

Existence and stability results for a Langevin system with Caputo–Hadamard fractional operators

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This paper is devoted to analyzing a new model of a coupled Langevin system with fractional operators under nonlocal antiperiodic integral boundary conditions. This model involves nonlinear Langevin fractional equations with Caputo–Hadamard and Caputo fractional operators. Also, the existence and uniqueness of solutions to the suggested model have been investigated by the fixed-point technique. Moreover, the Hyers–Ulam stability of the solutions has been discussed. Finally, we provide an illustrative example to support the theoretical results.

Keywords: Fractional derivatives; Langevin fractional equation; fixed point technique; stability analysis.

Mathematics Subject Classifications 2020: 47H10, 26A33, 34A08, 34B10, 34B15

1. Introduction

Recent studies have shown that some real-world processes cannot be effectively described by classical differential equations (DEs) formulated using standard derivatives and integrals, integer order ordinary DEs or partial DEs. Fractional calculus is a powerful tool for simulating these complex processes, since it extends the idea of

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integer order to arbitrary order. Due to its numerous applications across a variety of scientific domains, fractional calculus has attracted a lot of attention during the past few decades. It has been applied to a variety of scientific disciplines, including polymer rheology, biology, capacitor theory, the blood-Bow phenomenon, nonlinear seismic oscillation, image processing, aerodynamics, geophysics, and electrical circuits. We cite some publications [1–4] for other applications of fractional differential equations (FDEs).

The evolution of physical events in an environment that is fluctuating is described by the Langevin equation (LE). In 1908, Paul Langevin made the first mention of the idea of the classical version of the LE. This version of the mentioned equation describes the Brownian motion of particles [5]. For several regions of fractal disorder, the classical LE does not produce adequate results. Therefore, Kubo *et al.* [6, 7] presented several hereditary and fractal aspects of the FDEs to illustrate how they are used dynamically. Fractional LEs with two different fractional orders were first employed by Lim *et al.* [8] in 2008. In comparison to the solutions of single-order FDEs, one can find more flexible solutions for multi-order LE. On the other side, in recent years, we have seen comprehensive discussions on FDEs with different boundary conditions [9–19]. In between, many researchers have conducted an analysis on dynamics of the LE. Yu *et al.* [20] examine the existence results obtained using the fixed point criteria for FDEs in the Langevin settings with various types of boundary conditions. By employing upper and lower solution technique analyses, Baghani *et al.* [21] connected some results for coupled Langevin systems with various forms of boundary conditions. Discussion on LE with nonlocal fractional boundary conditions is provided by Fazli and Nieto [22] in 2018.

A close approximation of the exact solutions for FDEs is appeared in the notion of the Hyers–Ulam (HU) stability. Rizwan and Zada investigated the stability and existence results in relation to the LEs in their article [23]. Later in another study, the stability and existence of unique solution were addressed for the mentioned equation by Wang and Li [24]. After that, Wang and Wang [25] turned to similar theorems on the existence notion, but with strip conditions that include the p -Laplacian operator. Matar *et al.* [26] presented their stability theorems in the version of the fractional system of the coupled LEs. For more information about other studies, see [28, 29].

In this publication, the existence of solutions and stability results for the following BVP incorporating nonlinear coupled LEs with separate Caputo–Hadamard, Caputo fractional derivatives, and Hadamard integrals are investigated:

$$\begin{cases} {}^{\text{CH}}D^\xi ({}^C D^\omega + \gamma)\theta(\tau) = \Lambda(\tau, \theta(\tau), \vartheta(\tau)), & \tau \in [1, c], \\ {}^{\text{CH}}D^{\xi^*} ({}^C D^{\omega^*} + \gamma^*)\vartheta(\tau) = \Lambda^*(\tau, \vartheta(\tau), \theta(\tau)), & \tau \in [1, c], \\ \theta(0) = 0 = \vartheta(0), \quad {}^C D^\omega \theta(0) = 0 = {}^C D^{\omega^*} \vartheta(0), \\ {}^C D^\omega \theta(c) + \beta\theta(c) = 0, \quad {}^C D^{\omega^*} \vartheta(c) + \beta^*\vartheta(c) = 0, \\ {}^C D^\omega \theta(1) = I^\sigma \theta(\ell), \quad {}^C D^{\omega^*} \vartheta(1) = I^{\sigma^*} \vartheta(\ell), \end{cases} \quad (1.1)$$

where $\beta \neq \gamma, \beta^* \neq \gamma^*, {}^{\text{CH}}D^\xi, {}^{\text{CH}}D^{\xi^*}, {}^C D^\omega, {}^C D^{\omega^*}$ refer to the Caputo–Hadamard and Caputo derivatives of fractional orders $\xi, \xi^*, \omega, \omega^*$, respectively. Further $\xi + \omega \in (1, 3], \xi^* + \omega^* \in (1, 3], \gamma, \gamma^*, \sigma, \sigma^* > 0, \beta, \beta^* \in \mathbb{R}$, and $\ell \in (1, c)$. Also, $\Lambda, \Lambda^* \in C([1, c] \times \mathbb{R}^2, \mathbb{R})$. This version of BVP combines three fractional operators in the main LE and its boundary conditions. Also, the investigation of the qualitative properties of solutions for such a coupled system is the main goal of this paper. Despite the current interest in the topic of FDEs with boundary conditions, no articles specifically addressing this type of combined coupled Langevin system have been published.

The rest of the paper is organized as follows. Basic definitions and lemmas are found in Sec. 2. Section 3 contains the existence and uniqueness of the solution to the considered problem. HU-stability of the solution is discussed in the next section. We propose an application to provide a more transparent result in Sec. 5.

2. Preliminaries

First of all, the aim of this section is to present some important basics that help the readers in understanding our study.

Definition 2.1 ([2]). Let $\theta : [1, \infty) \rightarrow \mathbb{R}$ be a continuous and integrable function (CIF), then the Hadamard integral of fractional order $\sigma > 0$ is defined as

$${}^H I^\sigma \theta(\tau) = \frac{1}{\Gamma(\sigma)} \int_1^\tau \left(\log\left(\frac{\tau}{r}\right)\right)^{\sigma-1} \frac{\theta(r)}{r} dr.$$

Definition 2.2 ([2]). Let $\theta : [1, \infty) \rightarrow \mathbb{R}$ be a CIF, $\sigma \geq 0$ and $p = \sigma + 1$. The Hadamard derivative of fractional order σ is described as

$${}^H D^\sigma \theta(\tau) = \frac{1}{\Gamma(p - \sigma)} \left(\tau \frac{d}{d\tau}\right)^p \int_1^\tau \left(\log\left(\frac{\tau}{r}\right)\right)^{p-\sigma-1} dr.$$

Definition 2.3 ([30]). Let $\theta : [1, \infty) \rightarrow \mathbb{R}$ be a continuous function. The Caputo–Hadamard derivative of order $\sigma > 0$ is given by

$${}^{\text{CH}} D^\sigma \theta(\tau) = \frac{1}{\Gamma(p - \sigma)} \int_1^\tau \left(\log\left(\frac{\tau}{r}\right)\right)^{p-\sigma-1} \frac{\sigma^p \theta(r)}{r} dr, \quad p = \sigma + 1.$$

Definition 2.4 ([2]). Let $\theta : [1, \infty) \rightarrow \mathbb{R}$ be a p -times continuously differentiable function. The Caputo derivative of order $\sigma > 0$ is formulated as

$${}^C D^\sigma \theta(\tau) = \frac{1}{\Gamma(p - \sigma)} \int_0^\tau (\tau - r)^{p-\sigma-1} (\theta)^{(p)}(r) dr,$$

where $\sigma \in (p - 1, p)$ and $p = [\sigma] + 1$.

Lemma 2.5 ([2]). Suppose that $\varkappa > 0$, $\kappa = [\text{Re}(\varkappa)] + 1$ and $\text{Re}(\kappa) > 0$, then

$$\begin{cases} {}^{\text{CH}}D^{\varkappa}(\log(\tau))^{\kappa-1} = \frac{\Gamma(\kappa)}{\Gamma(\varkappa - \kappa)}(\log(\tau))^{\varkappa-\kappa-1}, \\ {}^{\text{H}}I^{\varkappa}(\log(\tau))^{\kappa-1} = \frac{\Gamma(\kappa)}{\Gamma(\kappa + \varkappa)}(\log(\tau))^{\kappa+\varkappa-1}. \end{cases}$$

The next theorem is a fixed point criterion for proving our main existence theorem.

Theorem 2.6 ([2]). Assume that $\Delta \neq \emptyset$ is a convex, closed and bounded set in a Banach space Ξ . Assume also that $\Omega_1, \Omega_2 : \Delta \rightarrow \Xi$ are operators such that

- (1) for $\theta, \vartheta \in \Delta$, $\Omega_1\theta + \Omega_2\vartheta \in \Delta$;
- (2) Ω_2 is contraction;
- (3) Ω_2 is compact and continuous.

Then, $\exists \theta^* \in \Delta$ s.t. $\theta^* = \Omega_1\theta^* + \Omega_2\theta^*$.

3. Auxiliary Lemma and Assumptions

We begin this part with the proof of an auxiliary lemma which gives us the main integral form of the solutions of the coupled system of LEs (1.1).

Lemma 3.1. Assume that $\Lambda, \Lambda^* \in C([1, c] \times \mathbb{R}^2, \mathbb{R})$ and the BVP including the coupled Langevin fractional DEs is given by (1.1). Then the form of the solution to the problem (1.1) is

$$\begin{aligned} \theta(\tau) = & \frac{1}{\Gamma(\xi + \omega)} \int_1^\tau \left(\log\left(\frac{\tau}{r}\right)\right)^{\xi+\omega-1} \Lambda(r, \theta(r), \vartheta(r)) \frac{dr}{r} \\ & - \frac{\gamma}{\Gamma(\omega)} \int_1^\tau \left(\log\left(\frac{\tau}{r}\right)\right)^{\omega-1} \theta(r) \frac{dr}{r} + \frac{(\log(\tau))^{\omega+1}}{\Gamma(2 + \omega) - (\beta - \gamma)} \\ & \times \left[\frac{(\gamma - \beta)}{\Gamma(\xi + \omega)} \int_1^c \left(\log\left(\frac{c}{r}\right)\right)^{\xi+\omega-1} \Lambda(r, \theta(r), \vartheta(r)) \frac{dr}{r} \right. \\ & + \frac{\gamma(\beta - \gamma)}{\Gamma(\omega)} \int_1^c \left(\log\left(\frac{c}{r}\right)\right)^{\omega-1} \theta(r) \frac{dr}{r} \\ & - \frac{1}{\Gamma(\xi)} \int_1^c \left(\log\left(\frac{c}{r}\right)\right)^{\xi-1} \Lambda(r, \theta(r), \vartheta(r)) \frac{dr}{r} \\ & \left. - \frac{1}{\Gamma(\sigma)} \int_1^\ell \left(\log\left(\frac{\ell}{r}\right)\right)^{\sigma-1} \theta(r) \frac{dr}{r} \right], \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} \vartheta(\tau) &= \frac{1}{\Gamma(\xi^* + \omega^*)} \int_1^\tau \left(\log\left(\frac{\tau}{r}\right)\right)^{\xi^* + \omega^* - 1} \Lambda^*(r, \vartheta(r), \theta(r)) \frac{dr}{r} \\ &\quad - \frac{\gamma^*}{\Gamma(\omega^*)} \int_1^\tau \left(\log\left(\frac{\tau}{r}\right)\right)^{\omega^* - 1} \vartheta(r) \frac{dr}{r} + \frac{(\log(\tau))^{\omega^* + 1}}{\Gamma(2 + \omega^*) - (\beta^* - \gamma^*)} \\ &\quad \times \left[\frac{(\gamma^* - \beta^*)}{\Gamma(\xi^* + \omega^*)} \int_1^c \left(\log\left(\frac{c}{r}\right)\right)^{\xi^* + \omega^* - 1} \Lambda^*(r, \vartheta(r), \theta(r)) \frac{dr}{r} \right. \\ &\quad \left. + \frac{\gamma(\beta^* - \gamma^*)}{\Gamma(\omega^*)} \int_1^c \left(\log\left(\frac{c}{r}\right)\right)^{\omega^* - 1} \vartheta(r) \frac{dr}{r} - \frac{1}{\Gamma(\xi^*)} \int_1^c \left(\log\left(\frac{c}{r}\right)\right)^{\xi^* - 1} \right. \\ &\quad \left. \times \Lambda^*(r, \vartheta(r), \theta(r)) \frac{dr}{r} - \frac{1}{\Gamma(\sigma^*)} \int_1^\ell \left(\log\left(\frac{\ell}{r}\right)\right)^{\sigma^* - 1} \vartheta(r) \frac{dr}{r} \right]. \end{aligned} \tag{3.2}$$

Proof. Applying ${}^H I^\xi$ to both sides of the first equation of (1.1), one can get

$${}^C D^\omega \theta(\tau) = I^\xi \Lambda(\tau) - \gamma \theta(\tau) + A_1 \log \tau + A_0, \tag{3.3}$$

with the real constants A_0, A_1 . Again, using ${}^H I^\omega$ to both sides of (3.3), one can get

$$\theta(\tau) = I^{\omega + \xi} \Lambda(\tau) - \gamma I^\omega \theta(\tau) + A_1 \frac{(\log \tau)^{\omega + 1}}{\Gamma(\omega + 2)} + A_0 \frac{\tau^\omega}{\Gamma(\omega + 1)} + A_2, \tag{3.4}$$

where A_0, A_1 and A_2 are arbitrary constants. From the conditions $\theta(0) = 0$ and ${}^C D^\omega \theta(0) = 0$, we have $A_0 = A_2 = 0$. Hence, Eq. (3.4) reduces to

$$\theta(\tau) = I^{\omega + \xi} \Lambda(\tau) - \gamma I^\omega \theta(\tau) + A_1 \frac{(\log \tau)^{\omega + 1}}{\Gamma(\omega + 2)}. \tag{3.5}$$

Applying boundary conditions ${}^C D^\omega \theta(c) + \beta \theta(c) = 0$ and ${}^C D^\omega \theta(1) = {}^H I^\sigma \theta(\ell)$, we obtain that

$$\begin{aligned} A_1 &= \frac{\Gamma(2 + \omega)}{\Gamma(2 + \omega) - (\beta - \gamma)} \left[\frac{(\gamma - \beta)}{\Gamma(\xi + \omega)} \int_1^c \left(\log\left(\frac{c}{r}\right)\right)^{\xi + \omega - 1} \Lambda(r, \theta(r), \vartheta(r)) \frac{dr}{r} \right. \\ &\quad \left. + \frac{\gamma(\beta - \gamma)}{\Gamma(\omega)} \int_1^c \left(\log\left(\frac{c}{r}\right)\right)^{\omega - 1} \theta(r) \frac{dr}{r} - \frac{1}{\Gamma(\xi)} \int_1^c \left(\log\left(\frac{c}{r}\right)\right)^{\xi - 1} \right. \\ &\quad \left. \times \Lambda(r, \theta(r), \vartheta(r)) \frac{dr}{r} - \frac{1}{\Gamma(\sigma)} \int_1^\ell \left(\log\left(\frac{\ell}{r}\right)\right)^{\sigma - 1} \theta(r) \frac{dr}{r} \right]. \end{aligned} \tag{3.6}$$

In view of (3.6) and inserting it in (3.5), we obtain (3.1). Similarly, using the same method as above and using conditions $\vartheta(0) = 0, {}^C D^{\omega^*} \vartheta = 0, {}^C D^{\omega^*} \vartheta(c) + \beta^* \vartheta(c) = 0$, and ${}^C D^{\omega^*} \vartheta(1) = {}^H I^{\sigma^*} \vartheta(\ell)$, we get (3.2). This completes the proof. \square

Assume that $\mathcal{U} = (C[1, c], \mathbb{R})$ is the space of all continuous functions from $[1, c]$ to \mathbb{R} . Define the space $\Theta = \{\theta(\tau) : \theta(\tau) \in \mathcal{U}\}$ equipped with $\|\theta\| = \sup\{|\theta(\tau)|, \tau \in [1, c]\}$. Also, consider $\Upsilon = \{\vartheta(\tau) : \vartheta(\tau) \in \mathcal{U}\}$ equipped with $\|\vartheta\| = \sup\{|\vartheta(\tau)|, \tau \in [1, c]\}$. Obviously, both $(\Theta, \|\cdot\|)$ and $(\Upsilon, \|\cdot\|)$ are Banach spaces. Clearly, the product $(\Theta \times \Upsilon, \|\cdot\|)$ is a Banach space endowed with $\|(\theta, \vartheta)\| = \|\theta\| + \|\vartheta\|$.

Using Lemma 3.1, we apply the operator $\Omega : \Theta \times \Upsilon \rightarrow \Theta \times \Upsilon$ as follows:

$$\Omega(\theta, \vartheta)(\tau) = (\Omega_1(\theta, \vartheta)(\tau), \Omega_2(\theta, \vartheta)(\tau)), \tag{3.7}$$

where

$$\begin{aligned} \Omega_1(\theta, \vartheta)(\tau) &= \frac{1}{\Gamma(\xi + \omega)} \int_1^\tau \left(\log\left(\frac{\tau}{r}\right)\right)^{\xi + \omega - 1} \Lambda(r, \theta(r), \vartheta(r)) \frac{dr}{r} \\ &\quad - \frac{\gamma}{\Gamma(\omega)} \int_1^\tau \left(\log\left(\frac{\tau}{r}\right)\right)^{\omega - 1} \theta(r) \frac{dr}{r}, \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} \Omega_2(\theta, \vartheta)(\tau) &= \frac{(\log(\tau)^{\omega+1})}{\Gamma(2 + \omega) - (\beta - \gamma)} \left[\frac{(\gamma - \beta)}{\Gamma(\xi + \omega)} \int_1^c \left(\log\left(\frac{c}{r}\right)\right)^{\xi + \omega - 1} \Lambda(r, \theta(r), \vartheta(r)) \frac{dr}{r} \right. \\ &\quad + \frac{\gamma(\beta - \gamma)}{\Gamma(\omega)} \int_1^c \left(\log\left(\frac{c}{r}\right)\right)^{\omega - 1} \theta(r) \frac{dr}{r} - \frac{1}{\Gamma(\xi)} \int_1^c \left(\log\left(\frac{c}{r}\right)\right)^{\xi - 1} \\ &\quad \left. \times \Lambda(r, \theta(r), \vartheta(r)) \frac{dr}{r} - \frac{1}{\Gamma(\sigma)} \int_1^\ell \left(\log\left(\frac{\ell}{r}\right)\right)^{\sigma - 1} \theta(r) \frac{dr}{r} \right]. \end{aligned} \tag{3.9}$$

Thus, the existence of the solution for the nonlinear coupled Langevin system (1.1) is limited to finding the fixed point of Eq. (3.7).

In order to facilitate computation, we set

$$\begin{aligned} \Upsilon_1 &= \frac{1}{\Gamma(1 + \xi + \omega)} + \frac{1}{|\Gamma(2 + \omega) - (\beta - \gamma)|\Gamma(1 + \xi)} \\ &\quad + \frac{|\gamma - \beta|}{|\Gamma(2 + \omega) - (\beta - \gamma)|\Gamma(1 + \xi + \omega)}, \\ \Upsilon_1^* &= \frac{1}{\Gamma(1 + \xi^* + \omega^*)} + \frac{1}{|\Gamma(2 + \omega^*) - (\beta^* - \gamma^*)|\Gamma(1 + \xi^*)} \\ &\quad + \frac{|\gamma^* - \beta^*|}{|\Gamma(2 + \omega^*) - (\beta^* - \gamma^*)|\Gamma(1 + \xi^* + \omega^*)}, \\ \Upsilon_2 &= \frac{\gamma}{\Gamma(1 + \omega)} + \frac{\gamma|\gamma - \beta|}{|\Gamma(2 + \omega) - (\beta - \gamma)|\Gamma(1 + \omega)} + \frac{\log(\ell)^\sigma}{|\Gamma(2 + \omega) - (\beta - \gamma)|\Gamma(1 + \xi)}, \end{aligned}$$

$$\begin{aligned}
 \Upsilon_2^* &= \frac{\gamma^*}{\Gamma(1 + \omega^*)} + \frac{\gamma^*|\gamma^* - \beta^*|}{|\Gamma(2 + \omega^*) - (\beta^* - \gamma^*)\Gamma(1 + \omega^*)|} \\
 &\quad + \frac{\log(\ell)^{\sigma^*}}{|\Gamma(2 + \omega^*) - (\beta^* - \gamma^*)\Gamma(1 + \xi^*)|}, \\
 \Upsilon_3 &= \left\{ \frac{\zeta_\Lambda|\gamma - \beta|}{|\Gamma(2 + \omega) - (\beta - \gamma)\Gamma(1 + \xi + \omega)|} + \frac{\zeta_\Lambda|\gamma - \beta|}{|\Gamma(2 + \omega) - (\beta - \gamma)\Gamma(1 + \xi)|} \right. \\
 &\quad \left. \times \frac{\gamma|\gamma - \beta|}{|\Gamma(2 + \omega) - (\beta - \gamma)\Gamma(1 + \omega)|} + \frac{\log(\ell)^\sigma}{|\Gamma(2 + \omega) - (\beta - \gamma)\Gamma(1 + \xi)|} \right\}, \\
 \Upsilon_3^* &= \left\{ \frac{\zeta_{\Lambda^*}|\gamma^* - \beta^*|}{|\Gamma(2 + \omega^*) - (\beta^* - \gamma^*)\Gamma(1 + \xi^* + \omega^*)|} + \frac{\zeta_{\Lambda^*}|\gamma^* - \beta^*|}{|\Gamma(2 + \omega^*) - (\beta^* - \gamma^*)\Gamma(1 + \xi^*)|} \right. \\
 &\quad \left. \times \frac{\gamma^*|\gamma^* - \beta^*|}{|\Gamma(2 + \omega^*) - (\beta^* - \gamma^*)\Gamma(1 + \omega^*)|} + \frac{\log(\ell)^{\sigma^*}}{|\Gamma(2 + \omega^*) - (\beta^* - \gamma^*)\Gamma(1 + \xi^*)|} \right\}. \tag{3.10}
 \end{aligned}$$

Next, we state the necessary hypotheses for continuing this study.

- (S₁) $\Lambda : [1, c] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ is continuous.
- (S₂) For all $\tau \in [1, c]$, and $\theta_1, \theta_2, \bar{\theta}_1, \bar{\theta}_2, \vartheta_1, \vartheta_2, \bar{\vartheta}_1, \bar{\vartheta}_2 \in \mathcal{U}$, there exist constants $\zeta_\Lambda, \zeta_{\Lambda^*} > 0$ such that

$$\begin{aligned}
 |\Lambda(\tau, \theta_1, \theta_2) - \Lambda(\tau, \bar{\theta}_1, \bar{\theta}_2)| &\leq \zeta_\Lambda[|\theta_1 - \bar{\theta}_1| + |\theta_2 - \bar{\theta}_2|], \\
 |\Lambda^*(\tau, \vartheta_1, \vartheta_2) - \Lambda^*(\tau, \bar{\vartheta}_1, \bar{\vartheta}_2)| &\leq \zeta_{\Lambda^*}[|\vartheta_1 - \bar{\vartheta}_1| + |\vartheta_2 - \bar{\vartheta}_2|].
 \end{aligned}$$

- (S₃) For $(\tau, \theta(\tau), \vartheta(\tau)) \in [1, c] \times \mathbb{R} \times \mathbb{R}$, there exist $\Xi(\tau), \Xi^*(\tau) \in (C[1, c], \mathbb{R}^+)$ such that

$$\begin{aligned}
 |\Lambda(\tau, \theta(\tau), \vartheta(\tau))| &\leq |\Xi(\tau)| \quad \text{and} \\
 |\Lambda(\tau, \vartheta(\tau), \theta(\tau))| &\leq |\Xi^*(\tau)|.
 \end{aligned}$$

- (S₄) For all $\tau \in [1, c]$, there exist $N, N^* > 0$ such that

$$\begin{aligned}
 N &= \sup_{\tau \in [1, c]} |\Lambda(\tau, 0, 0)| < \infty \quad \text{and} \\
 N^* &= \sup_{\tau \in [1, c]} |\Lambda^*(\tau, 0, 0)| < \infty.
 \end{aligned}$$

4. Existence Theorem

Now, everything is ready to turn to the main existence theorem.

Theorem 4.1. *Assume that the hypotheses (S₁)–(S₃) hold, then there exists at least one solution to the nonlinear coupled Langevin system (1.1) on [1, c] if $\Upsilon_3, \Upsilon_3^* < 1$ (given by (3.10)).*

Proof. Define a closed ball $\mathfrak{R}_y = \{(\theta, \vartheta) \in \Theta \times \Upsilon : \|\theta\| \leq y, \|\vartheta\| \leq y\}$ and consider $\frac{\Upsilon_1 \Xi(\tau)}{1-\Upsilon_2} \leq y$ and $\frac{\Upsilon_1^* \Xi^*(\tau)}{1-\Upsilon_2^*} \leq y$. We split the proof into the following steps:

Step 1: $\Omega \in \mathfrak{R}_y: \forall (\theta, \vartheta) \in \mathfrak{R}_y$, estimate

$$\|\Omega(\theta, \vartheta)\| = \|\Omega_1(\theta, \vartheta)\| + \|\Omega_2(\theta, \vartheta)\|.$$

From (3.8), we get

$$\begin{aligned} |\Omega_1(\theta, \vartheta)(\tau)| &= \left| \frac{1}{\Gamma(\xi + \omega)} \int_1^\tau \left(\log\left(\frac{\tau}{r}\right)\right)^{\xi + \omega - 1} \Lambda(r, \theta(r), \vartheta(r)) \frac{dr}{r} \right. \\ &\quad \left. - \frac{\gamma}{\Gamma(\omega)} \int_1^\tau \left(\log\left(\frac{\tau}{r}\right)\right)^{\omega - 1} \theta(r) \frac{dr}{r} \right| \\ &\leq \frac{1}{\Gamma(\xi + \omega)} \int_1^\tau \left(\log\left(\frac{\tau}{r}\right)\right)^{\xi + \omega - 1} |\Lambda(r, \theta(r), \vartheta(r))| \frac{dr}{r} \\ &\quad + \frac{\gamma}{\Gamma(\omega)} \int_1^\tau \left(\log\left(\frac{\tau}{r}\right)\right)^{\omega - 1} |\theta(r)| \frac{dr}{r} \\ &\leq \frac{|\Xi(\tau)|}{\Gamma(\xi + \omega)} \int_1^\tau \left(\log\left(\frac{\tau}{r}\right)\right)^{\xi + \omega - 1} \frac{dr}{r} \\ &\quad + \frac{\gamma}{\Gamma(\omega)} \int_1^\tau \left(\log\left(\frac{\tau}{r}\right)\right)^{\omega - 1} |\theta(r)| \frac{dr}{r}. \end{aligned}$$

Applying sup to both sides, we have

$$\|\Omega_1(\theta, \vartheta)\| \leq \|\Xi\| \left(\frac{1}{\Gamma(\xi + \omega + 1)} \right) + \frac{y\gamma}{\Gamma(\omega + 1)}. \tag{4.1}$$

Similarly, one can write

$$\|\Omega_1(\vartheta, \theta)\| \leq \|\Xi^*\| \left(\frac{1}{\Gamma(\xi^* + \omega^* + 1)} \right) + \frac{y\gamma}{\Gamma(\omega^* + 1)}. \tag{4.2}$$

Again, using (3.9), we get

$$\begin{aligned} &\|\Omega_2(\theta, \vartheta)\| \\ &= \sup_{\tau \in [1, c]} \left| \frac{(\log(\tau))^{\omega+1}}{\Gamma(2 + \omega) - (\beta - \gamma)} \left[\frac{(\gamma - \beta)}{\Gamma(\xi + \omega)} \int_1^c \left(\log\left(\frac{c}{r}\right)\right)^{\xi + \omega - 1} \Lambda(r, \theta(r), \vartheta(r)) \frac{dr}{r} \right. \right. \\ &\quad + \frac{\gamma(\beta - \gamma)}{\Gamma(\omega)} \int_1^c \left(\log\left(\frac{c}{r}\right)\right)^{\omega - 1} \theta(r) \frac{dr}{r} - \frac{1}{\Gamma(\xi)} \int_1^c \left(\log\left(\frac{c}{r}\right)\right)^{\xi - 1} \\ &\quad \left. \left. \times \Lambda(r, \theta(r), \vartheta(r)) \frac{dr}{r} - \frac{1}{\Gamma(\sigma)} \int_1^\ell \left(\log\left(\frac{\ell}{r}\right)\right)^{\sigma - 1} \theta(r) \frac{dr}{r} \right] \right| \end{aligned}$$

$$\begin{aligned} &\leq \sup_{\tau \in [1, c]} \left\{ \frac{|\log(\tau)|^{\omega+1}}{|\Gamma(2+\omega) - (\beta - \gamma)|} \left[\frac{|\gamma - \beta|}{\Gamma(\xi + \omega)} \int_1^c \left(\log\left(\frac{c}{r}\right)\right)^{\xi+\omega-1} \right. \right. \\ &\quad \times |\Lambda(r, \theta(r), \vartheta(r))| \frac{dr}{r} + \frac{|\gamma(\beta - \gamma)|}{\Gamma(\omega)} \int_1^c \left(\log\left(\frac{c}{r}\right)\right)^{\omega-1} |\theta(r)| \frac{dr}{r} \\ &\quad + \frac{1}{\Gamma(\xi)} \int_1^c \left(\log\left(\frac{c}{r}\right)\right)^{\xi-1} |\Lambda(r, \theta(r), \vartheta(r))| \frac{dr}{r} \\ &\quad \left. \left. + \frac{1}{\Gamma(\sigma)} \int_1^\ell \left(\log\left(\frac{\ell}{r}\right)\right)^{\sigma-1} |\theta(r)| \frac{dr}{r} \right] \right\}, \end{aligned}$$

which implies that

$$\begin{aligned} \|\Omega_2(\theta, \vartheta)\| &\leq \|\Xi\| \left(\frac{1}{|\Gamma(2+\omega) - (\beta - \gamma)|\Gamma(\xi + 1)} \right. \\ &\quad \left. + \frac{|\gamma - \beta|}{|\Gamma(2+\omega) - (\beta - \gamma)|\Gamma(\xi + \omega + 1)} \right) \\ &\quad + y \left(\frac{\gamma|\beta - \gamma|}{|\Gamma(2+\omega) - (\beta - \gamma)|\Gamma(\omega + 1)} \right. \\ &\quad \left. + \frac{\log(\tau)^\sigma}{|\Gamma(2+\omega) - (\beta - \gamma)|\Gamma(\sigma + 1)} \right). \end{aligned} \tag{4.3}$$

Obviously, one has

$$\begin{aligned} \|\Omega_2(\vartheta, \theta)\| &\leq \|\Xi^*\| \left(\frac{1}{|\Gamma(2+\omega^*) - (\beta^* - \gamma^*)|\Gamma(\xi^* + 1)} \right. \\ &\quad \left. + \frac{|\gamma^* - \beta^*|}{|\Gamma(2+\omega^*) - (\beta^* - \gamma^*)|\Gamma(\xi^* + \omega^* + 1)} \right) \\ &\quad + y \left(\frac{\gamma^*|\beta^* - \gamma^*|}{|\Gamma(2+\omega^*) - (\beta^* - \gamma^*)|\Gamma(\omega^* + 1)} \right. \\ &\quad \left. + \frac{\log(\tau)^{\sigma^*}}{|\Gamma(2+\omega^*) - (\beta^* - \gamma^*)|\Gamma(\sigma^* + 1)} \right). \end{aligned} \tag{4.4}$$

Combining (4.1) and (4.3), we have

$$\begin{aligned} \|\Omega(\theta, \vartheta)\| &= \|\Omega_1(\theta, \vartheta)\| + \|\Omega_2(\theta, \vartheta)\| \\ &\leq \|\Xi\| \left(\frac{1}{\Gamma(\xi + \omega + 1)} + \frac{1}{|\Gamma(2+\omega) - (\beta - \gamma)|\Gamma(\xi + 1)} \right. \\ &\quad \left. + \frac{|\gamma - \beta|}{|\Gamma(2+\omega) - (\beta - \gamma)|\Gamma(\xi + \omega + 1)} \right) \end{aligned}$$

$$\begin{aligned}
 &+ y \left(\frac{\gamma}{\Gamma(\omega + 1)} + \frac{\gamma|\beta - \gamma|}{|\Gamma(2 + \omega) - (\beta - \gamma)|\Gamma(\omega + 1)} \right. \\
 &\quad \left. + \frac{\log(\ell)^\sigma}{|\Gamma(2 + \omega) - (\beta - \gamma)|\Gamma(\sigma + 1)} \right) \\
 &= \|\Xi\| \Upsilon_1 + y \Upsilon_2 \leq y.
 \end{aligned}$$

Similarly, combining (4.2) and (4.4), we get

$$\|\Omega(\theta, \vartheta)\| \leq \|\Xi^*\| \Upsilon_1^* + y \Upsilon_2^* \leq y.$$

Hence, $\Omega \in \mathfrak{R}_y$, that is, the condition (1) of Theorem 2.6 is fulfilled.

Step 2: Ω_2 is contraction: For this regard, let $\theta, \vartheta, \bar{\theta}, \bar{\vartheta} \in \mathfrak{R}_y$. We have

$$\begin{aligned}
 &\|\Omega_2(\theta, \vartheta) - \Omega_2(\bar{\theta}, \bar{\vartheta})\| \\
 &= \sup_{\tau \in [1, c]} |\Omega_2(\theta, \vartheta)(\tau) - \Omega_2(\bar{\theta}, \bar{\vartheta})(\tau)| \\
 &\leq \sup_{\tau \in [1, c]} \left\{ \frac{|\log(\tau)|^{\omega+1}}{|\Gamma(2 + \omega) - (\beta - \gamma)|} \right. \\
 &\quad \times \left[\frac{|\gamma - \beta|}{\Gamma(\xi + \omega)} \int_1^c \left(\log\left(\frac{c}{r}\right)\right)^{\xi + \omega - 1} |\Lambda(r, \theta(r), \vartheta(r)) - \Lambda(r, \bar{\theta}(r), \bar{\vartheta}(r))| \frac{dr}{r} \right. \\
 &\quad + \frac{|\gamma(\beta - \gamma)|}{\Gamma(\omega)} \int_1^c \left(\log\left(\frac{c}{r}\right)\right)^{\omega - 1} |\theta(r) - \bar{\theta}(r)| \frac{dr}{r} \\
 &\quad + \frac{1}{\Gamma(\xi)} \int_1^c \left(\log\left(\frac{c}{r}\right)\right)^{\xi - 1} |\Lambda(r, \theta(r), \vartheta(r)) - \Lambda(r, \bar{\theta}(r), \bar{\vartheta}(r))| \frac{dr}{r} \\
 &\quad \left. \left. + \frac{1}{\Gamma(\sigma)} \int_1^\ell \left(\log\left(\frac{\ell}{r}\right)\right)^{\sigma - 1} |\theta(r) - \bar{\theta}(r)| \frac{dr}{r} \right] \right\},
 \end{aligned}$$

which implies that

$$\begin{aligned}
 &\|\Omega_2(\theta, \vartheta) - \Omega_2(\bar{\theta}, \bar{\vartheta})\| \\
 &\leq \left(\frac{\zeta_\Lambda |\gamma - \beta|}{|\Gamma(2 + \omega) - (\beta - \gamma)|\Gamma(\xi + \omega + 1)} + \frac{\gamma|(\beta - \gamma)|}{|\Gamma(2 + \omega) - (\beta - \gamma)|\Gamma(\omega + 1)} \right. \\
 &\quad \left. + \frac{\zeta_\Lambda}{|\Gamma(2 + \omega) - (\beta - \gamma)|\Gamma(\xi + 1)} + \frac{\log(\ell)^\sigma}{|\Gamma(2 + \omega) - (\beta - \gamma)|\Gamma(\sigma + 1)} \right) \|\theta - \bar{\theta}\| \\
 &\quad + \left(\frac{\zeta_\Lambda |\gamma - \beta|}{|\Gamma(2 + \omega) - (\beta - \gamma)|\Gamma(\xi + \omega + 1)} \right. \\
 &\quad \left. + \frac{\zeta_\Lambda}{|\Gamma(2 + \omega) - (\beta - \gamma)|\Gamma(\xi + 1)} \right) \|\vartheta - \bar{\vartheta}\|
 \end{aligned}$$

$$\begin{aligned}
 &\leq \left(\frac{\zeta_\Lambda |\gamma - \beta|}{|\Gamma(2 + \omega) - (\beta - \gamma)|\Gamma(\xi + \omega + 1)} + \frac{\gamma |(\beta - \gamma)|}{|\Gamma(2 + \omega) - (\beta - \gamma)|\Gamma(\omega + 1)} \right. \\
 &\quad + \frac{\zeta_\Lambda}{|\Gamma(2 + \omega) - (\beta - \gamma)|\Gamma(\xi + 1)} \\
 &\quad \left. + \frac{\log(\ell)^\sigma}{|\Gamma(2 + \omega) - (\beta - \gamma)|\Gamma(\sigma + 1)} \right) (\|\theta - \bar{\theta}\| + \|\vartheta - \bar{\vartheta}\|) \\
 &= \Upsilon_3 \|(\theta, \vartheta) - (\bar{\theta}, \bar{\vartheta})\|. \tag{4.5}
 \end{aligned}$$

Similarly, one can obtain that

$$\|\Omega_2(\vartheta, \theta) - \Omega_2(\bar{\vartheta}, \bar{\theta})\| \leq \Upsilon_3^* \|(\vartheta, \theta) - (\bar{\vartheta}, \bar{\theta})\|. \tag{4.6}$$

Since $\Upsilon_3, \Upsilon_3^* < 1$, then by (4.5) and (4.6), we conclude that the mapping Ω_2 is contraction. This completes the condition (2) in Theorem 2.6.

Step 3: Ω_1 is continuous and compact: Since $\Lambda(\tau, \theta(\tau), \vartheta(\tau)) \in C([1, c] \times \mathbb{R}^2, \mathbb{R})$, then it is continuous on $\tau \in [1, c]$. Hence, Ω_1 is continuous. To show that Ω_1 is compact, it is sufficient to claim that Ω_1 is relatively compact and uniformly bounded on \mathfrak{R}_y . Now, for $(\theta, \vartheta) \in \mathfrak{R}_y$, we have

$$\begin{aligned}
 \|\Omega_1(\vartheta, \theta)\| &= \sup_{\tau \in [1, c]} |\Omega_1(\theta, \vartheta)(\tau)| \\
 &\leq \sup_{\tau \in [1, c]} \left\{ \frac{1}{\Gamma(\xi + \omega)} \int_1^\tau \left(\log\left(\frac{\tau}{r}\right)\right)^{\xi + \omega - 1} |\Lambda(r, \theta(r), \vartheta(r))| \frac{dr}{r} \right. \\
 &\quad \left. + \frac{|\gamma|}{\Gamma(\omega)} \int_1^\tau \left(\log\left(\frac{\tau}{r}\right)\right)^{\omega - 1} |\theta(r)| \frac{dr}{r} \right\} \\
 &\leq \frac{\|\Xi\|}{\Gamma(\xi + \omega + 1)} + \frac{y|\gamma|}{\Gamma(\omega + 1)} < \infty. \tag{4.7}
 \end{aligned}$$

Similarly, we can write

$$\|\Omega_1(\vartheta, \theta)\| \leq \frac{\|\Xi^*\|}{\Gamma(\xi^* + \omega^* + 1)} + \frac{y|\gamma|}{\Gamma(\omega^* + 1)} < \infty. \tag{4.8}$$

It follows from (4.7) and (4.8) that Ω_1 is uniformly bounded on \mathfrak{R}_y . Consider

$$\begin{aligned}
 &|\Omega_1(\theta, \vartheta)(\tau_2) - \Omega_1(\theta, \vartheta)(\tau_1)| \\
 &\leq \left| \frac{1}{\Gamma(\xi + \omega)} \int_1^{\tau_2} \left(\log\left(\frac{\tau_2}{r}\right)\right)^{\xi + \omega - 1} \Lambda(r, \theta(r), \vartheta(r)) \frac{dr}{r} \right. \\
 &\quad - \frac{\gamma}{\Gamma(\omega)} \int_1^{\tau_2} \left(\log\left(\frac{\tau_2}{r}\right)\right)^{\omega - 1} \theta(r) \frac{dr}{r} - \frac{1}{\Gamma(\xi + \omega)} \int_1^{\tau_1} \left(\log\left(\frac{\tau_1}{r}\right)\right)^{\xi + \omega - 1} \\
 &\quad \left. \times \Lambda(r, \theta(r), \vartheta(r)) \frac{dr}{r} + \frac{\gamma}{\Gamma(\omega)} \int_1^{\tau_1} \left(\log\left(\frac{\tau_1}{r}\right)\right)^{\omega - 1} \theta(r) \frac{dr}{r} \right|
 \end{aligned}$$

$$\begin{aligned} &\leq \left| \frac{1}{\Gamma(\xi + \omega)} \int_1^{\tau_1} \left(\left(\log \left(\frac{\tau_2}{r} \right) \right)^{\xi + \omega - 1} - \left(\log \left(\frac{\tau_1}{r} \right) \right)^{\xi + \omega - 1} \right) \Lambda(r, \theta(r), \vartheta(r)) \frac{dr}{r} \right. \\ &\quad + \frac{1}{\Gamma(\xi + \omega)} \int_{\tau_1}^{\tau_2} \left(\log \left(\frac{\tau_2}{r} \right) \right)^{\xi + \omega - 1} \Lambda(r, \theta(r), \vartheta(r)) \frac{dr}{r} \\ &\quad - \frac{\gamma}{\Gamma(\omega)} \int_1^{\tau_1} \left(\left(\log \left(\frac{\tau_2}{r} \right) \right)^{\omega - 1} - \left(\log \left(\frac{\tau_1}{r} \right) \right)^{\omega - 1} \right) \theta(r) \frac{dr}{r} \\ &\quad \left. + \frac{\gamma}{\Gamma(\omega)} \int_{\tau_1}^{\tau_2} \left(\log \left(\frac{\tau_1}{r} \right) \right)^{\omega - 1} \theta(r) \frac{dr}{r} \right|, \end{aligned}$$

which yields

$$\begin{aligned} &|\Omega_1(\theta, \vartheta)(\tau_2) - \Omega_1(\theta, \vartheta)(\tau_1)| \\ &\leq \frac{\|\Xi\|}{\Gamma(\xi + \omega)} \left\{ \int_1^{\tau_1} \left| \left(\log \left(\frac{\tau_2}{r} \right) \right)^{\xi + \omega - 1} - \left(\log \left(\frac{\tau_1}{r} \right) \right)^{\xi + \omega - 1} \right| \frac{dr}{r} \right. \\ &\quad + \left. \int_{\tau_1}^{\tau_2} \left(\log \left(\frac{\tau_2}{r} \right) \right)^{\xi + \omega - 1} \frac{dr}{r} \right\} + \frac{\gamma y}{\Gamma(\omega)} \left\{ \int_1^{\tau_1} \left| \left(\log \left(\frac{\tau_2}{r} \right) \right)^{\omega - 1} \right. \right. \\ &\quad \left. \left. - \left(\log \left(\frac{\tau_1}{r} \right) \right)^{\omega - 1} \right| \frac{dr}{r} + \int_{\tau_1}^{\tau_2} \left(\log \left(\frac{\tau_1}{r} \right) \right)^{\omega - 1} \frac{dr}{r} \right\}. \end{aligned} \tag{4.9}$$

From Lagrange mean value theorem, we can write

$$\begin{cases} \left| \left(\log \left(\frac{\tau_2}{r} \right) \right)^{\xi + \omega - 1} - \left(\log \left(\frac{\tau_1}{r} \right) \right)^{\xi + \omega - 1} \right| \leq u_1 |\tau_2 - \tau_1|, \\ \left| \left(\log \left(\frac{\tau_2}{r} \right) \right)^{\omega - 1} - \left(\log \left(\frac{\tau_1}{r} \right) \right)^{\omega - 1} \right| \leq u_2 |\tau_2 - \tau_1|, \end{cases} \tag{4.10}$$

where u_1 and u_2 are independent of τ . Applying (4.10) in (4.9) and letting $\tau_2 \rightarrow \tau_1$, we conclude that

$$|\Omega_1(\theta, \vartheta)(\tau_2) - \Omega_1(\theta, \vartheta)(\tau_1)| \rightarrow 0.$$

By the same technique, we have

$$|\Omega_1(\vartheta, \theta)(\tau_2) - \Omega_1(\vartheta, \theta)(\tau_1)| \rightarrow 0.$$

Therefore, Ω_1 is equicontinuous. Thank to Arzela–Ascoli theorem, Ω_1 is compact on \mathfrak{R}_y . This completes the condition (3) in Theorem 2.6. Then there exists point $(\theta, \vartheta) \in \mathfrak{R}_y$ such that $(\theta, \vartheta) = \Omega_1(\theta, \vartheta) + \Omega_2(\theta, \vartheta)$ and $(\vartheta, \theta) = \Omega_1(\vartheta, \theta) + \Omega_2(\vartheta, \theta)$, for $(\vartheta, \theta) \in \mathfrak{R}_y$. Thus, there exists at least one solution to the nonlinear coupled Langevin system (1.1). \square

5. Uniqueness Theorem

We now turn to the uniqueness property for solutions of the nonlinear coupled Langevin system (1.1).

Theorem 5.1. *Suppose that (S_1) – (S_4) to be held. Then the nonlinear coupled Langevin system (1.1) has a unique solution on $[1, c]$ if $\zeta_\Lambda \Upsilon_1 + \Upsilon_2 < 1$ and $\zeta_\Lambda^* \Upsilon_1^* + \Upsilon_2^* < 1$.*

Proof. Let $\mathfrak{R}_\mathfrak{S} = \left\{ \begin{matrix} (\theta, \vartheta) \in \Theta \times \Upsilon : \|(\theta, \vartheta)\| \leq \mathfrak{S}, \\ (\vartheta, \theta) \in \Upsilon \times \Theta : \|(\vartheta, \theta)\| \leq \mathfrak{S} \end{matrix} \right\}$ be closed bounded and convex subsets of $\Theta \times \Upsilon$, where

$$\mathfrak{S} \geq \frac{\zeta_\Lambda \Upsilon_1}{1 - \Upsilon_1 - \Upsilon_2} \quad \text{and} \quad \mathfrak{S} \geq \frac{\zeta_\Lambda^* \Upsilon_1^*}{1 - \Upsilon_1^* - \Upsilon_2^*}.$$

To finish the proof, it is enough to investigate the boundedness of $\Omega \mathfrak{R}_\mathfrak{S}$, that is, $\Omega \mathfrak{R}_\mathfrak{S} \subset \mathfrak{R}_\mathfrak{S}$ and Ω is a contraction mapping. For any $(\theta, \vartheta) \in \mathfrak{R}_y$, estimate

$$\begin{aligned} |\Lambda(r, \theta(r), \vartheta(r))| &\leq |\Lambda(r, \theta(r), \vartheta(r)) - \Lambda(r, 0, 0)| + |\Lambda(r, 0, 0)| \\ &\leq \zeta_\Lambda[|\theta| + |\vartheta|] + |\Lambda(r, 0, 0)| \\ &\leq \zeta_\Lambda \|(\theta, \vartheta)\| + N \\ &\leq \zeta_\Lambda \mathfrak{S} + N, \end{aligned} \tag{5.1}$$

and for $(\vartheta, \theta) \in \mathfrak{R}_y$,

$$\begin{aligned} |\Lambda(r, \vartheta(r), \theta(r))| &\leq |\Lambda^*(r, \vartheta(r), \theta(r)) - \Lambda(r, 0, 0)| + |\Lambda(r, 0, 0)| \\ &\leq \zeta_\Lambda^*[|\vartheta| + |\theta|] + |\Lambda(r, 0, 0)| \\ &\leq \zeta_\Lambda^* \|(\vartheta, \theta)\| + N^* \\ &\leq \zeta_\Lambda^* \mathfrak{S} + N^*. \end{aligned} \tag{5.2}$$

Now, we have

$$\begin{aligned} |\Omega(\theta, \vartheta)(\tau)| &= |\Omega_1(\theta, \vartheta)(\tau) + \Omega_2(\theta, \vartheta)(\tau)| \\ &\leq \sup_{\tau \in [1, c]} \left\{ \frac{1}{\Gamma(\xi + \omega)} \int_1^\tau \left(\log \left(\frac{\tau}{r} \right) \right)^{\xi + \omega - 1} |\Lambda(r, \theta(r), \vartheta(r))| \frac{dr}{r} \right. \\ &\quad + \frac{|\gamma|}{\Gamma(\omega)} \int_1^\tau \left(\log \left(\frac{\tau}{r} \right) \right)^{\omega - 1} |\theta(r)| \frac{dr}{r} + \frac{|\log(\tau)|^{\omega + 1}}{|\Gamma(2 + \omega) - (\beta - \gamma)|} \\ &\quad \times \left[\frac{|\gamma - \beta|}{\Gamma(\xi + \omega)} \int_1^c \left(\log \left(\frac{c}{r} \right) \right)^{\xi + \omega - 1} |\Lambda(r, \theta(r), \vartheta(r))| \frac{dr}{r} \right. \\ &\quad \times \frac{|\gamma(\beta - \gamma)|}{\Gamma(\omega)} \int_1^c \left(\log \left(\frac{c}{r} \right) \right)^{\omega - 1} |\theta(r)| \frac{dr}{r} + \frac{1}{\Gamma(\xi)} \int_1^c \left(\log \left(\frac{c}{r} \right) \right)^{\xi - 1} \\ &\quad \left. \left. \times |\Lambda(r, \theta(r), \vartheta(r))| \frac{dr}{r} + \frac{1}{\Gamma(\sigma)} \int_1^\ell \left(\log \left(\frac{\ell}{r} \right) \right)^{\sigma - 1} |\theta(r)| \frac{dr}{r} \right] \right\}. \end{aligned} \tag{5.3}$$

Applying (5.1) in (5.3), we have

$$\begin{aligned}
 & |\Omega(\theta, \vartheta)(\tau)| \\
 & \leq (\zeta_\Lambda \mathfrak{S} + N) \sup_{\tau \in [1, c]} \left\{ \frac{1}{\Gamma(\xi + \omega)} \int_1^\tau \left(\log \left(\frac{\tau}{r} \right) \right)^{\xi + \omega - 1} \frac{dr}{r} + \frac{|\log(\tau)|^{\omega + 1}}{|\Gamma(2 + \omega) - (\beta - \gamma)|} \right. \\
 & \quad \times \left[\frac{|\gamma - \beta|}{\Gamma(\xi + \omega)} \int_1^c \left(\log \left(\frac{c}{r} \right) \right)^{\xi + \omega - 1} \frac{dr}{r} + \frac{1}{\Gamma(\xi)} \int_1^c \left(\log \left(\frac{c}{r} \right) \right)^{\xi - 1} \frac{dr}{r} \right] \Big\} \\
 & \quad + \|\theta\| \sup_{\tau \in [1, c]} \left\{ \frac{|\gamma|}{\Gamma(\omega)} \int_1^\tau \left(\log \left(\frac{\tau}{r} \right) \right)^{\omega - 1} \frac{dr}{r} + \frac{|\log(\tau)|^{\omega + 1}}{|\Gamma(2 + \omega) - (\beta - \gamma)|} \right. \\
 & \quad \times \left[\frac{|\gamma(\beta - \gamma)|}{\Gamma(\omega)} \int_1^c \left(\log \left(\frac{c}{r} \right) \right)^{\omega - 1} \frac{dr}{r} + \frac{1}{\Gamma(\sigma)} \int_1^\ell \left(\log \left(\frac{\ell}{r} \right) \right)^{\sigma - 1} \frac{dr}{r} \right] \Big\} \\
 & \leq (\zeta_\Lambda \mathfrak{S} + N) \sup_{\tau \in [1, c]} \left\{ \frac{1}{\Gamma(\xi + \omega)} \int_1^\tau \left(\log \left(\frac{\tau}{r} \right) \right)^{\xi + \omega - 1} \frac{dr}{r} + \frac{|\log(\tau)|^{\omega + 1}}{|\Gamma(2 + \omega) - (\beta - \gamma)|} \right. \\
 & \quad \times \left[\frac{|\gamma - \beta|}{\Gamma(\xi + \omega)} \int_1^c \left(\log \left(\frac{c}{r} \right) \right)^{\xi + \omega - 1} \frac{dr}{r} + \frac{1}{\Gamma(\xi)} \int_1^c \left(\log \left(\frac{c}{r} \right) \right)^{\xi - 1} \frac{dr}{r} \right] \Big\} \\
 & \quad + \|(\theta, \vartheta)\| \sup_{\tau \in [1, c]} \left\{ \frac{|\gamma|}{\Gamma(\omega)} \int_1^\tau \left(\log \left(\frac{\tau}{r} \right) \right)^{\omega - 1} \frac{dr}{r} + \frac{|\log(\tau)|^{\omega + 1}}{|\Gamma(2 + \omega) - (\beta - \gamma)|} \right. \\
 & \quad \times \left[\frac{|\gamma(\beta - \gamma)|}{\Gamma(\omega)} \int_1^c \left(\log \left(\frac{c}{r} \right) \right)^{\omega - 1} \frac{dr}{r} + \frac{1}{\Gamma(\sigma)} \int_1^\ell \left(\log \left(\frac{\ell}{r} \right) \right)^{\sigma - 1} \frac{dr}{r} \right] \Big\},
 \end{aligned}$$

which implies that

$$\begin{aligned}
 |\Omega(\theta, \vartheta)(\tau)| & \leq (\zeta_\Lambda \mathfrak{S} + N) \left\{ \frac{1}{\Gamma(\xi + \omega + 1)} + \frac{|\gamma - \beta|}{|\Gamma(2 + \omega) - (\beta - \gamma)|\Gamma(\xi + \omega + 1)} \right. \\
 & \quad \left. + \frac{1}{|\Gamma(2 + \omega) - (\beta - \gamma)|\Gamma(\xi + 1)} \right\} \\
 & \quad + \mathfrak{S} \left\{ \frac{|\gamma|}{\Gamma(\omega + 1)} + \frac{|\gamma||\beta - \gamma|}{|\Gamma(2 + \omega) - (\beta - \gamma)|\Gamma(\omega + 1)} \right. \\
 & \quad \left. + \frac{\log(\ell)^\sigma}{|\Gamma(2 + \omega) - (\beta - \gamma)|\Gamma(\sigma + 1)} \right\} \\
 & = (\zeta_\Lambda \mathfrak{S} + N)\Upsilon_1 + \mathfrak{S}\Upsilon_2 \leq \mathfrak{S}.
 \end{aligned}$$

Repeating the same steps and using (5.2), we have

$$|\Omega(\vartheta, \theta)(\tau)| \leq (\zeta_{\Lambda^*} \mathfrak{S} + N^*)\Upsilon_1^* + \mathfrak{S}\Upsilon_2^* \leq \mathfrak{S}.$$

Thus, $\|\Omega(\theta, \vartheta)\| \leq \mathfrak{S}$ and $\|\Omega(\vartheta, \theta)\| \leq \mathfrak{S}$. Hence $\Omega\mathfrak{R}_\mathfrak{S} \subset \mathfrak{R}_\mathfrak{S}$.

Finally, for completing the next part of the proof, for each $\theta, \vartheta, \bar{\theta}, \bar{\vartheta} \in \mathcal{U}$, we have

$$\begin{aligned}
 & |\Omega(\theta, \vartheta)(\tau) - \Omega(\bar{\theta}, \bar{\vartheta})(\tau)| \\
 &= |(\Omega_1(\theta, \vartheta)(\tau) - \Omega_1(\bar{\theta}, \bar{\vartheta})(\tau)) + (\Omega_2(\theta, \vartheta)(\tau) - \Omega_2(\bar{\theta}, \bar{\vartheta})(\tau))| \\
 &\leq |(\Omega_1(\theta, \vartheta)(\tau) - \Omega_1(\bar{\theta}, \bar{\vartheta})(\tau))| + |(\Omega_2(\theta, \vartheta)(\tau) - \Omega_2(\bar{\theta}, \bar{\vartheta})(\tau))| \\
 &\leq \sup_{\tau \in [1, c]} \left\{ \frac{1}{\Gamma(\xi + \omega)} \int_1^\tau \left(\log\left(\frac{\tau}{r}\right)\right)^{\xi + \omega - 1} |\Lambda(r, \theta(r), \vartheta(r)) - \Lambda(r, \bar{\theta}(r), \bar{\vartheta}(r))| \frac{dr}{r} \right. \\
 &\quad + \frac{|\gamma|}{\Gamma(\omega)} \int_1^\tau \left(\log\left(\frac{\tau}{r}\right)\right)^{\omega - 1} |\theta(r) - \bar{\theta}(r)| \frac{dr}{r} + \frac{|\log(\tau)|^{\omega + 1}}{|\Gamma(2 + \omega) - (\beta - \gamma)|} \\
 &\quad \times \left[\frac{|\gamma - \beta|}{\Gamma(\xi + \omega)} \int_1^c \left(\log\left(\frac{c}{r}\right)\right)^{\xi + \omega - 1} |\Lambda(r, \theta(r), \vartheta(r)) - \Lambda(r, \bar{\theta}(r), \bar{\vartheta}(r))| \frac{dr}{r} \right. \\
 &\quad + \frac{|\gamma(\beta - \gamma)|}{\Gamma(\omega)} \int_1^c \left(\log\left(\frac{c}{r}\right)\right)^{\omega - 1} |\theta(r) - \bar{\theta}(r)| \frac{dr}{r} \\
 &\quad + \frac{1}{\Gamma(\xi)} \int_1^c \left(\log\left(\frac{c}{r}\right)\right)^{\xi - 1} |\Lambda(r, \theta(r), \vartheta(r)) - \Lambda(r, \bar{\theta}(r), \bar{\vartheta}(r))| \frac{dr}{r} \\
 &\quad \left. \left. + \frac{1}{\Gamma(\sigma)} \int_1^\ell \left(\log\left(\frac{\ell}{r}\right)\right)^{\sigma - 1} |\theta(r) - \bar{\theta}(r)| \frac{dr}{r} \right] \right\},
 \end{aligned}$$

it follows that

$$\begin{aligned}
 & |\Omega(\theta, \vartheta)(\tau) - \Omega(\bar{\theta}, \bar{\vartheta})(\tau)| \\
 &\leq \left(\frac{\zeta_\Lambda}{\Gamma(\xi + \omega + 1)} + \frac{\gamma}{\Gamma(\omega + 1)} + \frac{\zeta_\Lambda |\gamma - \beta|}{|\Gamma(2 + \omega) - (\beta - \gamma)| \Gamma(\xi + \omega + 1)} \right. \\
 &\quad + \frac{\gamma |\beta - \gamma|}{|\Gamma(2 + \omega) - (\beta - \gamma)| \Gamma(\omega + 1)} + \frac{\zeta_\Lambda}{|\Gamma(2 + \omega) - (\beta - \gamma)| \Gamma(\xi + 1)} \\
 &\quad \left. + \frac{\log(\ell)^\sigma}{|\Gamma(2 + \omega) - (\beta - \gamma)| \Gamma(\sigma + 1)} \right) \|\theta - \bar{\theta}\| \\
 &\quad + \left(\frac{\zeta_\Lambda}{\Gamma(\xi + \omega + 1)} + \frac{\zeta_\Lambda |\gamma - \beta|}{|\Gamma(2 + \omega) - (\beta - \gamma)| \Gamma(\xi + \omega + 1)} \right. \\
 &\quad \left. + \frac{\zeta_\Lambda}{|\Gamma(2 + \omega) - (\beta - \gamma)| \Gamma(\xi + 1)} \right) \|\vartheta - \bar{\vartheta}\| \\
 &= (\zeta_\Lambda \Upsilon_1 + \Upsilon_2) \|\theta - \bar{\theta}\| + \zeta_\Lambda \Upsilon_1 \|\vartheta - \bar{\vartheta}\| \\
 &\leq (\zeta_\Lambda \Upsilon_1 + \Upsilon_2) (\|\theta - \bar{\theta}\| + \|\vartheta - \bar{\vartheta}\|) \\
 &= (\zeta_\Lambda \Upsilon_1 + \Upsilon_2) \|(\theta, \vartheta) - (\bar{\theta}, \bar{\vartheta})\|.
 \end{aligned}$$

Similarly

$$|\Omega(\vartheta, \theta)(\tau) - \Omega(\bar{\vartheta}, \bar{\theta})(\tau)| \leq (\zeta_{\Lambda}^* \Upsilon_1^* + \Upsilon_2^*) \|(\vartheta, \theta) - (\bar{\vartheta}, \bar{\theta})\|.$$

Since $\zeta_{\Lambda} \Upsilon_1 + \Upsilon_2 < 1$ and $\zeta_{\Lambda}^* \Upsilon_1^* + \Upsilon_2^* < 1$, then Ω is a contraction. Consequently, an operator Ω has a unique fixed point when utilizing the Banach contraction theorem. Therefore, there is a unique solution to the nonlinear coupled Langevin system (1.1). \square

6. Stability Analysis

The HU-stability of the nonlinear coupled Langevin system (1.1) will be discussed in this step. First, the HU-stability of the considered nonlinear coupled Langevin system is defined as follows.

Definition 6.1. The solution of the nonlinear coupled Langevin system (1.1) is called HU-stable if $\exists b > 0$ s.t. for every $\epsilon, \epsilon^* > 0$ and $(\theta, \vartheta) \in \mathcal{U} \times \mathcal{U}$ as a solution to the system

$$\begin{cases} |{}^{\text{HC}}D^{\xi}({}^C D^{\omega} + \gamma)\theta(\tau) - \Lambda(\tau, \theta(\tau), \vartheta(\tau))| \leq \epsilon, & 1 < \xi + \omega \leq 3, \\ |{}^{\text{HC}}D^{\xi^*}({}^C D^{\omega^*} + \gamma^*)\vartheta(\tau) - \Lambda^*(\tau, \vartheta(\tau), \theta(\tau))| \leq \epsilon, & 1 < \xi^* + \omega^* \leq 3, \end{cases} \quad (6.1)$$

for all $\tau \in [1, c]$, there exists a unique solution $(\bar{\theta}, \bar{\vartheta}) \in \mathcal{U} \times \mathcal{U}$ to the nonlinear coupled Langevin system (1.1) s.t.

$$\|(\theta, \vartheta) - (\bar{\theta}, \bar{\vartheta})\| \leq b\epsilon.$$

Remark 6.2. The pair $(\theta, \vartheta) \in \mathcal{U} \times \mathcal{U}$ is called a solution to the nonlinear coupled Langevin system (1.1) if and only if there exist $W, D \in \mathcal{U}$ s.t. for all $\tau \in [1, c]$,

- (i) $|W(\tau)| \leq \epsilon$ and $|D(\tau)| \leq \epsilon$;
- (ii) ${}^{\text{HC}}D^{\xi}({}^C D^{\omega} + \gamma)\theta(\tau) - \Lambda(\tau, \theta(\tau), \vartheta(\tau)) - W(\tau) = 0, 1 < \xi + \omega \leq 3$;
- (iii) ${}^{\text{HC}}D^{\xi^*}({}^C D^{\omega^*} + \gamma^*)\vartheta(\tau) - \Lambda^*(\tau, \vartheta(\tau), \theta(\tau)) \leq \epsilon^*, 1 < \xi^* + \omega^* \leq 3$.

Lemma 6.3. Based on Remark 6.2, for each $\tau \in [1, c]$, the solution of the system

$$\begin{cases} {}^{\text{HC}}D^{\xi}({}^C D^{\omega} + \gamma)\theta(\tau) = \Lambda(\tau, \theta(\tau), \vartheta(\tau)) + W(\tau), & 1 < \xi + \omega \leq 3, \\ {}^{\text{HC}}D^{\xi^*}({}^C D^{\omega^*} + \gamma^*)\vartheta(\tau) = \Lambda^*(\tau, \vartheta(\tau), \theta(\tau)) + D(\tau), & 1 < \xi^* + \omega^* \leq 3, \end{cases}$$

where $\beta \neq \gamma, \beta^* \neq \gamma^*, \gamma, \gamma^*, \sigma, \sigma^* > 0, \beta, \beta^* \in \mathbb{R}$ and $\ell \in (1, c)$, under boundary conditions

$$\begin{cases} \theta(0) = 0 = \vartheta(0), & {}^C D^{\omega}\theta(0) = 0 = {}^C D^{\omega^*}\vartheta(0), \\ {}^C D^{\omega}\theta(c) + \beta\theta(c) = 0, & {}^C D^{\omega^*}\vartheta(c) + \beta^*\vartheta(c) = 0, \\ {}^C D^{\omega}\theta(1) = I^{\sigma}\theta(\ell), & {}^C D^{\omega^*}\vartheta(1) = I^{\sigma^*}\vartheta(\ell) \end{cases}$$

is given by

$$\begin{aligned} \theta(\tau) = & \frac{1}{\Gamma(\xi + \omega)} \int_1^\tau \left(\log\left(\frac{\tau}{r}\right)\right)^{\xi + \omega - 1} \Lambda(r, \theta(r), \vartheta(r)) \frac{dr}{r} \\ & - \frac{\gamma}{\Gamma(\omega)} \int_1^\tau \left(\log\left(\frac{\tau}{r}\right)\right)^{\omega - 1} \theta(r) \frac{dr}{r} + \frac{(\log(\tau))^{\omega + 1}}{\Gamma(2 + \omega) - (\beta - \gamma)} \\ & \times \left[\frac{(\gamma - \beta)}{\Gamma(\xi + \omega)} \int_1^c \left(\log\left(\frac{c}{r}\right)\right)^{\xi + \omega - 1} \Lambda(r, \theta(r), \vartheta(r)) \frac{dr}{r} \right. \\ & + \frac{\gamma(\beta - \gamma)}{\Gamma(\omega)} \int_1^c \left(\log\left(\frac{c}{r}\right)\right)^{\omega - 1} \theta(r) \frac{dr}{r} - \frac{1}{\Gamma(\xi)} \int_1^c \left(\log\left(\frac{c}{r}\right)\right)^{\xi - 1} \\ & \times \Lambda(r, \theta(r), \vartheta(r)) \frac{dr}{r} - \frac{1}{\Gamma(\sigma)} \int_1^\ell \left(\log\left(\frac{\ell}{r}\right)\right)^{\sigma - 1} \theta(r) \frac{dr}{r} \left. \right] + \frac{1}{\Gamma(\xi + \omega)} \\ & \times \int_1^\tau \left(\log\left(\frac{\tau}{r}\right)\right)^{\xi + \omega - 1} W(\tau) \frac{dr}{r} - \frac{\gamma}{\Gamma(\omega)} \int_1^\tau \left(\log\left(\frac{\tau}{r}\right)\right)^{\omega - 1} W(\tau) \frac{dr}{r} \\ & + \frac{(\log(\tau))^{\omega + 1}}{\Gamma(2 + \omega) - (\beta - \gamma)} \left[\frac{(\gamma - \beta)}{\Gamma(\xi + \omega)} \int_1^c \left(\log\left(\frac{c}{r}\right)\right)^{\xi + \omega - 1} W(\tau) \frac{dr}{r} \right. \\ & + \frac{\gamma(\beta - \gamma)}{\Gamma(\omega)} \int_1^c \left(\log\left(\frac{c}{r}\right)\right)^{\omega - 1} W(\tau) \frac{dr}{r} - \frac{1}{\Gamma(\xi)} \int_1^c \left(\log\left(\frac{c}{r}\right)\right)^{\xi - 1} W(\tau) \frac{dr}{r} \\ & \left. - \frac{1}{\Gamma(\sigma)} \int_1^\ell \left(\log\left(\frac{\ell}{r}\right)\right)^{\sigma - 1} W(\tau) \frac{dr}{r} \right], \end{aligned}$$

and

$$\begin{aligned} \vartheta(\tau) = & \frac{1}{\Gamma(\xi^* + \omega^*)} \int_1^\tau \left(\log\left(\frac{\tau}{r}\right)\right)^{\xi^* + \omega^* - 1} \Lambda^*(r, \vartheta(r), \theta(r)) \frac{dr}{r} \\ & - \frac{\gamma^*}{\Gamma(\omega^*)} \int_1^\tau \left(\log\left(\frac{\tau}{r}\right)\right)^{\omega^* - 1} \vartheta(r) \frac{dr}{r} + \frac{(\log(\tau))^{\omega^* + 1}}{\Gamma(2 + \omega^*) - (\beta^* - \gamma^*)} \\ & \times \left[\frac{(\gamma^* - \beta^*)}{\Gamma(\xi^* + \omega^*)} \int_1^c \left(\log\left(\frac{c}{r}\right)\right)^{\xi^* + \omega^* - 1} \Lambda^*(r, \vartheta(r), \theta(r)) \frac{dr}{r} \right. \\ & + \frac{\gamma(\beta^* - \gamma^*)}{\Gamma(\omega^*)} \int_1^c \left(\log\left(\frac{c}{r}\right)\right)^{\omega^* - 1} \vartheta(r) \frac{dr}{r} - \frac{1}{\Gamma(\xi^*)} \int_1^c \left(\log\left(\frac{c}{r}\right)\right)^{\xi^* - 1} \\ & \times \Lambda^*(r, \vartheta(r), \theta(r)) \frac{dr}{r} - \frac{1}{\Gamma(\sigma^*)} \int_1^\ell \left(\log\left(\frac{\ell}{r}\right)\right)^{\sigma^* - 1} \vartheta(r) \frac{dr}{r} \left. \right] \\ & + \frac{1}{\Gamma(\xi^* + \omega^*)} \int_1^\tau \left(\log\left(\frac{\tau}{r}\right)\right)^{\xi^* + \omega^* - 1} D(\tau) \frac{dr}{r} \end{aligned}$$

$$\begin{aligned}
 & - \frac{\gamma^*}{\Gamma(\omega^*)} \int_1^\tau \left(\log\left(\frac{\tau}{r}\right)\right)^{\omega^*-1} D(\tau) \frac{dr}{r} + \frac{(\log(\tau))^{\omega^*+1}}{\Gamma(2+\omega^*) - (\beta^* - \gamma^*)} \\
 & \times \left[\frac{(\gamma^* - \beta^*)}{\Gamma(\xi^* + \omega^*)} \int_1^c \left(\log\left(\frac{c}{r}\right)\right)^{\xi^* + \omega^* - 1} D(\tau) \frac{dr}{r} + \frac{\gamma(\beta^* - \gamma^*)}{\Gamma(\omega^*)} \right. \\
 & \times \int_1^c \left(\log\left(\frac{c}{r}\right)\right)^{\omega^* - 1} D(\tau) \frac{dr}{r} - \frac{1}{\Gamma(\xi^*)} \int_1^c \left(\log\left(\frac{c}{r}\right)\right)^{\xi^* - 1} D(\tau) \frac{dr}{r} \\
 & \left. - \frac{1}{\Gamma(\sigma)} \int_1^\ell \left(\log\left(\frac{\ell}{r}\right)\right)^{\sigma - 1} D(\tau) \frac{dr}{r} \right].
 \end{aligned}$$

Proof. From Lemma 3.1, the desired relations are proved. □

Theorem 6.4. *If the assumptions (S₁)–(S₃) hold, then the solution to the nonlinear coupled Langevin system (1.1) is HU-stable if*

$$\nabla_1 \nabla_1^* \neq 1,$$

where the latter constants will be introduced later.

Proof. Thank to Lemma 6.3, if (θ, ϑ) is a solution to (6.1), and $(\bar{\theta}, \bar{\vartheta})$ is a solution to (1.1), then, we get

$$\begin{aligned}
 |\theta(\tau) - \bar{\theta}(\tau)| & \leq \frac{1}{\Gamma(\xi + \omega)} \int_1^\tau \left(\log\left(\frac{\tau}{r}\right)\right)^{\xi + \omega - 1} |\Lambda(r, \theta(r), \vartheta(r)) - \Lambda(r, \bar{\theta}(r), \bar{\vartheta}(r))| \frac{dr}{r} \\
 & + \frac{\gamma}{\Gamma(\omega)} \int_1^\tau \left(\log\left(\frac{\tau}{r}\right)\right)^{\omega - 1} |\theta(r) - \bar{\theta}(r)| \frac{dr}{r} + \frac{(\log(\tau))^{\omega + 1}}{\Gamma(2 + \omega) - (\beta - \gamma)} \\
 & \times \left[\frac{(\gamma - \beta)}{\Gamma(\xi + \omega)} \int_1^c \left(\log\left(\frac{c}{r}\right)\right)^{\xi + \omega - 1} \right. \\
 & \times |\Lambda(r, \theta(r), \vartheta(r)) - \Lambda(r, \bar{\theta}(r), \bar{\vartheta}(r))| \frac{dr}{r} \\
 & + \frac{\gamma(\beta - \gamma)}{\Gamma(\omega)} \int_1^c \left(\log\left(\frac{c}{r}\right)\right)^{\omega - 1} |\theta(r) - \bar{\theta}(r)| \frac{dr}{r} \\
 & + \frac{1}{\Gamma(\xi)} \int_1^c \left(\log\left(\frac{c}{r}\right)\right)^{\xi - 1} |\Lambda(r, \theta(r), \vartheta(r)) - \Lambda(r, \bar{\theta}(r), \bar{\vartheta}(r))| \frac{dr}{r} \\
 & \left. + \frac{1}{\Gamma(\sigma)} \int_1^\ell \left(\log\left(\frac{\ell}{r}\right)\right)^{\sigma - 1} |\theta(r) - \bar{\theta}(r)| \frac{dr}{r} \right] \\
 & + \frac{1}{\Gamma(\xi + \omega)} \int_1^\tau \left(\log\left(\frac{\tau}{r}\right)\right)^{\xi + \omega - 1} |W(\tau)| \frac{dr}{r}
 \end{aligned}$$

$$\begin{aligned}
 & -\frac{\gamma}{\Gamma(\omega)} \int_1^\tau \left(\log\left(\frac{\tau}{r}\right)\right)^{\omega-1} |W(\tau)| \frac{dr}{r} \\
 & \times \frac{(\log(\tau))^{\omega+1}}{\Gamma(2+\omega) - (\beta - \gamma)} \left[\frac{(\gamma - \beta)}{\Gamma(\xi + \omega)} \int_1^c \left(\log\left(\frac{c}{r}\right)\right)^{\xi+\omega-1} |W(\tau)| \frac{dr}{r} \right. \\
 & \times \frac{\gamma(\beta - \gamma)}{\Gamma(\omega)} \int_1^c \left(\log\left(\frac{c}{r}\right)\right)^{\omega-1} |W(\tau)| \frac{dr}{r} \\
 & + \frac{1}{\Gamma(\xi)} \int_1^c \left(\log\left(\frac{c}{r}\right)\right)^{\xi-1} |W(\tau)| \frac{dr}{r} \\
 & \left. + \frac{1}{\Gamma(\sigma)} \int_1^\ell \left(\log\left(\frac{\ell}{r}\right)\right)^{\sigma-1} W(\tau) \frac{dr}{r} \right],
 \end{aligned}$$

which implies that

$$\begin{aligned}
 \|\theta - \bar{\theta}\| & \leq \left(\frac{\zeta_\Lambda}{\Gamma(\xi + \omega + 1)} + \frac{\gamma}{\Gamma(\omega + 1)} + \frac{\zeta_\Lambda |\gamma - \beta|}{|\Gamma(2 + \omega) - (\beta - \gamma)| \Gamma(\xi + \omega + 1)} \right. \\
 & + \frac{\gamma |\beta - \gamma|}{|\Gamma(2 + \omega) - (\beta - \gamma)| \Gamma(\omega + 1)} + \frac{\zeta_\Lambda}{|\Gamma(2 + \omega) - (\beta - \gamma)| \Gamma(\xi + 1)} \\
 & \left. + \frac{\log(\ell)^\sigma}{|\Gamma(2 + \omega) - (\beta - \gamma)| \Gamma(\sigma + 1)} \right) \|\theta - \bar{\theta}\| \\
 & + \left(\frac{\zeta_\Lambda}{\Gamma(\xi + \omega + 1)} + \frac{\zeta_\Lambda |\gamma - \beta|}{|\Gamma(2 + \omega) - (\beta - \gamma)| \Gamma(\xi + \omega + 1)} \right. \\
 & \left. + \frac{\zeta_\Lambda}{|\Gamma(2 + \omega) - (\beta - \gamma)| \Gamma(\xi + 1)} \right) \|\vartheta - \bar{\vartheta}\| \\
 & \times \left(+ \frac{2}{\Gamma(\xi + \omega + 1)} + \frac{\gamma}{\Gamma(\omega + 1)} + \frac{2|\gamma - \beta|}{|\Gamma(2 + \omega) - (\beta - \gamma)| \Gamma(\xi + \omega + 1)} \right. \\
 & + \frac{\gamma |\beta - \gamma|}{|\Gamma(2 + \omega) - (\beta - \gamma)| \Gamma(\omega + 1)} + \frac{2}{|\Gamma(2 + \omega) - (\beta - \gamma)| \Gamma(\xi + 1)} \\
 & \left. + \frac{\log(\ell)^\sigma}{|\Gamma(2 + \omega) - (\beta - \gamma)| \Gamma(\sigma + 1)} \right) \epsilon,
 \end{aligned}$$

hence

$$\|\theta - \bar{\theta}\| \leq (\zeta_\Lambda \Upsilon_1 + \Upsilon_2) \|\theta - \bar{\theta}\| + \zeta_\Lambda \Upsilon_1 \|\vartheta - \bar{\vartheta}\| + (2\Upsilon_1 + \Upsilon_2) \epsilon,$$

and so

$$(1 - (\zeta_\Lambda \Upsilon_1 + \Upsilon_2)) \|\theta - \bar{\theta}\| \leq \zeta_\Lambda \Upsilon_1 \|\vartheta - \bar{\vartheta}\| + (2\Upsilon_1 + \Upsilon_2) \epsilon.$$

Hence

$$\|\theta - \bar{\theta}\| - \frac{\zeta_\Lambda \Upsilon_1}{1 - (\zeta_\Lambda \Upsilon_1 + \Upsilon_2)} \|\vartheta - \bar{\vartheta}\| \leq \frac{(2\Upsilon_1 + \Upsilon_2)}{1 - (\zeta_\Lambda \Upsilon_1 + \Upsilon_2)} \epsilon.$$

Setting $\nabla_1 = \frac{\zeta_\Lambda \Upsilon_1}{1 - (\zeta_\Lambda \Upsilon_1 + \Upsilon_2)}$ and $\nabla_2 = \frac{(2\Upsilon_1 + \Upsilon_2)}{1 - (\zeta_\Lambda \Upsilon_1 + \Upsilon_2)}$, we can write the above inequality as

$$\|\theta - \bar{\theta}\| - \nabla_1 \|\vartheta - \bar{\vartheta}\| \leq \nabla_2 \epsilon. \tag{6.2}$$

Similarly, we have

$$\|\vartheta - \bar{\vartheta}\| - \nabla_1^* \|\theta - \bar{\theta}\| \leq \nabla_2^* \epsilon, \tag{6.3}$$

where $\nabla_1^* = \frac{\zeta_\Lambda^* \Upsilon_1^*}{1 - (\zeta_\Lambda^* \Upsilon_1^* + \Upsilon_2^*)}$ and $\nabla_2^* = \frac{(2\Upsilon_1^* + \Upsilon_2^*)}{1 - (\zeta_\Lambda^* \Upsilon_1^* + \Upsilon_2^*)}$.

The inequalities (6.2) and (6.3) can be written as

$$\begin{bmatrix} 1 & -\nabla_1 \\ -\nabla_1^* & 1 \end{bmatrix} \begin{bmatrix} \|\theta - \bar{\theta}\| \\ \|\vartheta - \bar{\vartheta}\| \end{bmatrix} \leq \begin{bmatrix} \nabla_2 \epsilon \\ \nabla_2^* \epsilon \end{bmatrix}.$$

Using the inverse of the matrix, we get

$$\begin{bmatrix} \|\theta - \bar{\theta}\| \\ \|\vartheta - \bar{\vartheta}\| \end{bmatrix} \leq \begin{bmatrix} \frac{1}{M} & \frac{\nabla_1}{M} \\ \frac{\nabla_1^*}{M} & \frac{1}{M} \end{bmatrix} \begin{bmatrix} \nabla_2 \epsilon \\ \nabla_2^* \epsilon \end{bmatrix}, \tag{6.4}$$

where $M = 1 - \nabla_1 \nabla_1^*$. Based on (6.4), we have

$$\|\theta - \bar{\theta}\| \leq \left(\frac{\nabla_2}{M} + \frac{\nabla_1 \nabla_2^*}{M} \right) \epsilon, \quad \text{and} \quad \|\vartheta - \bar{\vartheta}\| \leq \left(\frac{\nabla_1^* \nabla_2}{M} + \frac{\nabla_2^*}{M} \right) \epsilon.$$

Hence

$$\|\theta - \bar{\theta}\| + \|\vartheta - \bar{\vartheta}\| \leq \left(\frac{\nabla_2 + \nabla_2^* + \nabla_1 \nabla_2^* + \nabla_1^* \nabla_2}{M} \right) \epsilon.$$

Putting

$$b = \left(\frac{\nabla_2 + \nabla_2^* + \nabla_1 \nabla_2^* + \nabla_1^* \nabla_2}{M} \right),$$

then we have $\|(\theta, \vartheta) - (\bar{\theta}, \bar{\vartheta})\| \leq b\epsilon$. This proves that the solution of the nonlinear coupled Langevin system (1.1) is HU-stable. \square

7. An Application

This part is devoted to presenting an illustrative example in the form of the given nonlinear coupled Langevin system to support the results.

Example 7.1. Consider the nonlinear coupled Langevin system

$$\begin{cases} {}^{\text{HC}}D^{\frac{4}{3}} \left({}^C D^{\frac{1}{2}} + \frac{1}{4} \right) e^{-\tau} = \frac{1}{25 + \tau} \left(\frac{|\theta - \vartheta|}{5 + |\theta - \vartheta|} \right), & \tau \in [1, c], \\ {}^{\text{HC}}D^{\frac{5}{2}} \left({}^C D^{\frac{1}{3}} + \frac{1}{5} \right) \sin(\tau) = \frac{1}{30 + \tau^2} \left(\frac{|\theta - \vartheta|}{9 + |\theta - \vartheta|} \right), & \tau \in [1, c], \\ \theta(0) = 0 = \vartheta(0), \quad {}^C D^{\frac{1}{2}}\theta(0) = 0 = {}^C D^{\frac{1}{3}}\vartheta(0), \\ {}^C D^{\frac{1}{2}}\theta(c) + \frac{1}{2}\theta(c) = 0, \quad {}^C D^{\frac{1}{3}}\vartheta(c) + \frac{1}{4}\vartheta(c) = 0, \\ {}^C D^{\frac{1}{2}}\theta(1) = I^{\frac{4}{3}}\theta \left(\frac{c}{3} \right), \quad {}^C D^{\frac{1}{3}}\vartheta(1) = I^{\frac{8}{3}}\vartheta \left(\frac{c}{3} \right). \end{cases} \tag{7.1}$$

Problem (7.1) is a special case of (1.1) with $\frac{1}{2} = \beta \neq \gamma = \frac{1}{4}$, $\frac{1}{4} = \beta^* \neq \gamma^* = \frac{1}{5}$, $\xi = \frac{4}{3}$, $\omega = \frac{1}{2}$, $\xi^* = \frac{5}{2}$, $\omega^* = \frac{1}{3}$, $\xi + \omega \in (1, 3]$, $\xi^* + \omega^* \in (1, 3]$, $\gamma = \frac{1}{4} > 0$, $\gamma^* = \frac{1}{5} > 0$, $\sigma = \frac{4}{6} > 0$, $\sigma^* = \frac{8}{3} > 0$, $\ell \in \frac{c}{3}$, $\theta(\tau) = e^{-\tau}$, $\vartheta(\tau) = \sin(\tau)$. Moreover

$$\begin{aligned} \Lambda(\tau, \theta(\tau), \vartheta(\tau)) &= \frac{1}{25 + \tau} \left(\frac{|\theta - \vartheta|}{1 + |\theta - \vartheta|} \right), \\ \Lambda^*(\tau, \vartheta(\tau), \theta(\tau)) &= \frac{1}{30 + \tau^2} \left(\frac{|\theta - \vartheta|}{1 + |\theta - \vartheta|} \right). \end{aligned}$$

Clearly

$$\begin{aligned} |\Lambda(\tau, \theta(\tau), \vartheta(\tau)) - \Lambda(\tau, \bar{\theta}(\tau), \bar{\vartheta}(\tau))| &\leq \frac{1}{25} (|\theta - \bar{\theta}| + |\vartheta - \bar{\vartheta}|), \\ |\Lambda(\tau, \vartheta(\tau), \theta(\tau)) - \Lambda(\tau, \bar{\vartheta}(\tau), \bar{\theta}(\tau))| &\leq \frac{1}{30} (|\vartheta - \bar{\vartheta}| + |\theta - \bar{\theta}|), \\ |\Lambda(\tau, \theta(\tau), \vartheta(\tau))| &\leq \frac{1}{25 + \tau}, \quad \text{and} \quad |\Lambda^*(\tau, \vartheta(\tau), \theta(\tau))| \leq \frac{1}{30 + \tau^2}, \end{aligned}$$

with $\Xi(\tau) = \frac{1}{25 + \tau}$, $\Xi^*(\tau) = \frac{1}{30 + \tau^2}$, $\zeta_{\Lambda} = \frac{1}{25} > 0$, $\zeta_{\Lambda^*} = \frac{1}{30} > 0$, $N = \sup_{\tau \in [1, c]} |\Lambda(\tau, 0, 0)| = 0 < \infty$ and $N^* = \sup_{\tau \in [1, c]} |\Lambda^*(\tau, 0, 0)| = 0 < \infty$. Hence the assumptions (S_1) – (S_4) are satisfied. By doing some calculations, we find that

$$\Upsilon_1 \approx 0.909, \quad \Upsilon_2 \approx 0.676, \quad \Upsilon_1^* \approx 0.477, \quad \Upsilon_2^* \approx 0.270.$$

Hence, $\zeta_{\Lambda} \Upsilon_1 + \Upsilon_2 \approx 0.712 < 1$ and $\zeta_{\Lambda^*} \Upsilon_1^* + \Upsilon_2^* \approx 0.286 < 1$. Therefore all requirements of Theorem 5.1 are fulfilled. Then the nonlinear coupled Langevin system (7.1) has a unique solution on $[1, c]$. Moreover


$$\nabla_1 = \frac{\zeta_{\Lambda} \Upsilon_1}{1 - (\zeta_{\Lambda} \Upsilon_1 + \Upsilon_2)} \approx 0.126, \quad \nabla_1^* = \frac{\zeta_{\Lambda^*} \Upsilon_1^*}{1 - (\zeta_{\Lambda^*} \Upsilon_1^* + \Upsilon_2^*)} \approx 0.022,$$


such that $\nabla_1 \nabla_1^* \approx 0.003 \neq 1$. Thus, Theorem 6.4 is satisfied. Then the unique solution of (7.1) is HU-stable.

8. Conclusion

In some applicable areas of sciences like physics, chemistry, and electrical engineering, fractional LEs have been an important subject. In fact, this type of equation does a good job of describing how physical events evolve in a changing environment. It's vital to remember that in these equations, for defining the Brownian motion, random deviations are appeared as the Gaussian noises. Many researchers substituted FDEs instead of the standard classical DEs in order to reduce noise and staircase effects. Therefore, in this paper, by defining a coupled system of such LEs in its fractional form, we discussed on some qualitative properties. With the help of the Krasnoselskii's fixed point theorem and the Banach principle, the existence and uniqueness results of the Langevin fractional DEs have been investigated under the Caputo–Hadamard and Caputo derivatives. Despite the current interest in the topic of FDEs with boundary conditions, no articles specifically addressing this type of combined coupled Langevin system have been published. Stability property was also checked completely in relation to the solutions of the mentioned system. An application that describes and illustrates the validity of the results was provided as a numerical example.

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