2 + v_3 + \cdots \tag{7}$$

3. Padè approximation

For the function $u(x)$ $[26, 27]$, the Padé approximation of order $\left[\frac{L}{M}\right]$, can be formulated as follows,

$$\left[\frac{L}{M}\right] = \frac{P_L(x)}{Q_M(x)},$$

where $P_L(x)$ and $Q_M(x)$, are two polynomials of the highest degree L and M. The power series is given in form of

$$u(x) = \sum_{i=1}^{\infty} a_i x^i.$$

The coefficients of the polynomials $P_L(x)$ and $Q_M(x)$, can be obtained from

$$u(t) - \frac{P_L(x)}{Q_M(x)} = O(x^{L+M+1}). \tag{8}$$

When the denominator and numerator's functions $\frac{P_L(x)}{Q_M(x)}$ is multiplied by a constant that is not zero, the fractional values stay the same, such that we can set up the normalization requirement as

$$\mathrm{Q_M}(0) = 1. \tag{9}$$

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It observed that the polynomial for the functions $P_L(x)$ and $Q_M(x)$ has no public factors. The coefficients of the polynomial $Q_M(x)$ and $P_L(x)$ aregiven by

$$P_L(t) = P_0 + P_1 t + P_2 t^2 + \cdots + P_L t^L,$$
$$Q_M(t) = q_0 + q_1 t + q_2 t^2 + \cdots + q_M t^M, \tag{10}$$

the following linear systems of coefficients can be obtained by multiplying Eq. (8) by $Q_M(x)$ in light of Eq. (8).

$$\left\{\begin{array}{c} a_{L+1} + a_L q_1 + \cdots + a_{L-M+1} q_M = 0 \\ a_{L+2} + a_{L+1} q_1 + \cdots + a_{L-M+2} q_M = 0 \\ . \\ . \\ a_{L+M} + a_{L+M-1} q_1 + \cdots + a_L q_M = 0 \end{array}\right\}, \tag{11}$$

$$\left\{\begin{array}{c} a_0 = P_0 \\ a_0 + a_0 q_1 = P_1 \\ a_2 + a_1 q_1 + a_0 q_2 = P_2 \\ . \\ . \\ a_L + a_{L-1} q_1 + \cdots + a_0 q_L = P_L \end{array}\right\}, \tag{12}$$

These equations will be solved using Eq. (11), It is seen as a set of linear formulas for the unidentified $q`s$ . When the $q's$ are identified, then (12) have an explicit formula for the unknown p`s, this concludes the solution to the problem. If (11) and (12) are non-singular, then we can solve them directly and get Eq. (13) .

$$\left[\frac{L}{M}\right] = \frac{\det\begin{bmatrix} a_{L-M+1} & a_{L-M+2} & \cdots & a_{L+1} \\ . & . & \cdots & . \\ . & . & . & . \\ a_L & a_{L+1} & . & a_{L+M} \\ \sum_{j=M}^{L} a_{j-M} X^j & \sum_{j=M-1}^{L} a_{j-M+1} X^j & \cdots & \sum_{j=0}^{L} a_j X^j \end{bmatrix}}{\det\begin{bmatrix} a_{L-M+1} & a_{L-M+2} & \cdots & a_{L+1} \\ . & . & . & . \\ . & . & . & . \\ a_L & a_{L+1} & \cdots & a_{L+M} \\ X^M & X^{M-1} & \cdots & 1 \end{bmatrix}}, \tag{13}$$

4. Applications of HPM

In the following section, we present two illustrative examples of systems of ordinary differential equations (ODEs), both linear and nonlinear. The purpose of displaying these examples is to demonstrate the effectiveness and reliability of the Modified HPM procedure.

Example 4. 1 Consider the following system of first-order linear equations [23].

$$u_1'(t) = u_2(t),$$
$$u_2'(t) = u_3(t),$$
$$u_3'(t) = \frac{1}{t} u_1(t) + u_3(t), \tag{14}$$

Subject to

$u_1(0) = 0, u_2(0) = 1,$ and $u_3(0) = 2$, with exact solutions $u = (u_1(t), u_2(t)) = \left(te^t, \mathrm{e}^t(1+t), \mathrm{e}^t(2+t)\right),$

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We will now formulate the following set of homotopy equations, following the algorithm outlined in Section 2.

$$(1-q)\frac{dv_1(t;p)}{dt} = (h;q)(\frac{dv_1(t;p)}{dt} - v_2(t;p)$$

$(1-q)\frac{dv_2(t;p)}{dt}(h;q)\left[\frac{dv_1(t;p)}{dt} - v_3(t;p)\right],$

$$(1-p)[\frac{dv_3(t;p)}{dt} = (h;q)\left[\frac{dv_3(t;p)}{dt} - \frac{1}{t}v_1(t;p) - v_3(t;p)\right] \quad (15)$$

Following the same process in example one, we have the 5th -order HPM approximate solution

$$\tilde{u}_1(t) = t + t^2 + \frac{t^3}{2} + \frac{t^4}{6} + \frac{t^5}{24} + \frac{t^6}{120} + \frac{167t^7}{151200} + \frac{t^8}{20160},$$

$$\tilde{u}_2(t) = 1 + 2t + \frac{3t^2}{2} + \frac{2t^3}{3} + \frac{5t^4}{24} + \frac{t^5}{20} + \frac{13t^6}{1350} + \frac{59t^7}{50400} + \frac{t^8}{20160}.$$

$$\tilde{u}_3(t) = 2 + 3t + 2t^2 + \frac{5t^3}{6} + \frac{t^4}{4} + \frac{7t^5}{120} + \frac{1403t^6}{129600} + \frac{433t^7}{352800} + \frac{t^8}{20160}. \quad (16)$$

Tables (1), (2) and(2) present a comparison between HPM process and exact solutions. We observed that the accuracy of the results varied depending on the order of the approximations. Therefore, to enhance the precision of the HPM procedure, we will begin by applying the Laplace transformation to the initial terms in the HPM series solutions. Next, we will utilize the Pade approximants and finally, we will conclude by implementing the inverse Laplace transformation. The process is outlined below

$$L(\tilde{u}_1(t)) = \frac{2}{s^9} + \frac{167}{30s^8} + \frac{6}{s^7} + \frac{5}{s^6} + \frac{4}{s^5} + \frac{3}{s^4} + \frac{2}{s^3} + \frac{1}{s^2},$$

$$L(\tilde{u}_2(t)) = \frac{2}{s^9} + \frac{59}{10s^8} + \frac{104}{15s^7} + \frac{6}{s^6} + \frac{5}{s^5} + \frac{4}{s^4} + \frac{3}{s^3} + \frac{2}{s^2} + \frac{1}{s},$$

$$L(\tilde{u}_3(t)) = \frac{2}{s^9} + \frac{433}{70s^8} + \frac{1403}{180s^7} + \frac{7}{s^6} + \frac{6}{s^5} + \frac{5}{s^4} + \frac{4}{s^3} + \frac{3}{s^2} + \frac{2}{s}, \quad (17)$$

Use $s = \frac{1}{z}$, leads to

$$L(\tilde{u}_1(t)) = z^2 + 2z^3 + 3z^4 + 4z^5 + 5z^6 + 6z^7 + \frac{167z^8}{30} + 2z^9,$$

$$L(\tilde{u}_2(t)) = z + 2z^2 + 3z^3 + 4z^4 + 5z^5 + 6z^6 + \frac{104z^7}{15} + \frac{59z^8}{10} + 2z^9,$$

$$L(\tilde{u}_3(t)) = 2z + 3z^2 + 4z^3 + 5z^4 + 6z^5 + 7z^6 + \frac{1403z^7}{180} + \frac{433z^8}{70} + 2z^9, \quad (18)$$

The Pade approximates of order $\left[\frac{3}{3}\right]$ in term of $x = \frac{1}{s}$, gives

$$\left[\frac{3}{3}\right] = \frac{1}{\left(1 + \frac{1}{s^2} - \frac{2}{s}\right)s^2},$$

$$\left[\frac{3}{3}\right] = \frac{1}{\left(1 + \frac{1}{s^2} - \frac{2}{s}\right)s},$$

$$\left[\frac{3}{3}\right] = -\frac{1}{\left(1+\frac{1}{s^2}-\frac{2}{s}\right)s^2} + \frac{2}{\left(1+\frac{1}{s^2}-\frac{2}{s}\right)s} \quad (19)$$

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The exact solutions $u = (u_1(t), u_2(t), u_3(t)) = \left(te^t, \mathrm{e}^t(1+t), \mathrm{e}^t(2+t)\right).$, are obtained by applying the inverse Laplace transform to the $\left[\frac{3}{3}\right]$ Pade approximate.

Example 4.2: Given the system of nonlinear first-order equations [24],

$$u_1'(t) = 2u_2^2(t),$$
$$u_2'(t) = e^{-t}u_1(t),$$
$$u_3'(t) = u_2(t) + u_3(t), \tag{20}$$

subject to

$u_1(0) = 1, u_2(0) = 1$, and $u_3(0) = 0$, with exact solutions $u = (u_1(t), u_2(t), u_3(t)) = (e^{2t}, e^t, t\, e^t)$.

We will now construct the following homotopy equation, which is based on the algorithm presented in Section 2.

$$(1-p)[\frac{dv_1(t;p)}{dt} = (h;q)\left[\frac{dv_1(t;p)}{dt} - 2v_2^2(t;p)\right],$$
$$(1-p)[\frac{dv_2(t;p)}{dt} = (h;q)\left[\frac{dv_2(t;p)}{dt} - e^{-t}v_1(t;p)\right]$$
$$(1-p)[\frac{dv_3(t;p)}{dt} = (h;q)\left[\frac{dv_3(t;p)}{dt} - v_{21}(t;p) - v_3(t;p)\right] \tag{21}$$

The problem of zeroth order is expressed in Eqs. (22) as given below

$$u'_{1,0}(t) = 0, \qquad u_{1,0}(0) = 1,$$
$$u'_{2,0}(t) = 0, \qquad u_{2,0}(0) = 1,$$
$$u'_{3,0}(t) = 0, \qquad u_{3,0}(0) = 0, \tag{22}$$

and their solutions are

$$u_{1,0}(t) = 1,$$
$$u_{2,0}(t) = 1,$$
$$u_{3,0}(t) = 0. \tag{23}$$

According to the HPM procedure, we have the approximate solution of the 8'th order.

$$\tilde{u}_1(t) = 1 + 2t + 2t^2 + \frac{4t^3}{3} + \frac{2t^4}{3} + \frac{4t^5}{15} + \frac{4t^6}{45} + \frac{8t^7}{315} + \frac{2t^8}{315},$$
$$\tilde{u}_2(t) = 1 + t + \frac{t^2}{2} + \frac{t^3}{6} + \frac{t^4}{24} + \frac{t^5}{120} + \frac{t^6}{720} + \frac{t^7}{5040} + \frac{t^8}{40320},$$
$$\tilde{u}_3(t) = t + t^2 + \frac{t^3}{2} + \frac{t^4}{6} + \frac{t^5}{24} + \frac{t^6}{120} + \frac{t^7}{720} + \frac{t^8}{5040}, \tag{24}$$

which provides the precise solution of Eq. (20) as the number of terms approaches infinity, i.e. $\lim_{n\to\infty} \tilde{u}_n(t) = (e^{2t}, e^t, te^t)$, which are the. In order to enhance the accuracy of the HPM solution, we apply the Laplace transform to the first few terms of HPM solutions (24), as follows:

$$L(\tilde{u}_1(t)) = \frac{128}{s^8} + \frac{64}{s^7} + \frac{32}{s^6} + \frac{16}{s^5} + \frac{8}{s^4} + \frac{4}{s^3} + \frac{2}{s^2} + \frac{1}{s},$$
$$L(\tilde{u}_2(t)) = \frac{1}{s^9} + \frac{1}{s^8} + \frac{1}{s^7} + \frac{1}{s^6} + \frac{1}{s^5} + \frac{1}{s^4} + \frac{1}{s^3} + \frac{1}{s^2} + \frac{1}{s},$$
$$L(\tilde{u}_3(t)) = \frac{8}{s^9} + \frac{7}{s^8} + \frac{6}{s^7} + \frac{5}{s^6} + \frac{4}{s^5} + \frac{3}{s^4} + \frac{2}{s^3} + \frac{1}{s^2}. \tag{25}$$

Taking $s = \frac{1}{z}$, gives

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$$L(\tilde{u}_1(t)) = z + 2z^2 + 4z^3 + 8z^4 + 16z^5 + 32z^6 + 64z^7 + 128z^8,$$

$$L(\tilde{u}_2(t)) = z + z^2 + z^3 + z^4 + z^5 + z^6 + z^7 + z^8 + z^9,$$

$$L(\tilde{u}_3(t)) = z^2 + 2z^3 + 3z^4 + 4z^5 + 5z^6 + 6z^7 + 7z^8 + 8z^9, \qquad (26)$$

in term of $z = \frac{1}{s}$, the Pade approximates of $\left[\frac{3}{3}\right]$, are

$\left[\frac{3}{3}\right] \frac{1}{\left(1-\frac{2}{s}\right)s}$,

$$\left[\frac{3}{3}\right] = \frac{1}{\left(1-\frac{1}{s}\right)s},$$

$$\left[\frac{3}{3}\right] = \frac{1}{\left(1+\frac{1}{s^2}-\frac{2}{s}\right)s^2}, \qquad (27)$$

By applying the inverse, Laplace transforms into Eqs. (27), respectively, we obtain the exact solutions.

5. Results and Discussion

From the HPM solutions, we tabulated and formulated numerical results and discussions in Tables 1-6 and Figures 1-6. Our observations indicate that the accuracy of the solutions depends approximately the approximation terms, and the solutions converge to the exact ones when an infinitely large number of terms are considered. This implies that achieving higher accuracy necessitates additional computational work and effort. Consequently, to obtain accurate results, modifications should be made to the HPM procedure by incorporating Laplace transformations and Pade approximations, which provides exact solutions without the need to increase the number of approximation terms in the standard HPM.

Table 1: Numerical result of example 1

x	Exact Solution $u_1(t) = te^t$	Approximate Solution	HPM Absolute Error
0.0	0.0000000000	0.0000000000	0.00000000
0.2	0.2442805516	0.2442805476	4.04×10^{-9}
0.4	0.5967298791	0.5967293088	5.70×10^{-7}
0.6	1.0932712802	1.0932605520	1.07×10^{-5}
0.8	1.7804327428	1.7803444852	8.83×10^{-5}
1.0	2.7182818285	2.7178207672	4.61×10^{-4}

Table 2: Numerical result of example 1

t	Exact Solution $u_1(t) = (t+1)e^t$	Approximate Solution	HPM Absolute Error
0.0	1.0000000000	1.0000000000	0.00000000
0.2	1.4656833098	1.465683299	1.17×10^{-8}
0.4	2.0885545767	2.0885533934	1.18×10^{-6}
0.6	2.9153900806	2.9153708834	1.92×10^{-5}
0.8	4.0059736713	4.0058288383	1.45×10^{-4}
1.0	5.4365636569	5.4358498677	7.14×10^{-4}

Table 3: Numerical result of example 1

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t	Exact Solution $u_1(t) = (t+2)e^t$	Approximate Solution	HPM Absolute Error
0.0	2.0000000000	2.0000000000	0.00000000
0.2	2.6870860680	2.687086042	2.59×10^{-8}
0.4	3.5803792743	3.580377052	2.22×10^{-6}
0.6	4.737508881	4.737476270	3.26×10^{-5}
0.8	6.2315145998	6.231284915	2.30×10^{-4}
1.0	8.1548454854	8.153769211	1.07×10^{-3}

Table 4: Numerical result of example2

t	Exact Solution $u_1(t) = e^{2t}$	Approximate Solution	HPM Absolute Error
0.0	1.00000000000	1.00000000000	0.00
0.2	1.4918246976	1.4918246969	7.52×10^{-10}
0.4	2.2255409285	2.2255405267	4.02×10^{-7}
0.6	3.3201169227	3.3201007909	1.61×10^{-5}
0.8	4.9530324244	4.9528076759	2.25×10^{-5}
1.0	7.3890560989	7.3873015873	1.75×10^{-3}

Table 5: Numerical result of example 2

t	Exact Solution $u_1(t) = e^t$	Approximate Solution	HPM Absolute Error
0.0	1.000000000	1.000000000	0.0
0.2	1.2214027581	1.2214027581	1.44×10^{-12}
0.4	1.4918246976	1.4918246969	7.53×10^{-10}
0.6	1.8221188004	1.8221187709	2.95×10^{-8}
0.8	2.2255409285	2.2255405267	4.02×10^{-7}
1.0	2.7182818285	2.7182787698	3.06×10^{-6}

Table 6: Numerical result of example 1

t	Exact Solution $\mathrm{u}_1(\mathrm{t}) = \mathrm{te}^{\mathrm{t}}$	Approximate Solution	HPM Absolute Error
0.0	0.0000000000	0.0000000000	0.00
0.2	0.2442805516	0.2442805516	1.30×10^{-11}
0.4	0.5967298791	0.5967298723	6.80×10^{-9}
0.6	1.0932712802	1.0932710126	2.67×10^{-7}
0.8	1.7804327428	1.7804290926	3.65×10^{-6}
1.0	2.7182818285	2.7182539683	2.79×10^{-5}

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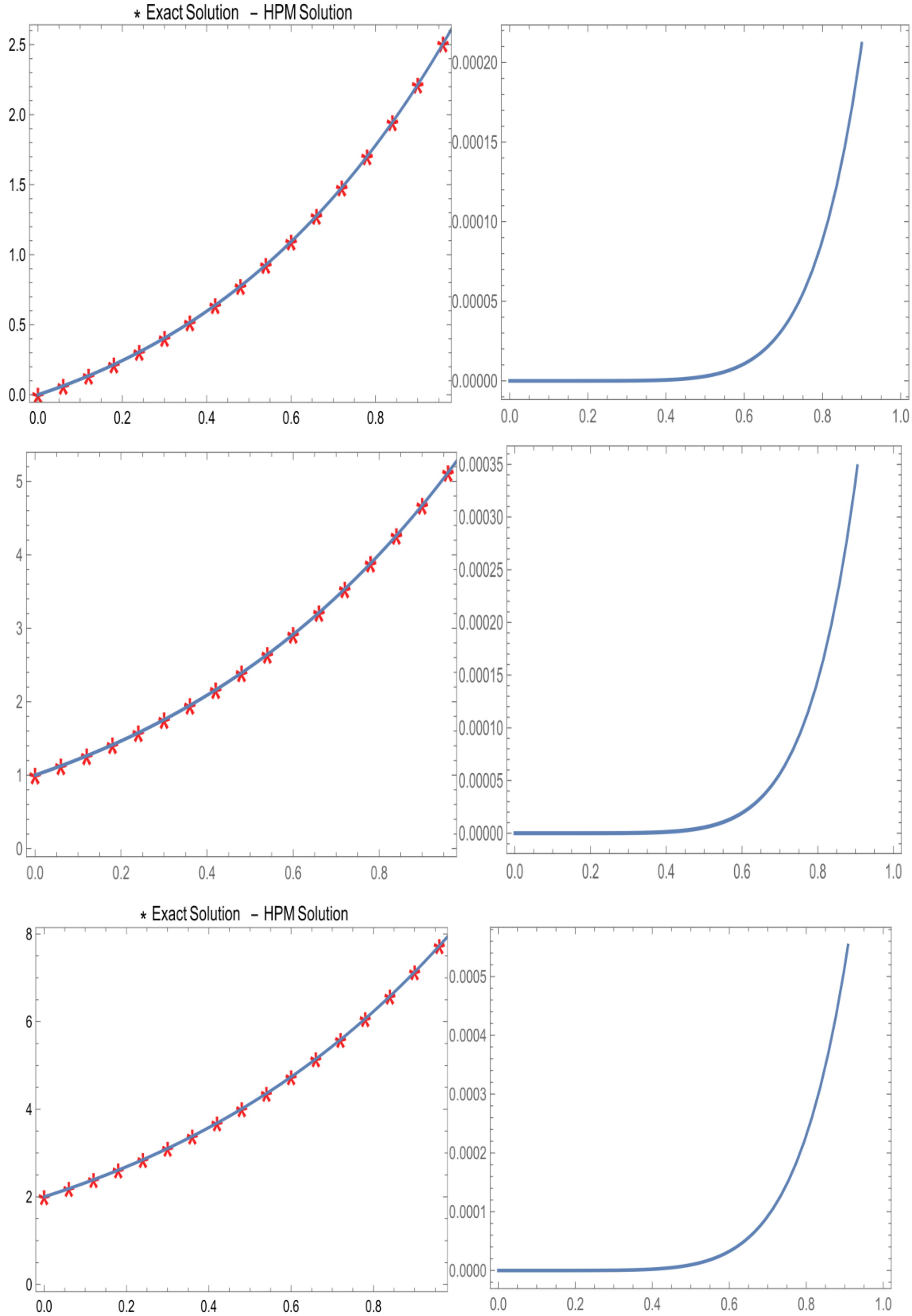

Figure 1. Plot of a) Exact and approximate solutions b) Absolute errors for example 1

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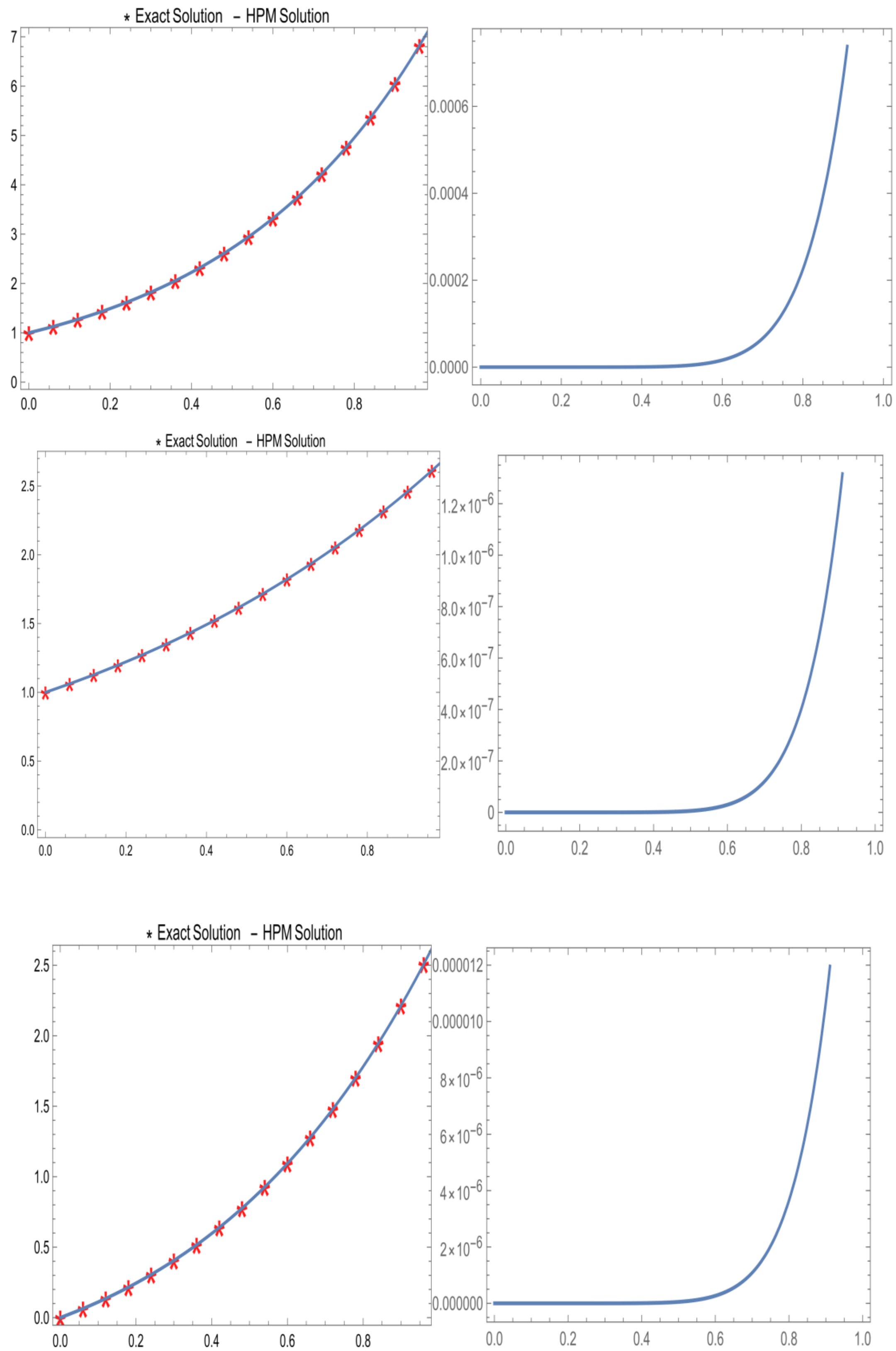

Figure 2. Plot of a) Exact and approximate solutions b) Absolute errors for example 2

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6. Conclusion

In this research study, we present a novel methodology based on the HPM for solving a system of ordinary differential equations. This methodology not only demonstrates effectiveness and reliability, but also offers distinct advantages over alternative techniques. It is capable of accurately providing solutions for complex systems, making it a valuable tool for researchers and practitioners involved in the analysis of dynamic phenomena governed by such systems. Through illustrative examples and comparisons with numerical outcomes obtained from existing literature, we demonstrate that this methodology can achieve the exact analytical solution by using only a limited number of terms from the HPM truncated series solution. In conclusion, we firmly assert that this methodology proposes a robust and promising approach for handling various types of differential equations.

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