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Research article

New insights into rough approximations of a fuzzy set inspired by soft relations with decision making applications

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Abstract: In the fields of mathematics and information sciences, binary relations are vital. Fuzzy sets (FSs), rough sets (RSs), and soft sets (SSs) are mathematical strategies that effectively handle ambiguous and imprecise data in practical situations. This work presents various properties of the roughness of fuzzy sets regarding foresets (F-sets) and aftersets (A-sets) using soft binary relations (SBRs). Initially, two pairs of fuzzy soft sets (FSSs) are obtained by approximating a fuzzy subset using an SBR, and their distinctive axiomatic systems are explored. Additionally, two types of fuzzy topologies that result from soft reflexive relations (SRRs) are examined. Numerous similarity relations allied with SBRs are also investigated. In addition, we present the accuracy measure and roughness measure for a fuzzy subset based on the mass assignment of the fuzzy subset through soft relations. Next, we outline a decision-making (DM) approach within the context of the proposed method. In addition, we provide two algorithms and decision phases. Ultimately, an applied example is used to evaluate the reliability of the decision processes. An extensive comparison study confirms the proposed method's feasibility and superiority over current DM methods.

Keywords: fuzzy set; rough set; soft set; decision-making; soft relation; accuracy measure; roughness measure
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1. Introduction

Conventional tools for reasoning and computation are typically precise. However, most real-world problems across diverse fields such as social sciences, engineering, environmental studies, and medicine involve imprecise and ambiguous data. These problems, characterized by varying degrees of uncertainty, cannot be effectively addressed using traditional mathematical tools. Probability theory has historically been regarded as a powerful approach to handling uncertainty. However, its application often requires systems to be stochastically stable, a fundamental prerequisite that necessitates a substantial number of validation trials. This process can be time-consuming, which is impractical in today's fast-paced world where efficiency is paramount. To address this challenge, unconventional methods must be explored. Researchers are increasingly focused on developing strategies to extract meaningful insights from ambiguous and uncertain information. These efforts have garnered significant attention from experts across various scientific and technological domains. In this context, Zadeh [1] made a pivotal contribution by introducing the FS theory, which provides a mathematical framework for managing uncertainty and interpreting linguistic terms in human language. This groundbreaking innovation marked a significant leap forward, accelerating advancements in the field.

FSs provide a more flexible framework than traditional crisp sets by permitting elements to belong to multiple sets with varying membership grades. This capability enables FSs to circumvent the inherent ambiguity in real-world dilemmas. By representing uncertainty and gradation, FS theory proves advantageous for modelling complex systems and human decision-making processes, where boundaries are often indistinct. Over time, various generalizations of FS theory have emerged, including interval-valued FSs, intuitionistic FSs, vague sets, and neutrosophic FSs [2]. While these theories have proven valuable for managing uncertainty, each presents unique challenges. As Molodtsov [3] highlighted, these challenges underscore the need for further refinement and innovation in this domain.

Molodtsov [3] presented the SS theory as an essential idea for another numerical tool designed to handle ambiguity. One of the most promising characteristics of SSs is their capability to circumvent uncertainty using a parametric manner. In practical applications, information is often incomplete or ambiguous, making it challenging to represent using classical set theory. However, SSs offer a more flexible framework, enabling the effective modelling of such imprecise and uncertain dilemmas. Additionally, SSs offer a wide range of actions that are quite helpful in handling many kinds of scenarios. The author of [4] provided some exercises in SS theory and hypotheses. Maji et al. [5] employed the SS hypothesis in a DM problem. FSSs were also developed by the same creators as classical SSs [6]. In light of FSSs, Roy and Maji demonstrated a technique for question affirmation based on unclear multi-eyewitness information and related it to decision-based problems. Many authors have considered the idea of parametric reduction in SSs [7]. Numerous scholars have contemplated the paradigm of SSs in alternate points of view [8]. Alcantud et al. [9] conducted a comprehensive systematic literature survey on SS theory, providing an in-depth analysis of its development, applications, and theoretical advancements. Bayram et al. [10] fostered an advanced encryption scheme based on SSs. SS theory has been fruitfully applied across numerous realms, including decision analysis [11], data mining [12], image processing, machine learning, coding theory, group theory, and cryptography, as reported in [13, 14].

RS theory was first proposed by Pawlak [15] as another approach to classifying incomplete information. The RS efficiently incorporates uncertainty by offering lower and upper approximations. The philosophy of RS is founded on the premise that every object in the universe is associated with specific information. Objects sharing similar information are considered indistinguishable from one another. This indistinguishability, formalized as the indiscernibility relation, serves as the mathematical cornerstone of RS theory. In most cases, the indiscernibility relation is also titled an equivalence relation (ER). Even though the RS paradigm has been applied successfully in various realms, this is not generally the case, as ERs have been replaced with arbitrary binary relations to avoid the restrictive conditions of ERs when addressing practical problems. Several authors have deliberated rough approximation operators using neighborhood systems inspired by arbitrary binary relations, such as Overlapping containment rough neighborhoods [16, 17], cardinality rough neighborhoods [18, 19, 20], intersection rough neighborhoods [21], subset rough neighborhoods [22], and topological neighborhoods [23]. Zhu [24] introduced the concept of generalized RS based on relational structures. She et al. [25] incorporated logical operators into RS theory, further extending its applicability. Dubois and Prade [26] developed the fuzzy RS (FRS) framework by replacing crisp relations with fuzzy relations. Gul et al. [27] proposed an RS variant grounded in fuzzy preference relations. The authors in [28] introduced a novel approach for the fuzzification of RSs using α-indiscernibility. Meanwhile, Greco and Slowinski [29] presented a dominance-based RS model, further diversifying the applications of RS theory. Zhan and Zhu [30] conducted a systematic review of DM methods based on SSs, FSSs, and rough SS.

Binary relations play a vital role in the creation of topological structures across various disciplines. Topology, a significant branch of mathematics, extends beyond its theoretical foundations to influence numerous mathematical domains and practical applications. Particularly, the generation of topologies through relations and the representation of topological notions via relational contexts serve to bridge the gap between abstract topology and its real-world applicability. Salama et al. [31] familiarized topological approaches for generalized RSs, while Shabir and Naz [32] founded the concept of soft topological spaces, characterized by an initial universe defined by a fixed set of parameters. Li et al. [33] explored the interrelations among SSs, soft rough sets (SRSs), and topological structures. Furthermore, Riaz et al. [34] explored soft multi-rough set topology and its applications in DM.

A SBR is a parameterized collection of binary relations in the universe. Rough approximations in RS theory only take into account one binary relation. Rough approximations can handle multiple binary relations while taking into account SBRs. If we predetermine the set of parameters and let each parameter match the same binary relation, then this binary relation (BR) becomes an SBR. Thus, every ordinary BR on a set is thought to be an SBR, emphasizing the fact that SBRs deserve additional investigation. Qurashi et al. [35] inspected rough fuzzy substructures of quantale module under SBRs and related DM applications. Additionally, Feng et al. [36] deployed SBRs to semigroups. The authors of [37] exposed multigranulation roughness of a set via SBRs. Soft relations were used to take a different approach to DM in [38]. Mehmood et al. [39] disclosed the roughness of FSs by BRs induced from SBRs with application in DM. The authors of [40, 41] propounded a fusion among SSs, FSs, and RSs. Kanwal and Shabir [42] studied the approximation of ideals in semigroups by SBRs. The authors of [43] studied the reduction of an information system via SBRs.

Maji et al. [5] began the possibility of SS applications in DM. Some errors in this early work were pointed out by Chen et al. [7]. The authors of [24] explored novel operations within the framework of SSs. Maji et al. [44] introduced the concept of FSSs, establishing a significant extension of the SS theory. Feng et al. [45] investigated the relationship between SSs and RSs, leading to the development of a hybrid model known as soft RSs (SRSs). The authors of [46] refined this concept further by proposing a modified version of SRS. The authors of [47] defined the notion of a soft rough fuzzy covering using soft neighborhoods and studied its properties in detail. Li and Xie [48] examined the interconnections between SSs, SRSs, and topologies, highlighting their theoretical implications. Ayub et al. [49] analyzed modules of fractions under the frameworks of FSs and SSs. Many researchers have proposed a roughness measure for FSs by mass assignment [50]. Banerjee and Pal [51] presented an FS roughness measure. Many authors have delivered different ways of DM within the context of the SS paradigm [52, 53]. Çağman and Enginoğlu [54] proposed the idea of soft matrix theory and its decision-making.

It is well-recognized that many problems involve two distinct universes, including the relationship between customer objections and their corresponding solutions in enterprise management, the alignment of customer characteristics with product features in personalized marketing, and the association between mechanical defects and their remedies in machine diagnostics. To address these problems, the RS variant has been generalized over two universes. Liu [55] designed an RS version based on two universes with applications. Based on the interrelation between two universes, Liu et al. [56] developed a link between the graded RS and suitable parameters. The authors of [57] anticipated a probabilistic RS model to deal with uncertainty. Xu et al. [58] pioneered the FRS mechanism over two universes. Zhai and Wang [59] formulated a variable precision RS model over two universes and its properties. In [60], the authors devised an approach to emergency DM based on decision-theoretic RS over two universes.

1.1. Motivations and research gaps

The knowledge gaps, motivations, and originality of this article are encapsulated as:

(1) From the examined reviewed literature, it becomes evident that numerous researchers have proposed various hybrid models integrating FSs, RSs, and SSs. Despite abundant research and the quick growth of FSs, RSs, and SSs, many real-world situations remain unexplored. To the best of our knowledge, there does not exist any appropriate study where the roughness of an FS using SBRs is explored. As a result, the purpose of this work is to provide a new roughness mechanism for FSs in the context of SBRs. This strategy is fundamentally different from all former approaches, in which we approximate an FS by using SBRs over two universes concerning the A-sets and F-sets.

(2) Moreover, many real applications involve the consideration of two universes. For example, patients in a hospital frequently have several symptoms at the same time. A single disease might have a diversity of clinical symptoms. Consequently, it might be difficult for a doctor to determine whether or not a patient is suffering from a particular disease. In such circumstances, RS paradigms are better in two universes: one for the clinical symptoms and the other for the patients. To address this situation, the RS contexts have been prolonged over dual universes. Based on the above literature survey, it becomes evident that the concept of the roughness of FSs has not been explored through SBRs in dual-universe settings, leaving a significant gap in the theoretical development of RS theory. Additionally, the roughness of SSs via SBRs over dual universes allows us to analyze objects from two universes and gain a more comprehensive understanding of their relationship.

(3) Besides, despite the potential of rough FSs in handling uncertainty, their application in decision-making in the context of SBRs remains underdeveloped, necessitating further research to bridge this gap and enhance practical implementations.

1.2. Main contributions

The main contributions of this work are outlined as follows:

- ❖ The main objective of this study is to construct a more comprehensive and robust approach to the roughness of FSs via SBRs. We concentrate on rough approximations of an FS through SBRs, which are titled the lower approximation and upper approximation concerning the A-sets and F-sets.
- ❖ To investigate several axiomatic systems of newly constructed lower and upper approximations with concrete illustrations.
- ❖ To frame two kinds of soft topologies induced from SRRs.
- ❖ To study several similarity relations associated with SBRs.
- ❖ To present accuracy and roughness measures based on the idea of mass assignment of an FS via SBRs.
- ❖ Given the assistance of the recommended mechanism, efforts are made to establish an FS-based DM approach regarding A-sets and F-sets, respectively.
- ❖ To validate the established work's dominance and performance by a comparative analysis between the projected method and some existing techniques.

1.3. Organization of this work

The remaining of this paper is organized as follows: Some basic ideas related to RSs, FSs, SSs, and SBRs are provided in Section 2. In Section 3, we formulate rough approximations of an FS based on an SBR in terms of A-sets and F-sets and explore their axiomatics systems in detail with several concrete illustrations. Section 4 examines two types of fuzzy topologies brought about by SRRs and analyzes their properties. Section 5 focuses on soft similarity relations associated with SBRs. In Section 6, we provide the membership functions for FSs regarding the A-sets and the F-sets, together with their corresponding degrees of accuracy and roughness. An approach to an FS's roughness measure using SBRs is given in Section 7. We offer a strategy for dealing with a DM problem using the proposed framework with two algorithms in Section 8. Also, a practical application of decision-making problems is provided. In Section 9, we execute an in-depth comparative analysis of our devised scheme with several preexisting approaches. Section 10 provides an overview of the conclusion and future research recommendations of this article.

2. Preliminaries

This segment introduces fundamental notions related to SSs, FSs, RSs, and SBRs. Unless otherwise stated, U and W are presumed to be non-empty finite sets throughout this work.

Definition 1. A subset of $U \times W$ is a binary relation from U to W, and a subset of $U \times U$ is stated to as a binary relation on U.

Definition 2. If π is a BR on U, it is considered as:

1) Reflexive if and only if $(\mathscr{f},\mathscr{f}) \in \pi$ $\forall \mathscr{f}$ in U.
2) Symmetric if $\forall$ $\mathscr{f}, \mathscr{w} \in U, (\mathscr{f},\mathscr{w}) \in \pi \Rightarrow (\mathscr{w},\mathscr{f}) \in \pi$.
3) Transitive if $\forall$ $\mathscr{f},\mathscr{g},\mathscr{w} \in U, (\mathscr{f},\mathscr{g}) \in \pi$ and $(\mathscr{g},\mathscr{w}) \in \pi \Rightarrow (\mathscr{f},\mathscr{w}) \in \pi$.
4) π is named an ER if it is reflexive, symmetric, and transitive.

Definition 3. [15] An approximation space is a pair $(\mathcal{U},\pi)$, where $\mathcal{U}$ is a non-empty finite set and π is an ER on $\mathcal{U}$. $\mathcal{X}$ is definable if $\mathcal{X} \subseteq \mathcal{U}$ is the union of some equivalence classes of $\mathcal{U}$. If not, it is entitled as undefinable. If $\mathcal{X}$ is undefinable, we can make two definable subsets that we refer to as lower and upper approximations of $\mathcal{X}$, which are respectively represented as follows:

$$\underline{\pi}(\mathcal{X}) = \cup \{[\mathscr{k}]_\pi : [\mathscr{k}]_\vartheta \subseteq \mathcal{X}\}, \quad (2.1)$$

$$\overline{\vartheta}(\mathcal{X}) = \cup \{[\mathscr{k}]_\pi : [\mathscr{k}]_\vartheta \cap \mathcal{X} \neq \emptyset\}. \quad (2.2)$$

An RS is denoted by a pair $(\underline{\pi}(\mathcal{X}), \overline{\pi}(\mathcal{X}))$. The set

$$Bnd(\mathcal{X}) = \overline{\pi}(\mathcal{X}) - \underline{\pi}(\mathcal{X}), \quad (2.3)$$

is said to be the boundary region. $\mathcal{X}$ is definable if $Bnd(\mathcal{X}) = \emptyset$; else, $\mathcal{X}$ is an RS.

According to the previous definition, we infer that:

- The lower approximation $\underline{\pi}(\mathcal{X})$ is a gathering of objects that can be classified with full guarantee as a member of set $\mathcal{X}$ with the information of π.
- The upper approximation $\overline{\pi}(\mathcal{X})$ is an assemblage of elements that may possibly be characterized as an element of set $\mathcal{X}$ using knowledge of π.

Definition 4. [1] A mapping $\lambda: \mathcal{U} \to [0,1]$ defines an FS λ in $\mathcal{U}$. The membership value $\lambda(x)$ for x in $\mathcal{U}$ effectively indicates the extent to which x is a member of the FS λ.

Let two FSs be in $\mathcal{U}\, as\, \lambda_1$ and λ_2. If $\lambda_1(u) \leq \lambda_2(u)$ for every u in $\mathcal{U}$, then $\lambda_1 \leq \lambda_2$. Additionally, $\lambda_1 = \lambda_2$ when $\lambda_1 \geq \lambda_2$ and $\lambda_1 \leq \lambda_2$.

The assemblage of all FSs in $\mathcal{U}$ is symbolized by $F(\mathcal{U})$.

Definition 5. [1] If $(u) = 0\ \forall u \in \mathcal{U}$, then an FS λ in $\mathcal{U}$ is referred to as null FS. An FS λ is considered full FS if $(u) = 1\ \forall u \in \mathcal{U}$. Typically, a null FS is represented by 0 and a full FS by 1.

FS intersection, union, and complement are defined component-wise using Zadeh's min-max system. With $\lambda, \mu \in F(\mathcal{U})$ and $x \in \mathcal{U}$, we have

1) $(\lambda \cap \mu)(x) = \lambda(x) \wedge \mu(x)$,
2) $(\lambda \cup \mu)(x) = \lambda(x) \vee \mu(x)$,
3) $\lambda^c\,(x) = 1 - \lambda(x)$.

Definition 6. The $\alpha\text{-}cut$ or α-level set of an FS λ in $\mathcal{U}$, with a number $\alpha \in (0,1)$, is described as:

$$\lambda_\alpha = \{x \in U : \lambda(x) \geq \alpha\}. \quad (2.4)$$

Let $\mathcal{U}$ be the non-empty universe. The family of all subsets of $\mathcal{U}$ (i.e., all FSs in $\mathcal{U}$) is represented as $P(\mathcal{U})$ (resp. $\vartheta(\mathcal{U})$).

Definition 7. [8] A pair (ϑ, A) is called an FSS over $\mathcal{U}$ if ϑ is a mapping given by $\vartheta: A \to Ғ(\mathcal{U})$ and A is a subset of E (the set of parameters). Thus, $\vartheta(e)$ is an FS in $\forall e \in A$. Hence, an FSS over $\mathcal{U}$ is a gathering of FSs in $\mathcal{U}$.

Definition 8. [8] If (1) $A \subseteq B$ and (2) $F(e)$ is an FS of (e) $\forall e \in A$, then we say that (ϑ, A) is a fuzzy soft subset of (G, B) for two FSSs (ϑ, A) and (G, B) over a universe U. If (F, A) is a fuzzy soft subset of (G, B) and (G, B) is a fuzzy soft subset of (ϑ, A), then two FSSs (ϑ, A) and (G, B) over a universe $\mathcal{U}$ are called fuzzy soft equal.

Definition 9. [8] For every $e \in A$ such that $H(e) = \vartheta(e) \cup G(e)$, the FSS (H, A) is the union of two FSSs (ϑ, A) and (G, A) over the universe $\mathcal{U}$ $\forall e \in A$ such that $H(e) = \vartheta(e) \cap G(e)$, the FSS (H, A) over the universe $\mathcal{U}$ is the intersection of two FSSs (ϑ, A) and (G, A).

Definition 10. [36] Assume that (ϑ, A) is an SBR from U to W, where $A \subseteq E$, if (ϑ, A) is a SS over $U \times W$, i.e., $\vartheta : A \to P(U \times W)$. A parameterized collection of BRs from U to W is denoted by (ϑ, A). In other words, for every parameter e in A, we have a $BR\ \vartheta(e)$ from U to W.

3. Approximation of a fuzzy set through soft binary relations

In this section, we use an SBR from a set U to W to define the rough approximations of an FS in terms of F-sets and A-sets. For this purpose, we approximate an FS of universe W in universe U and an FS of U in W using A-sets and F-sets of SBRs, respectively. In this way, we attain two FSSs corresponding to FSs in W (resp. U). We also scrutinize several axiomatic systems of these approximations. We discuss several concrete illustrations to better comprehend the proposed notions.

Definition 11. Let (ϑ, A) be an SBR from U to W and λ be an FS of W. Then, the lower approximation $\underline{\vartheta}^{\lambda}$ and upper approximation $\overline{\vartheta}^{\lambda}$ of λ concerning A-sets are postulated as follows:

$$\underline{\vartheta}^{\lambda}(e)(u) = \begin{cases} \bigwedge_{a' \in u\vartheta(e)} \lambda(a') & if\ u\vartheta(e) \neq \phi, \\ 0 & if\ u\vartheta(e) = \phi, \end{cases} \tag{3.1}$$

and

$$\overline{\vartheta}^{\lambda}(e)(u) = \begin{cases} \bigvee_{a' \in u\vartheta(e)} \lambda(a') & if\ u\vartheta(e) \neq \phi, \\ 0 & if\ u\vartheta(e) = \phi, \end{cases} \tag{3.2}$$

where $u\vartheta(e) = \{w \in W : (u, w) \in \vartheta(e)\}$ and is called the A-set of u for $u \in U$ and $e \in A$.

In Definition 11, a SBR from U to W is assumed, and a FS in W can be approximated as lower and upper approximations regarding A-sets. The resultant sets are two FSSs over U.

The information about the object u interpreted by lower and upper approximations is as follows:

- $\underline{\vartheta}^{\lambda}(e)(u)$ specifies the degree to which the object u certainly has the property e.
- $\overline{\vartheta}^{\lambda}(e)(u)$ shows the degree to which the object u possibly has the property e.

Definition 12. Let (ϑ, A) be an SBR from U to W. Then, the lower approximation $\delta_{\underline{\vartheta}}$ and upper approximation $\delta_{\overline{\vartheta}}$ of an FS δ of U regarding F-sets are characterized as follows:

$$\delta_{\underline{\vartheta}}(e)(w) = \begin{cases} \bigwedge_{a' \in \vartheta(e)w} \delta(a') & if\ \vartheta(e)w \neq \phi, \\ 0 & if\ \vartheta(e)w = \phi, \end{cases} \tag{3.3}$$

and

$$\delta_{\overline{\vartheta}}(e)(w) = \begin{cases} \bigvee_{a' \in \vartheta(e)w} \delta(a') & if\ \vartheta(e)w \neq \phi, \\ 0 & if\ \vartheta(e)w = \phi, \end{cases} \tag{3.4}$$

where $\vartheta(e)w = \{u \in U : (u, w) \in \vartheta(e)\}$ and is called the F-set of w for $w \in W$ and $e \in A$.

Moreover, $\underline{\vartheta}^{\lambda} : A \to \vartheta(U)$, $\overline{\vartheta}^{\lambda} : A \to \vartheta(U)$ and $\delta_{\underline{\vartheta}} : A \to \vartheta(W)$, $\delta_{\overline{\vartheta}} : A \to \vartheta(W)$ and we say (U, W, ϑ) is a generalized soft approximation space.

In the following, we elaborate on an example to better comprehend the above-described ideas.

Example 1. Assume $Mr.X$ wishes to purchase a bike for himself. Let $U = \{\text{the set of all bikes models}\} = \{d_1, d_2, d_3, d_4, d_5, d_6\}$ and $W = \{\text{the colors of all models}\} = \{c_1, c_2, c_3, c_4\}$ and the set of attributes be $A = \{e_1, e_2, e_3\} = \{\text{the set of stores near his home}\}$.

Describe $\vartheta : A \to P(U \times W)$ by

$$\vartheta(e_1) = \begin{Bmatrix} (d_1, c_1), (d_1, c_2), (d_1, c_3), (d_2, c_2), (d_2, c_4), \\ (d_4, c_2), (d_4, c_3), (d_5, c_3), (d_5, c_4), (d_6, c_1) \end{Bmatrix},$$

$$\vartheta(e_2) = \{(d_1, c_3), (d_2, c_3), (d_4, c_1), (d_5, c_1), (d_6, c_2), (d_6, c_3)\},$$

$$\vartheta(e_3) = \{(d_3, c_3), (d_3, c_1), (d_2, c_4), (d_5, c_3), (d_5, c_4)\},$$

signifies the relation between models and colors accessible on store e_i for $1 \leq i \leq 3$. Then

$$d_1\vartheta(e_1) = \{c_1, c_2, c_3\}, d_2\vartheta(e_1) = \{c_2, c_4\}, d_3\vartheta(e_1) = \phi,$$

$$d_4\vartheta(e_1) = \{c_2, c_3\}, d_5\vartheta(e_1) = \{c_3, c_4\}, d_6\vartheta(e_1) = \{c_1\},$$

and

$$d_1\vartheta(e_2) = \{c_3\}, d_2\vartheta(e_2) = \{c_3\}, d_3\vartheta(e_2) = \phi,$$

$$d_4\vartheta(e_2) = \{c_1\}, d_5\vartheta(e_2) = \{c_1\}, d_6\vartheta(e_2) = \{c_2, c_3\},$$

and

$$d_1\vartheta(e_3) = \phi, d_2\vartheta(e_3) = \{c_4\}, d_3\vartheta(e_3) = \{c_1, c_3\},$$

$$d_4\vartheta(e_3) = \phi, d_5\vartheta(e_3) = \{c_3, c_4\}, d_6\vartheta(e_3) = \phi.$$

Similarly

$$\vartheta(e_1)c_1 = \{d_1, d_6\}, \vartheta(e_1)c_2 = \{d_1, d_2, d_4\},$$

$$\vartheta(e_1)c_3 = \{d_1, d_4, d_5\}, \vartheta(e_1)c_4 = \{d_2, d_5\},$$

and

$$\vartheta(e_2)c_1 = \{d_4, d_5\}, \vartheta(e_2)c_2 = \{d_6\},$$

$$\vartheta(e_2)c_3 = \{d_1, d_2\}, \vartheta(e_2)c_4 = \phi,$$

and

$$\vartheta(e_3)c_1 = \{d_3\}, \vartheta(e_3)c_2 = \phi,$$

$$\vartheta(e_3)c_3 = \{d_3, d_5\}, \vartheta(e_3)c_4 = \{d_2, d_5\}.$$

Define $\lambda: W \to [0,1]$, which characterizes the preference of the color given by $Mr. X$ such that

$$\lambda(c_1) = 0.9, \lambda(c_2) = 0.8, \lambda(c_3) = 0.4, \lambda(c_4) = 0.$$

And define $\delta: U \to [0,1]$, which signifies the preference of the color given by $Mr. X$ such that

$$\delta(d_1) = 1, \delta(d_2) = 0.7\ , \delta(d_3) = 0.5\ , \delta(d_4) = 0.1,\ \delta(d_5) = 0, \delta(d_6) = 0.4.$$

Consequently, the lower and upper approximations regarding the F-sets and the A-sets are, respectively, given in Tables 1 and 2.

Table 1. Lower and upper approximations w.r.t. A-sets.

	d_1	d_2	d_3	d_4	d_5	d_6
$\underline{\vartheta}^{\lambda}(e_1)$	0.4	0	0	0.4	0	0.9
$\overline{\vartheta}^{\lambda}(e_1)$	0.9	0.8	0	0.8	0.4	0.9
$\underline{\vartheta}^{\lambda}(e_2)$	0.4	0.4	0	0.9	0.9	0.4
$\overline{\vartheta}^{\lambda}(e_2)$	0.4	0.4	0	0.9	0.9	0.8
$\underline{\vartheta}^{\lambda}(e_3)$	0	0	0.4	0	0	0
$\overline{\vartheta}^{\lambda}(e_3)$	0	0	0.9	0	0.4	0

Table 2. Lower and upper approximations w.r.t. F-sets.

	c_1	c_2	c_3	c_4
$\delta_{\underline{\vartheta}}(e_1)$	0.4	0.1	0	0
$\delta_{\overline{\vartheta}}(e_1)$	1	1	1	0.7
$\delta_{\underline{\vartheta}}(e_2)$	0	0.4	0.7	0
$\delta_{\overline{\vartheta}}(e_2)$	0.1	0.4	1	0
$\delta_{\underline{\vartheta}}(e_3)$	0.5	0	0	0
$\delta_{\overline{\vartheta}}(e_3)$	0.5	0	0.5	0.7

Hence, $\underline{\vartheta}^{\lambda}(e_i)(d)$ gives the degree of definite fulfilment of the objects of $d\vartheta(e_i)$ to λ on store e_i, and $\overline{\vartheta}^{\lambda}(e_i)(d)$ provides the degree of possible fulfilment of the objects of $d\vartheta(e_i)$ to λ on store e_i for $1 \leq i \leq 3$ regarding aftersets. Similarly, $\delta_{\underline{\vartheta}}(e_i)(d)$ gives the degree of definite fulfilment of the objects of $\vartheta(e_i)d$ to δ on store e_i, and $\delta_{\overline{\vartheta}}(e_i)(d)$ gives the degree of possible fulfilment of the objects of $\vartheta(e_i)d$ to δ on store e_i for $1 \leq i \leq 3$ regarding foresets.

Theorem 1. Given a generalized soft approximation space (U, W, ϑ) and an SBR $\vartheta: A \to P(U \times W)$ from U to W. Then, the following statements hold for the lower and upper approximations regarding A-sets for any $\lambda_1, \lambda_2, \lambda_3 \in \vartheta(W)$:

1) $\lambda_1 \leq \lambda_2 \Longrightarrow \underline{\vartheta}^{\lambda_1} \leq \underline{\vartheta}^{\lambda_2}$,
2) $\lambda_1 \leq \lambda_2 \Longrightarrow \overline{\vartheta}^{\lambda_1} \leq \overline{\vartheta}^{\lambda_2}$,
3) $\underline{\vartheta}^{\lambda_1} \cap \underline{\vartheta}^{\lambda_2} = \underline{\vartheta}^{\lambda_1 \cap \lambda_2}$,
4) $\overline{\vartheta}^{\lambda_1} \cap \overline{\vartheta}^{\lambda_2} \geq \overline{\vartheta}^{\lambda_1 \cap \lambda_2}$,
5) $\underline{\vartheta}^{\lambda_1} \cup \underline{\vartheta}^{\lambda_2} \leq \underline{\vartheta}^{\lambda_1 \cup \lambda_2}$,
6) $\overline{\vartheta}^{\lambda_1} \cup \overline{\vartheta}^{\lambda_2} = \overline{\vartheta}^{\lambda_1 \cup \lambda_2}$,
7) $\underline{\vartheta}^{1}(e)(u) = 1$ for all $e \in A$ if $u\vartheta(e) \neq \phi$,
8) $\overline{\vartheta}^{1}(e)(u) = 1$ for all $e \in A$ if $u\vartheta(e) \neq \phi$,
9) $\underline{\vartheta}^{\lambda} = \left(\overline{\vartheta}^{\lambda^c}\right)^c$ if $u\vartheta(e) \neq \phi$,
10) $\overline{\vartheta}^{\lambda} = \left(\underline{\vartheta}^{\lambda^c}\right)^c$ if $u\vartheta(e) \neq \phi$,
11) $\underline{\vartheta}^{0} = 0 = \overline{\vartheta}^{0}$.

Proof. For $u \in U$, we have two cases: (i) If $u\vartheta(e) = \phi$ and (ii) If $u\vartheta(e) \neq \phi$. If $u\vartheta(e) = \phi$, then all the above parts are trivial. So, we consider only the case when $u\vartheta(e) \neq \phi$.

1) Since, $\lambda_1 \leq \lambda_2$, so $\underline{\vartheta}^{\lambda_1}$ (e)(u) = $\bigwedge_{a' \in u\vartheta(e)} \lambda_1(a') \leq \bigwedge_{a' \in u\vartheta(e)} \lambda_2(a')$ = $\underline{\vartheta}^{\lambda_2}$ (e)(u). Hence, $\underline{\vartheta}^{\lambda_1} \leq \underline{\vartheta}^{\lambda_2}$.
2) Since, $\lambda_1 \leq \lambda_2$, so $\overline{\vartheta}^{\lambda_1}$ (e)(u) = $\bigvee_{a' \in u\vartheta(e)} \lambda_1(a') \leq \bigvee_{a' \in u\vartheta(e)} \lambda_2(a')$ = $\overline{\vartheta}^{\lambda_2}$ (e)(u). Hence, $\overline{\vartheta}^{\lambda_1} \leq \overline{\vartheta}^{\lambda_2}$.
3) Consider $\left(\underline{\vartheta}^{\lambda_1} \cap \underline{\vartheta}^{\lambda_2}\right)(e)(u) = \underline{\vartheta}^{\lambda_1}(e)(u) \wedge \underline{\vartheta}^{\lambda_2}(e)(u) =$
$\left(\bigwedge_{a \in u\vartheta(e)} \lambda_1(a)\right) \wedge \left(\bigwedge_{a \in u\vartheta(e)} \lambda_2(a)\right) = \bigwedge_{a \in u\vartheta(e)} \left(\lambda_1(a) \wedge \lambda_2(a)\right) = \bigwedge_{a \in u\vartheta(e)} (\lambda_1 \wedge \lambda_2)(a) =$ $\underline{\vartheta}^{\lambda_1 \cap \lambda_2}(e)(u)$. Hence, $\underline{\vartheta}^{\lambda_1} \cap \underline{\vartheta}^{\lambda_2} = \underline{\vartheta}^{\lambda_1 \cap \lambda_2}$.
4) Consider $\left(\overline{\vartheta}^{\lambda_1} \cap \overline{\vartheta}^{\lambda_2}\right)(e)(u) = \overline{\vartheta}^{\lambda_1}(e)(u) \wedge \overline{\vartheta}^{\lambda_2}(e)(u) =$

$\left(\vee_{a\in u\vartheta(e)}\ \lambda_1(a)\right) \wedge \left(\vee_{a\in u\vartheta(e)}\ \lambda_2(a)\right) \geq \vee_{a\in u\vartheta(e)}\ (\lambda_1 \wedge \lambda_2)(a) = \overline{\vartheta}^{\lambda_1\cap\lambda_2}(e)(u)$.
Hence, $\overline{\vartheta}^{\lambda_1} \cap \overline{\vartheta}^{\lambda_2} = \overline{\vartheta}^{\lambda_1\cap\lambda_2}$.

5) Consider $\left(\underline{\vartheta}^{\lambda_1} \cup \underline{\vartheta}^{\lambda_2}\right)(e)(u) = \underline{\vartheta}^{\lambda_1}(e)(u) \vee \underline{\vartheta}^{\lambda_2}(e)(u) =$
$\left(\wedge_{a\in u\vartheta(e)}\ \lambda_1(a)\right) \vee \left(\wedge_{b\in u\vartheta(e)}\ \lambda_2(b)\right) = \wedge_{a,b\in u\vartheta(e)}\ (\lambda_1(a) \vee \lambda_2(b)\ ) \leq \wedge_{a\in u\vartheta(e)}\ (\lambda_1\ \vee \lambda_2\ )(a) =$
$\underline{\vartheta}^{\lambda_1\cup\lambda_2}(e)(u)$. Hence, $\underline{\vartheta}^{\lambda_1} \cup \underline{\vartheta}^{\lambda_2} \leq \underline{\vartheta}^{\lambda_1\cup\lambda_2}$.

6) Consider $\left(\overline{\vartheta}^{\lambda_1} \cup \overline{\vartheta}^{\lambda_2}\right)(e)(u) = \overline{\vartheta}^{\lambda_1}(e)(u) \vee \overline{\vartheta}^{\lambda_2}(e)(u) =$
$\left(\vee_{a\in u\vartheta(e)}\ \lambda_1(a)\right) \vee \left(\vee_{a\in u\vartheta(e)}\ \lambda_2(a)\right) = \vee_{a\in u\vartheta(e)}\ (\lambda_1(a) \vee \lambda_2(a)\ ) = \vee_{a\in u\vartheta(e)}\ (\lambda_1 \vee \lambda_2\ )(a) =$
$\left(\overline{\vartheta}^{\lambda_1\cup\lambda_2}\right)(e)(u)$. Hence, $\overline{\vartheta}^{\lambda_1} \cup \overline{\vartheta}^{\lambda_2} = \overline{\vartheta}^{\lambda_1\cup\lambda_2}$.

7) Consider $\underline{\vartheta}^{1}(e)(u) = \wedge_{a\in u\vartheta(e)}\ 1(a) = \wedge_{a\in u\vartheta(e)}\ (1) = 1$, because $u\vartheta(e) \neq \phi$.

8) Consider $\overline{\vartheta}^{1}(e)(u) = \vee_{a\in u\vartheta(e)}\ 1(a) = \vee_{a\in u\vartheta(e)}\ (1) = 1$, because $u\vartheta(e) \neq \phi$.

9) Consider $\overline{\vartheta}^{\lambda^c}(e)(u) = \vee_{a\in u\vartheta(e)}\ \lambda^c(a) = \vee_{a\in u\vartheta(e)}\ \left(1-\lambda(a)\right) = \left(\wedge_{a\in u\vartheta(e)}\ \lambda(a)\right)^c =$
$\left(\underline{\vartheta}^{\lambda}(e)(u)\right)^c$. Therefore, $\left(\overline{\vartheta}^{\lambda^c}(e)(u)\right)^c = \underline{\vartheta}^{\lambda}(e)(u)$. Hence, $\underline{\vartheta}^{\lambda} = \left(\overline{\vartheta}^{\lambda^c}\right)^c$.

10) By part (9), $\underline{\vartheta}^{\lambda} = \left(\overline{\vartheta}^{\lambda^c}\right)^c$, thererfore, $\underline{\vartheta}^{\lambda^c} = \left(\overline{\vartheta}^{(\lambda^c)^c}\right)^c \Rightarrow \underline{\vartheta}^{\lambda^c} = \left(\overline{\vartheta}^{\lambda}\right)^c$.
Hence, $\overline{\vartheta}^{\lambda} = \left(\underline{\vartheta}^{\lambda^c}\right)^c$.

11) Straightforward.

Theorem 2. Given a generalized soft approximation space (U, W, ϑ) and an SBR $\vartheta: A \to P(U \times W)$ from U to W. Then, the subsequent features hold for the lower and upper approximations with regard to the F-sets for any $\delta, \delta_1, \delta_2, \in \vartheta(U)$:

1) $\delta_1 \leq \delta_2 \Rightarrow \delta_{1\underline{\vartheta}} \leq \delta_{2\underline{\vartheta}}$,
2) $\delta_1 \leq \delta_2 \Rightarrow \delta_{1\overline{\vartheta}} \leq \delta_{2\overline{\vartheta}}$,
3) $\delta_{1\underline{\vartheta}} \cap \delta_{2\underline{\vartheta}} = \delta_1 \cap \delta_{2\underline{\vartheta}}$,
4) $\delta_{1\overline{\vartheta}} \cap \delta_{2\overline{\vartheta}} \geq \delta_1 \cap \delta_{2\overline{\vartheta}}$,
5) $\delta_{1\underline{\vartheta}} \cup \delta_{2\underline{\vartheta}} \leq \delta_1 \cup \delta_{2\underline{\vartheta}}$,
6) $\delta_{1\overline{\vartheta}} \cup \delta_{2\overline{\vartheta}} = \delta_1 \cup \delta_{2\overline{\vartheta}}$,
7) $1_{\underline{\vartheta}}(e)(u) = 1$ for all $e \in A$ if $\vartheta(e)w \neq \phi$,
8) $1_{\overline{\vartheta}}(e)(u) = 1$ for all $e \in A$ if $\vartheta(e)w \neq \phi$.
9) $\delta_{\underline{\vartheta}} = \left(\delta^c{}_{\overline{\vartheta}}\right)^c$ if $\vartheta(e)(w) \neq \phi$,
10) $\delta_{\overline{\vartheta}} = \left(\delta^c{}_{\underline{\vartheta}}\right)^c$ if $\vartheta(e)(w) \neq \phi$.
11) $0_{\underline{\vartheta}} = 0 = 0_{\overline{\vartheta}}$.

Proof. Similar to the proof of Theorem 1.

In general, equality does not hold in the preceding Theorems 4 and 5 assertions, as demonstrated by the following illustration.

Example 2. Consider $W = \{m_1, m_2, m_3, m_4\}$, $U = \{n, u, o, b, w\}$ and $A = \{e_1, e_2\}$. Define $F: A \longrightarrow P(U \times W)$ by

$$\vartheta(e_1) = \left\{\begin{matrix}(n, m_1), (n, m_2), (o, m_3), (o, m_4), (u, m_1), (o, m_2), \\ (n, m_3), (u, m_4)\end{matrix}\right\},$$

$$\vartheta(e_2)=\begin{Bmatrix}(b,m_3),(b,m_1),(b,m_2),\\(w,m_1),(w,m_3),(w,m_4)\end{Bmatrix}.$$

Now,

$$n\vartheta(e_1)=\{m_1,m_2,m_3\},u\vartheta(e_1)=\{m_1,m_4\},ov(e_1)=\{m_2,m_3,m_4\},$$
$$b\vartheta(e_1)=\phi,w\vartheta(e_1)=\phi,$$

and

$$n\vartheta(e_2)=\phi,u\vartheta(e_2)=\phi,o\vartheta(e_2)=\phi$$
$$b\vartheta(e_2)=\{m_1,m_2,m_3\},w\vartheta(e_2)=\{m_1,m_3,m_4\}.$$

Moreover,

$$\vartheta(e_1)m_1=\{n,u\},\vartheta(e_1)m_2=\{n,o\},\vartheta(e_1)m_3=\{n,o\},$$
$$\vartheta(e_1)m_4=\{o,u\},$$

and

$$\vartheta(e_2)m_1=\{b,w\},\vartheta(e_2)m_2=\{b\},\vartheta(e_2)m_3=\{b,w\},$$
$$\vartheta(e_2)m_4=\{w\}.$$

Define $\lambda_1,\lambda_2,\lambda_1\cap\lambda_2,\lambda_1\cup\lambda_2\colon W\longrightarrow[0,1]$ as exhibited in Table 3.

Table 3. $\lambda_1,\lambda_2,\lambda_1\cap\lambda_2,\lambda_1\cup\lambda_2$.

	m_1	m_2	m_3	m_4
λ_1	0.1	0	0.5	0.4
λ_2	0.2	1	0.3	0.6
$\lambda_1\cap\lambda_2$	0.1	0	0.3	0.4
$\lambda_1\cup\lambda_2$	0.2	1	0.5	0.6

Define $\delta_1,\delta_2,\delta_1\cap\delta_2,\delta_1\cup\delta_2\colon U\longrightarrow[0,1]$, which are displayed in Table 4.

Table 4. $\delta_1,\delta_2,\delta_1\cap\delta_2,\delta_1\cup\delta_2$.

	n	u	o	b	w
δ_1	0.1	0.5	0.3	0.6	0.8
δ_2	0	0.1	0.4	1	0.7
$\delta_1\cap\delta_2$	0	0.1	0.3	0.6	0.7
$\delta_1\cup\delta_2$	0.1	0.5	0.4	1	0.8

Thus, lower and upper approximations of $\lambda_1,\lambda_2,\lambda_1\cap\lambda_2,\lambda_1\cup\lambda_2$ w.r.t. A-sets are tabulated in Table 5.

Table 5. Lower and upper approximations.

	$(e_1)(o)$
$\overline{\vartheta}^{\lambda_1}$	0.5
$\overline{\vartheta}^{\lambda_2}$	1
$\underline{\vartheta}^{\lambda_1}$	0
$\underline{\vartheta}^{\lambda_2}$	0.3
$\overline{\vartheta}^{\lambda_1 \cap \lambda_2}$	0.4
$\underline{\vartheta}^{\lambda_1 \cup \lambda_2}$	0.5

Also, lower and upper approximations of $\delta_1, \delta_2, \delta_1 \cap \delta_2, \delta_1 \cup \delta_2$ w.r.t. F-sets are demonstrated in Table 6.

Table 6. Lower and upper approximations.

	$(e_1)(o)$
$\delta_{1\overline{\vartheta}}$	0.4
$\delta_{2\overline{\vartheta}}$	0.5
$\delta_{1\underline{\vartheta}}$	0.1
$\delta_{2\underline{\vartheta}}$	0.3
$\delta_1 \cap \delta_{2\overline{\vartheta}}$	0.3
$\delta_1 \cup \delta_{2\underline{\vartheta}}$	0.4

Hence, $\overline{\vartheta}^{\lambda_1}(e_1)(o) \cap \overline{\vartheta}^{\lambda_2}(e_1)(o) \nleq \overline{\vartheta}^{\lambda_1 \cap \lambda_2}(e_1)(o)$ and $\underline{\vartheta}^{\lambda_1 \cup \lambda_2}(e_1)(o) \nleq \underline{\vartheta}^{\lambda_1}(e_1)(o) \cup \underline{\vartheta}^{\lambda_2}(e_1)(o)$.

Similarly, $\delta_{1\overline{\vartheta}}(e_1)(m_4) \cap \delta_{2\overline{\vartheta}}(e_1)(m_4) \nleq \delta_1 \cap \delta_{2\overline{\vartheta}}(e_1)(m_4)$ and $\delta_1 \cup \delta_{2\underline{\vartheta}}(e_1)(m_4) \nleq \delta_{1\underline{\vartheta}}(e_1)(m_4) \cup \delta_{2\underline{\vartheta}}(e_1)(m_4)$.

Theorem 3. Let (U, W, ϑ) be a generalized soft approximation space. Then, subsequent axioms hold for both lower and upper approximations regarding A-sets for $\{i \in I: \lambda_i\} \subseteq \vartheta(U)$:

1) $\underline{\vartheta}^{(\cap_{i \in I} \lambda_i)}(e) = \cap_{i \in I} \underline{\vartheta}^{\lambda_i}(e)$ for all $e \in A$.
2) $\underline{\vartheta}^{(\cup_{i \in I} \lambda_i)}(e) \supseteq \cup_{i \in I} \underline{\vartheta}^{\lambda_i}(e)$ for all $e \in A$.
3) $\overline{v}^{(\cup_{i \in I} \lambda_i)}(e) = \cup_{i \in I} \overline{\vartheta}^{\lambda_i}(e)$ for all $e \in A$.
4) $\overline{\vartheta}^{(\cap_{i \in I} \lambda_i)}(e) \subseteq \cap_{i \in I} \overline{\vartheta}^{\lambda_i}(e)$ for all $e \in A$.

Proof.

1) Take $\lambda_i \in \vartheta(U)$, where $i \in I$. Then we have

$\underline{\vartheta}^{(\cap_{i \in I} \lambda_i)}(e) = \wedge_{a \in u\vartheta(e)}(\wedge_{i \in I} \lambda_i)(a) = \wedge_{i \in I}\left(\wedge_{a \in u\vartheta(e)} \lambda_i(a)\right) = \cap_{i \in I} \underline{\vartheta}^{\lambda_i}(e)$ for all $e \in A$.

2) Take $\lambda_i \in \vartheta(U)$, where $i \in I$. Then we have
$\underline{\vartheta}^{(\bigcup_{i\in I}\lambda_i)}(e) = \bigwedge_{a\in u\vartheta(e)}(\bigvee_{i\in I}\lambda_i)(a) \geq \bigvee_{i\in I}\left(\bigwedge_{a\in u\vartheta(e)}\lambda_i(a)\right) = \bigcup_{i\in I}\underline{\vartheta}^{\lambda_i}(e)$ for all $e \in A$.

3) The proof is analogous to (1).

4) The proof is similar to (2).

Theorem 4. Presume that (U, W, ϑ) is a generalized soft approximation space. Then, the following properties for lower and upper approximations regarding F-sets hold for $\{i \in I: \delta_i\} \subseteq \vartheta(W)$:

1) $\bigcap_{i\in I} \delta_{i_{\underline{\vartheta}(e)}} = \bigcap_{i\in I}\delta_{i_{\underline{\vartheta}(e)}}$ $\forall e \in A$;
2) $\bigcup_{i\in I}\delta_{i_{\underline{\vartheta}(e)}} \supseteq \bigcup_{i\in I}\delta_{i_{\underline{\vartheta}(e)}}$ $\forall e \in A$;
3) $\bigcup_{i\in I}\delta_{i_{\overline{\vartheta}(e)}} = \bigcup_{i\in I}\delta_{i_{\overline{\vartheta}(e)}}$ $\forall e \in A$;
4) $\bigcap_{i\in I} \delta_{i_{\overline{\vartheta}(e)}} \subseteq \bigcap_{i\in I}\delta_{i_{\overline{\vartheta}(e)}}$ $\forall e \in A$.

Proof. The proof is analogous to the proof of Theorem 3.

Definition 13. An SBR on U is defined as (ϑ, A) if $(\vartheta$,A) is a SS over $U \times U$. Actually, a parameterized set of BRs on U is represented by (ϑ, A). In other words, for every parameter e in A, we have a BR $\vartheta(e)$ on U.

Definition 14. If for every e in A, $\vartheta(e)$ is a reflexive relation on U, then $\vartheta(e)$ is an SBR (ϑ, A) on U. Each $u\vartheta(e)$ (resp. $\vartheta(e)u$) in this particular case is non-empty, and $u \in u\vartheta(e)$ (resp. $\vartheta(e)u$).

Definition 15. If an SBR (ϑ, A) on U is also a soft reflexive, soft symmetric, and soft transitive relation on U, then it is a soft equivalence relation (SER) on U.

Definition 16. If $\vartheta(e)$ for every e in A is an ER on U, then a SBR (ϑ, A) on U is a SER on U. Every $\vartheta(e)$ on U is an ER if (ϑ, A) is a SER on U. The set U is thereby split into equivalence classes $u\vartheta(e)$ by $\vartheta(e). u\vartheta(e) = \vartheta(e)u$ in this instance, and $\{u\vartheta(e): u \in U\}$ is a partition of U. Additionally, $\underline{\vartheta}^{\lambda}(e) = \lambda_{\underline{\vartheta}}(e)$ and $\overline{\vartheta}^{\lambda}(e) = \lambda_{\overline{\vartheta}}(e)$ apply in this scenario. Additional features of the approximation operators regarding SRR are as follows:

Theorem 5. For $\lambda \in \vartheta(U)$, the following features for lower and upper approximations w.r.t. A-sets hold:

1) $\underline{\vartheta}^{\lambda}(e) \leq \lambda$ for all $e \in A$;
2) $\lambda \leq \overline{\vartheta}^{\lambda}(e)$ for all $e \in A$;
3) $\underline{\vartheta}^{\lambda}(e) \leq \overline{\vartheta}^{\lambda}(e)$ for all $e \in A$.

Proof. For $u \in U$,

1) Consider $\underline{\vartheta}^{\lambda}(e)(u) = \bigwedge_{a\in u\vartheta(e)}\lambda(a) \leq \lambda(u)$, because $u \in uF(e)$, therefore $\underline{\vartheta}^{\lambda}(e)(u) \leq \lambda(u)$. Hence, $\underline{\vartheta}^{\lambda}(e) \leq \lambda$.
2) Consider $\lambda(u) \leq \bigvee_{a\in u\vartheta(e)}\lambda(a) = \overline{\vartheta}^{\lambda}(e)(u)$. Hence, $\lambda \leq \overline{\vartheta}^{\lambda}(e)$.
3) It directly follows from (1) and (2).

Theorem 6. For $\delta \in \vartheta(W)$, the succeeding characteristics for lower and upper approximations regarding the F-sets hold:

1) $\delta_{\underline{\vartheta}(e)} \leq \delta$ for all $e \in A$;
2) $\delta \leq \delta_{\overline{\vartheta}(e)}$ for all $e \in A$;
3) $\delta_{\underline{\vartheta}(e)} \leq \delta_{\overline{\vartheta}(e)}$ for all $e \in A$.

Proof. Identical to the proof of above theorem.

Theorem 7. Let (ϑ, A) and (σ, A) be two $SRRs$ on a non-empty set U such that $\vartheta(e) \subseteq \sigma(e)$ for all $e \in A$. Then, $\underline{\sigma}^{\mu}(e) \leq \underline{\vartheta}^{\mu}(e)$ and $\overline{\vartheta}^{\mu}(e) \leq \overline{\sigma}^{\mu}(e)$ for all $\mu \in \vartheta(U)$ and $e \in A$ regarding A-sets.

Proof. Let $\mu \in \vartheta(U)$. Since $\vartheta(e) \subseteq \sigma(e)$, we have $u\vartheta(e) \subseteq u\sigma(e)$ for all $u \in U$ and $e \in A$. Therefore, $\bigwedge_{a\in u\vartheta(e)}\mu(a) \geq \bigwedge_{a\in u\sigma(e)}\mu(a)$ and $\bigvee_{a\in u\vartheta(e)}\mu(a) \leq \bigvee_{a\in u\sigma(e)}\mu(a)$ for all $u \in U$. By

Definition 11, $\underline{\sigma}^{\mu}(e) \leq \underline{\vartheta}^{\mu}(e)$ and $\overline{\vartheta}^{\mu}(e) \leq \overline{\sigma}^{\mu}(e)$ regarding A-sets.

Theorem 8. Let (ϑ, A) and (σ, A) be two $SRRs$ on U such that $\vartheta(e) \subseteq \sigma(e)$ for all $e \in A$. Then, $\mu_{\underline{\sigma}}(e) \leq \mu_{\underline{\vartheta}}(e)$ and $\vartheta_{\overline{\sigma}}(e) \leq \mu_{\overline{\sigma}}(e)$ for all $\mu \in \vartheta(U)$ and $e \in A$ regarding F-sets.

Proof. Analogous to the proof of Theorem 7.

Corollary 1. Let (ϑ, A) and (σ, A) be two $SRRs$ on U. Then, the following claims hold for all $\lambda \in \vartheta(U)$ and $e \in A$ regarding A-sets.

1) $\left(\overline{\vartheta \cap \sigma}\right)^{\lambda}(e) \leq \left(\overline{\vartheta}\right)^{\lambda}(e) \cap (\overline{\sigma})^{\lambda}(e)$;
2) $\left(\underline{\vartheta \cap \sigma}\right)^{\lambda}(e) \geq \left(\underline{\vartheta}\right)^{\lambda}(e) \cap \left(\underline{\sigma}\right)^{\lambda}(e)$.

Proof.

1) Let (ϑ, A) and (σ, A) be two $SRRs$ on U. Then, $(\vartheta \cap \sigma, A)$ is also a SRR on U. Also, $(\vartheta \cap \sigma)(e) \subseteq \vartheta(e)$ and $(\vartheta \cap \sigma)(e) \subseteq \sigma(e)$. By Theorem 7, $\left(\overline{\vartheta \cap \sigma}\right)^{\lambda}(e) \leq (\overline{\vartheta})^{\lambda}(e)$ and $\left(\overline{\vartheta \cap \sigma}\right)^{\lambda}(e) \leq (\overline{\sigma})^{\lambda}(e)$ for any $\lambda \in \vartheta(U)$. This proves that $\left(\overline{\vartheta \cap \sigma}\right)^{\lambda}(e) \leq \left(\overline{\vartheta}\right)^{\lambda}(e) \cap (\overline{\sigma})^{\lambda}(e)$ for all $\lambda \in \vartheta(U)$ and $e \in A$.
2) This can be proved as (1).

Corollary 2. Let (ϑ, A) and (σ, A) be two $SRRs$ on a non-empty set U. Then, the following assertions hold for all $\delta \in \vartheta(U)$ and $e \in A$ with respect to F-sets.

1) $\delta_{\overline{(\vartheta \cap \sigma)}}(e) \leq \delta_{\overline{\vartheta}}(e) \cap \delta_{\overline{\sigma}}(e)(e)$;
2) $\delta_{\underline{(\vartheta \cap \sigma)}}(e) \geq \delta_{\underline{\vartheta}}(e) \cup \delta_{\underline{\sigma}}(e)$.

Proof. Analogous to the proof of the above corollary.

4. Approximation of a fuzzy set through soft binary relations

This section examines two different types of fuzzy topologies brought about by SRRs, and it also takes certain related findings into consideration.

Definition 17. If a family of FSs on $T \subseteq \mathcal{F}(U)$ satisfies the following three axioms, it is referred to as a fuzzy topology for U:

1) $0,1 \in T$.
2) $\forall \lambda, \mu \in T \Rightarrow \lambda \wedge \mu \in T$.
3) $\forall \left(\lambda_j\right)_{j \in J} \in T \Longrightarrow \vee_{j \in J} \lambda_j \in T$.

Moreover, the pair (U, T) is named a fuzzy topological space and the elements of T are called fuzzy open sets.

Theorem 9. If (ϑ, A) is a SRR on U, then $T_e = \left\{\lambda \in \mathcal{F}(U): \underline{\vartheta}^{\lambda}(e) = \lambda\right\}$ is a fuzzy topology on U for each $e \in A$.

Proof.

1) Take $e \in A$. By Theorem 1, $\underline{\vartheta}^{0}(e) = 0$ and $\underline{\vartheta}^{1}(e) = 1$. This implies that $0,1 \in T_e$.
2) Let $\lambda, \delta \in T_e$. This implies $\underline{\vartheta}^{\lambda}(e) = \lambda$ and $\underline{\vartheta}^{\delta}(e) = \delta$. Now, by using Theorem 1, $\underline{\vartheta}^{\lambda \wedge \delta}(e) = \underline{\vartheta}^{\lambda}(e) \cap \underline{\vartheta}^{\delta}(e) = \lambda \wedge \delta$. This implies that $\lambda \wedge \delta \in T_e$.
3) Let $\lambda_j \in T_e$. This implies $\underline{F}^{\lambda_j}(e) = \lambda_j$ for $j \in J$. Since, the relation is SR, so by Theorem 5, $\underline{\vartheta}^{\vee_{j \in J} \lambda_j}(e) \leq \vee_{j \in J} \lambda_j$. Since, $\lambda_j \leq \vee_{j \in J} \lambda_j$. By using Theorem 1, $\underline{\vartheta}^{\lambda_j}(e) \leq \underline{\vartheta}^{\vee_{j \in J} \lambda_j}(e)$. This implies $\vee_{j \in J} \underline{\vartheta}^{\lambda_j}(e) \leq \underline{\vartheta}^{\vee_{j \in J} \lambda_j}(e)$. This implies that $\vee_{j \in J} \lambda_j \leq \underline{\vartheta}^{\vee_{j \in J} \lambda_j}(e)$. Therefore, $\underline{\vartheta}^{\vee_{j \in J} \lambda_j}(e) = \vee_{j \in J} \lambda_j$. Hence, $\vee_{j \in J} \lambda_j \in T_e$.

Theorem 10. If (ϑ, A) is a SRR on U, then $T_e' = \left\{\mu \in \mathcal{F}(U): \mu_{\underline{\vartheta}}(e) = \mu\right\}$ is a fuzzy topology on U for $e \in A$.

Proof. Analogous to the proof of Theorem 9.

Remark 1. In the above two theorems, corresponding to each $e \in A$, we construct two fuzzy topologies on U. If we define $\underline{T_e} = \{\lambda \in \mathcal{F}\,(U): \underline{\vartheta}^{\lambda}(e) = \lambda \text{ for all } e \in A\}$, then $\underline{T_e}$ is a fuzzy topology on U and $\underline{T_e} = \bigcap_{e \in A} T_e$. Similarly, if we define $\underline{T'_e} = \{\mu \in \mathcal{F}\,(U): {}^{\mu}\underline{\vartheta}(e) = \mu \text{ for all } e \in A\}$, then $\underline{T'_e}$ is a fuzzy topology on U and $\underline{T'_e} = \bigcap_{e \in A} T'_e$.

Definition 18. Let (ϑ, A) be a $SRRs$ over U. Define a BR R_ϑ on U by $xR_\vartheta y \Leftrightarrow x\vartheta(e)y$ for some $e \in E$ where $x, y \in U$. Then, R_ϑ is called the BR induced by (ϑ, A).

Remark 2. (ϑ, A) is a SRR over $U \Rightarrow R_\vartheta$ is a reflexive relation over U. (ϑ, A) is a soft symmetric relation over $U \Rightarrow R_\vartheta$ is a symmetric relation over U.

Theorem 11. Let (U, W, R_ϑ) be a generalized approximation space and (ϑ, A) be a SBR over U. For $\lambda_1, \lambda_2 \in \mathcal{F}\,(U)$, the following properties for lower and upper approximations regarding A-sets hold:

1) $\lambda_1 \leq \lambda_2 \Rightarrow \underline{R_\vartheta}(\lambda_1) \leq \underline{R_\vartheta}(\lambda_2)$,
2) $\lambda_1 \leq \lambda_2 \Rightarrow \overline{R_\vartheta}(\lambda_1) \leq \overline{R_\vartheta}(\lambda_2)$,
3) $\underline{R_\vartheta}(\lambda_1) \cap \underline{R_\vartheta}(\lambda_2) = \underline{R_\vartheta}(\lambda_1 \cap \lambda_2)$,
4) $\overline{R_\vartheta}(\lambda_1) \cap \overline{R_\vartheta}(\lambda_2) \geq \overline{R_\vartheta}(\lambda_1 \cap \lambda_2)$,
5) $\underline{R_\vartheta}(\lambda_1) \cup \underline{R_\vartheta}(\lambda_2) \leq \underline{R_\vartheta}(\lambda_1 \cup \lambda_2)$,
6) $\overline{R_\vartheta}(\lambda_1) \cup \overline{R_\vartheta}(\lambda_2) = \overline{R_\vartheta}(\lambda_1 \cup \lambda_2)$,
7) $\underline{R_\vartheta}(1) = 1$ if $u\vartheta(e) \neq \phi$,
8) $\overline{R_\vartheta}(1) = 1$ if $u\vartheta(e) \neq \phi$,
9) $\underline{R_\vartheta}(\lambda) = \left(\overline{R_\vartheta}(\lambda^c)\right)^c$ if $u\vartheta(e) \neq \phi$,
10) $\overline{R_\vartheta}(\lambda) = \left(\underline{R_\vartheta}(\lambda^c)\right)^c$ if $u\vartheta(e) \neq \phi$,
11) $\underline{R_\vartheta}(0) = 0 = \overline{R_\vartheta}(0)$.

Proof. Similar to the proof of Theorem 1.

Theorem 12. Let (U, W, R_ϑ) be a generalized approximation space and (F, A) be a SBR over U. For $\delta_1, \delta_2 \in \mathcal{F}\,(U)$, the following properties for lower and upper approximations regarding F-sets hold:

1) $\delta_1 \leq \delta_2 \Rightarrow (\delta_1)\underline{R_\vartheta} \leq (\delta_2)\underline{R_\vartheta}$.
2) $\delta_1 \leq \delta_2 \Rightarrow (\delta_1)\overline{R_\vartheta} \leq (\delta_2)\overline{R_\vartheta}$.
3) $(\delta^1)\underline{R_\vartheta} \cap (\delta_2)\underline{R_\vartheta} = (\delta^1 \cap \delta_2)\underline{R_\vartheta}$.
4) $(\delta_1)\overline{R_\vartheta} \cap (\delta_2)\overline{R_\vartheta} \geq (\delta_1 \cap \delta_2)\overline{R_\vartheta}$.
5) $(\delta^1)\underline{R_\vartheta} \cup (\delta_2)\underline{R_\vartheta} \leq (\delta_1 \cup \delta_2)\underline{R_\vartheta}$.
6) $(\delta^1)\overline{R_\vartheta} \cup (\delta_2)\overline{R_\vartheta} = (\delta^1 \cup \delta_2)\overline{R_\vartheta}$.
7) $(1)\underline{R_\vartheta} = 1$ if $\vartheta(e)w \neq \phi$.
8) $(1)\overline{R_\vartheta} = 1$ if $\vartheta(e)w \neq \phi$.
9) $(\delta)\underline{R_\vartheta} = ((\delta^c)\overline{R_\vartheta})^c$ if $\vartheta(e)w \neq \phi$.
10) $(\delta)\overline{R_\vartheta} = ((\delta^c)\underline{R_\vartheta})^c$ if $\vartheta(e)w \neq \phi$.
11) $(0)\underline{R_\vartheta} = 1 = (0)\overline{R_v}$.

Proof. Analogous to the proof of Theorem 2.

Theorem 13. If (ϑ, A) is a SRR on U, then $T_{R_F} = \left\{\lambda \in \mathcal{F}\,(U): \underline{R_\vartheta}(\lambda) = \lambda\right\}$ is a fuzzy topology on U regarding A-sets for any $e \in A$.

Proof.

1) By Theorem 11, $\underline{R_\vartheta}(0) = 0$ and $\underline{R_\vartheta}(1) = 1$. This implies $0,1 \in T_{R_\vartheta}$.

2) Let $\lambda, \delta \in T_{R_\vartheta}$. This implies $\underline{R_\vartheta}(\lambda) = \lambda$ and $\underline{R_\vartheta}(\delta) = \delta$. Now, by using Theorem 11, $\underline{R_\vartheta}(\lambda \cap \delta) = \underline{R_\vartheta}(\lambda) \cap \underline{R_\vartheta}(\delta) = \lambda \wedge \delta$. This implies $\lambda \wedge \delta \in T_{R_\vartheta}$.
3) Let $\lambda_j \in T_{R_\vartheta}$. This implies, $\underline{R_\vartheta}(\lambda_j) = \lambda_j$ for $j \in J$. Since, the relation is SRR, by Theorem 5, $\underline{R_\vartheta}(\cup_{j\in J} \lambda_j) \leq (\vee_{j\in J} \lambda_j)$. Since $\lambda_j \leq \vee_{j\in J} \lambda_j$. By using Theorem 11, $\underline{R_\vartheta}(\lambda_j) \leq \underline{R_\vartheta}(\cup_{j\in J} \lambda_j)$. This implies $\cup_{j\in J} \underline{R_\vartheta}(\lambda_j) \leq \underline{R_\vartheta}(\cup_{j\in J} \lambda_j)$. Therefore, $\cup_{j\in J} \underline{R_\vartheta}(\lambda_j) = \underline{R_\vartheta}(\cup_{j\in J} \lambda_j)$. Hence, $\vee_{j\in J} \lambda_j \in T_{R_\vartheta}$.

Theorem 14. If (ϑ, A) is a SRR on U, then $T'_{R_\vartheta} = \left\{\lambda \in \mathcal{F}(U): (\mu)\underline{R_\vartheta} = \mu\right\}$ is a fuzzy topology on U regarding F-sets for $e \in A$.

Proof. Similar to the proof of Theorem 13.

Theorem 15. If (ϑ, A) is a SRR over U, then $\underline{T_e} = T_{R_\vartheta}$ regarding A-sets and $\underline{T'_e} = T'_{R_\vartheta}$ regarding F-sets.

Proof. Let $\lambda \in T_{R_\vartheta}$. This implies $\underline{R_\vartheta}(\lambda) = \lambda$. So $\wedge_{a\in uR_\vartheta(e)}\lambda(a) = \lambda$. Since $R_\vartheta = \cup_{e\in A}\vartheta(e)$
$\lambda \in \cap_{e\in A} T_e$. Hence, $\lambda \in \underline{T_e}$.

5. Similarity relations associated with soft binary relations

Some BRs between FSs are defined based on rough approximation, and their related characteristics are examined in this section.

Definition 19. Let (U, W, ϑ) be a generalized approximation space. For $\lambda_1, \lambda_2 \in \mathcal{F}(W)$, we define
(1). $\lambda_1 \backsimeq \lambda_2$ if and only if $\underline{\vartheta}^{\lambda_1} = \underline{\vartheta}^{\lambda_2}$,
(2). $\lambda_1 \eqsim \lambda_2$ if and only if $\overline{\vartheta}^{\lambda_1} = \overline{\vartheta}^{\lambda_2}$,
(3). $\lambda_1 \approx \lambda_2$ if and only if $\underline{\vartheta}^{\lambda_1} = \underline{\vartheta}^{\lambda_2}$ and $\overline{\vartheta}^{\lambda_1} = \overline{\vartheta}^{\lambda_2}$.

Definition 20. Let (U, W, ϑ) be a generalized approximation space. For $\delta_1, \delta_2 \in \mathcal{F}(U)$, we define
(1). $\delta_1 \backsimeq \delta_2$ if and only if ${}^{\delta_1}\underline{\vartheta} = {}^{\delta_2}\underline{\vartheta}$,
(2). $\delta_1 \eqsim \delta_2$ if and only if ${}^{\delta_1}\overline{\vartheta} = {}^{\delta_2}\overline{\vartheta}$,
(3). $\delta_1 \approx \delta_2$ if and only if ${}^{\delta_1}\underline{\vartheta} = {}^{\delta_2}\underline{\vartheta}$ and ${}^{\delta_1}\overline{\vartheta} = {}^{\delta_2}\overline{\vartheta}$.

The lower fuzzy similarity relation, upper fuzzy similarity relation, and fuzzy similarity relation are called for these binary relations, respectively. Obviously, $\underline{\vartheta}^{\lambda_1}$ and $\overline{\vartheta}^{\lambda_1}$ are similar if and only if they are both lower and upper similar for $\lambda \in \mathcal{F}(W)$, and they are both lower and upper similar for $\delta \in \mathcal{F}(U)$ if and only if ${}^{\delta_1}\underline{\vartheta}$ and ${}^{\delta_1}\overline{\vartheta}$ are similar.

Proposition 1. The relations $\backsimeq, \eqsim$ and $\approx$ are ERs on $\mathcal{F}(U)$.

Proof. Obvious.

Proposition 2. The relations $\backsimeq, \eqsim$ and $\approx$ are ERs on $\mathcal{F}(W)$.

Proof. Straightforward.

Theorem 16. Let (ϑ, A) be a SRR on U. For $\lambda_i \in \mathcal{F}(U)$, where $i = 1,2,3,4$, the subsequent assumptions are true:
1) $\lambda_1 \eqsim \lambda_2$ if and only if $\lambda_1 \eqsim (\lambda_1 \cup \lambda_2) \eqsim \lambda_2$;
2) $\lambda_1 \eqsim \lambda_2$ and $\lambda_3 \eqsim \lambda_4$ imply that $(\lambda_1 \cup \lambda_3) \eqsim (\lambda_2 \cup \lambda_4)$;
3) $\lambda_1 \leq \lambda_2$ and $\lambda_2 \eqsim 0$ imply that $\lambda_1 \eqsim 0$;
4) $(\lambda_1 \cup \lambda_2) \eqsim 0$ if and only if $\lambda_1 \eqsim 0$ and $\lambda_2 \eqsim 0$;
5) $\lambda_1 \leq \lambda_2$ and $\lambda_1 \eqsim 1$ imply that $\lambda_2 \eqsim 1$;
6) If $(\lambda_1 \cap \lambda_2) \eqsim 1$ then, $\lambda_1 \eqsim 1$ and $\lambda_2 \eqsim 1$.

Proof.

1) Let $\lambda_1 \eqsim \lambda_2$. Then $\overline{\vartheta}^{\lambda_1} = \overline{\vartheta}^{\lambda_2}$. By part (6) of Theorem 1, we get $\overline{\vartheta}^{\lambda_1 \cup \lambda_2} = \overline{\vartheta}^{\lambda_1} \cup \overline{\vartheta}^{\lambda_2} = \overline{\vartheta}^{\lambda_1} = \overline{\vartheta}^{\lambda_2}$ so $\lambda_1 \eqsim (\lambda_1 \cup \lambda_2) \eqsim \lambda_2$. The converse holds by transitivity of the relation $\eqsim$.
2) Given that $\lambda_1 \eqsim \lambda_2$ and $\lambda_3 \eqsim \lambda_4$. Then $\overline{\vartheta}^{\lambda_1} = \overline{\vartheta}^{\lambda_2}$ and $\overline{\vartheta}^{\lambda_3} = \overline{\vartheta}^{\lambda_4}$. By part (6) of Theorem 1, we get $\overline{\vartheta}^{\lambda_1 \cup \lambda_3} = \overline{\vartheta}^{\lambda_1} \cup \overline{\vartheta}^{\lambda_3} = \overline{\vartheta}^{\lambda_2} \cup \overline{\vartheta}^{\lambda_4} = \overline{\vartheta}^{\lambda_2 \vee \lambda_4}$. Thus, $(\lambda_1 \cup \lambda_3) \eqsim (\lambda_2 \cup \lambda_4)$.
3) Given $\lambda_2 \eqsim 0$. This implies $\overline{\vartheta}^{\lambda_2} = \overline{\vartheta}^{0}$. Also, $\lambda_1 \leq \lambda_2 \Rightarrow \overline{\vartheta}^{\lambda_1} \subseteq \overline{\vartheta}^{\lambda_2} = \overline{\vartheta}^{0}$. It follows that $\overline{\vartheta}^{\lambda_1} \subseteq \overline{\vartheta}^{0}$ but $\overline{\vartheta}^{0} \subseteq \overline{\vartheta}^{\lambda_1}$. Therefore, $\overline{\vartheta}^{\lambda_1} = \overline{\vartheta}^{0} \Rightarrow \lambda_1 \eqsim 0$.
4) Let $\lambda_1 \eqsim 0$ and $\lambda_2 \eqsim 0$. Then $\overline{\vartheta}^{\lambda_1} = \overline{\vartheta}^{0}$ and $\vartheta^{\lambda_2} = \overline{\vartheta}^{0}$. By part (6) of Theorem 1, we get $\overline{\vartheta}^{\lambda_1 \cup \lambda_2} = \overline{\vartheta}^{\lambda_1} \cup \overline{\vartheta}^{\lambda_2} = \overline{\vartheta}^{0} \cup \overline{\vartheta}^{0} = \overline{\vartheta}^{0}$. Thus, $(\lambda_1 \cup \lambda_2) \eqsim 0$. The converse follows from (3).
5) Given $\lambda_1 \eqsim 1$. This implies $\overline{\vartheta}^{\lambda_1} = \overline{\vartheta}^{1}$. Also, $\lambda_1 \leq \lambda_2 \Rightarrow \overline{\vartheta}^{\lambda_2} \supseteq \overline{\vartheta}^{\lambda_1} = \overline{\vartheta}^{1} = 1 \supseteq \overline{\vartheta}^{\lambda_2}$. Therefore, $\overline{\vartheta}^{\lambda_2} = \overline{\vartheta}^{1} \Rightarrow \lambda_2 \eqsim 1$.
6) It follows from (5).

Theorem 17. Let (ϑ, A) be a SRR on U. For $\delta_i \in \mathcal{F}\,(U)$, where $i = 1,2,3,4$, the subsequent assumptions are true:

1) $\delta_1 \eqsim \delta_2$ if and only if $\delta_1 \eqsim (\delta_1 \cup \delta_2) \eqsim \delta_2$;
2) $\delta_1 \eqsim \delta_2$ and $\delta_3 \eqsim \delta_4$ imply that $(\delta_1 \cup \delta_3) \eqsim (\delta_2 \cup \delta_4)$;
3) $\delta_1 \leq \delta_2$ and $\delta_2 \eqsim 0$ imply that $\delta_1 \eqsim 0$;
4) $(\delta_1 \cup \delta_2) \eqsim 0$ if and only if $\delta_1 \eqsim 0$ and $\delta_2 \eqsim 0$;
5) $\delta_1 \leq \delta_2$ and $\delta_1 \eqsim 1$ imply that $\delta_2 \eqsim 1$;
6) If $(\delta_1 \cap \delta_3) \eqsim 1$ then, $\delta_1 \eqsim 1$ and $\delta_2 \eqsim 1$.

Proof. Similar to the proof of Theorem 16.

Theorem 18. Let (ϑ, A) be a SRR on U. For $\lambda_i \in \mathcal{F}\,(U)$, where $i = 1,2,3,4$, the subsequent assumptions are true:

1) $\lambda_1 \,\underline{\smile}\, \lambda_2$ if and only if $\lambda_1 \,\underline{\smile}\, (\lambda_1 \cap \lambda_2) \,\underline{\smile}\, \lambda_2$;
2) $\lambda_1 \,\underline{\smile}\, \lambda_2$ and $\lambda_3 \,\underline{\smile}\, \lambda_4$ imply that $(\lambda_1 \cap \lambda_3) \,\underline{\smile}\, (\lambda_2 \cap \lambda_4)$;
3) $\lambda_1 \leq \lambda_2$ and $\lambda_2 \,\underline{\smile}\, 0$ imply that $\lambda_1 \,\underline{\smile}\, 0$;
4) $(\lambda_1 \cup \lambda_2) \,\underline{\smile}\, 0$ if and only if $\lambda_1 \,\underline{\smile}\, 0$ and $\lambda_2 \,\underline{\smile}\, 0$;
5) $\lambda_1 \leq \lambda_2$ and $\lambda_1 \,\underline{\smile}\, 1$ imply that $\lambda_2 \,\underline{\smile}\, 1$;
6) If $(\lambda_1 \cap \lambda_2) \,\underline{\smile}\, 1$ then, $\lambda_1 \,\underline{\smile}\, 1$ and $\lambda_2 \,\underline{\smile}\, 1$.

Proof. Analogous to the proof of Theorem 16.

Theorem 19. Let (ϑ, A) be a SRR on U. For $\delta_i \in \mathcal{F}\,(U)$, where $i = 1,2,3,4$, the succeeding assumptions are valid:

1) $\delta_1 \,\underline{\smile}\, \delta_2$ if and only if $\delta_1 \,\underline{\smile}\, (\delta_1 \cap \delta_2) \,\underline{\smile}\, \delta_2$;
2) $\delta_1 \,\underline{\smile}\, \delta_2$ and $\delta_3 \,\underline{\smile}\, \delta_4$ imply that $(\delta_1 \cap \delta_3) \,\underline{\smile}\, (\delta_2 \cap \delta_4)$;
3) $\delta_1 \leq \delta_2$ and $\delta_2 \,\underline{\smile}\, 0$ imply that $\delta_1 \,\underline{\smile}\, 0$;
4) $(\delta_1 \cup \delta_2) \,\underline{\smile}\, 0$ if and only if $\delta_1 \,\underline{\smile}\, 0$ and $\delta_2 \,\underline{\smile}\, 0$;
5) $\delta_1 \leq \delta_2$ and $\delta_1 \,\underline{\smile}\, 1$ imply that $\delta_2 \,\underline{\smile}\, 1$;
6) If $(\delta_1 \cap \delta_2) \,\underline{\smile}\, 1$ then, $\delta_1 \,\underline{\smile}\, 1$ and $\delta_2 \,\underline{\smile}\, 1$.

Proof. Similar to the proof of Theorem 16.

Theorem 20. Let (ϑ, A) be a SRR on U. For $\lambda_i \in \mathcal{F}\,(U)$, where $i = 1,2$, the following assumptions are true:

1) $\lambda_1 \leq \lambda_2$ and $\lambda_2 \approx 0$ imply that $\lambda_1 \approx 0$;
2) $\lambda_1 \leq \lambda_2$ and $\lambda_1 \approx 1$ imply that $\lambda_2 \approx 1$;

3) $(\lambda_1 \cup \lambda_2) \approx 0$, then $\lambda_1 \approx 0$ and $\lambda_2 \approx 0$;
4) $(\lambda_1 \cap \lambda_2) \approx 1$, then $\lambda_1 \approx 1$ and $\lambda_2 \approx 1$;
5) $\lambda_1 \approx \lambda_2$ if and only if $\lambda_1 \mathrel{\overline{\eqsim}} (\lambda_1 \cup \lambda_2) \mathrel{\overline{\eqsim}} \lambda_2$ and $\lambda_1 \mathrel{\underline{\backsim}} (\lambda_1 \cap \lambda_2) \mathrel{\underline{\backsim}} \lambda_2$.

Proof. It follows immediately from Theorems 1 and 18.

Theorem 21. Let (ϑ, A) be a SRR on U. For $\delta_i \in \mathcal{F}(U)$, where $i = 1,2,3$, the subsequent assumptions are valid:

1) $\delta_1 \leq \delta_2$ and $\delta_2 \approx 0$ imply that $\delta_1 \approx 0$;
2) $\delta_1 \leq \delta_2$ and $\delta_1 \approx 1$ imply that $\delta_2 \approx 1$;
3) $(\delta_1 \cup \delta_2) \approx 0$, then $\delta_1 \approx 0$ and $\delta_2 \approx 0$;
4) $(\delta_1 \cap \delta_2) \approx 1$, then $\delta_1 \approx 1$ and $\delta_2 \approx 1$;
5) $\delta_1 \approx \delta_2$ if and only if $\delta_1 \mathrel{\overline{\eqsim}} (\delta_1 \cup \delta_2) \mathrel{\overline{\eqsim}} \delta_2$ and $\delta_1 \mathrel{\underline{\backsim}} (\delta_1 \cap \delta_2) \mathrel{\underline{\backsim}} \delta_2$.

Proof. Analogous to the proof of Theorem 20.

6. Accuracy measures

An approach to examining the degree to which the membership functions of FSs accurately characterize the objects is provided by the approximation of FSs. This section presents the membership functions of FSs regarding. A-sets and F-sets, together with their corresponding degrees of accuracy and roughness. To achieve this, we first give a definition and some properties of the α-level cuts of an FS.

Definition 21. Let U be a non-empty universe and $\lambda \in \mathcal{F}(U)$. For $0 \lneqq \alpha \leq 1$, the α −level cut of λ is denoted and defined as:

$$\lambda_a = \{u \in U : \lambda(u) \geq \alpha\}. \tag{6.1}$$

Lemma 1. Let U be a non-empty universe and $\lambda, \mu \in \mathcal{F}(U)$. For $0 \lneqq \alpha \leq 1$, $\lambda \leq \mu$ implies that $\lambda_\alpha \subseteq \mu_\alpha$.

Proof. It follows directly from Definition 21.

Lemma 2. Let U be a non-empty universe and $\lambda \in \mathcal{F}(U)$ and $0 \lneqq \beta \leq \alpha \leq 1$. Then $\lambda_\alpha \subseteq \lambda_\beta$.

Proof. It follows immediately from Definition 21.

Note that $\underline{\vartheta}^{(\lambda_\alpha)}$ is the lower approximation of the crisp set λ_α while $\left(\underline{\vartheta}^{\lambda}(e)\right)_\alpha$ is the α −level cut of $\underline{\vartheta}^{\lambda}(e)$ regarding A-sets. Therefore,

$$\left(\underline{\vartheta}^{\lambda}\ (\mathrm{e})\right)_\alpha = \left\{u \in U : \underline{\vartheta}^{\lambda}(e)(u) \geq \alpha\right\} = \left\{u \in U : \wedge_{a \in u\vartheta(e)} \lambda(a) \geq \alpha\right\}$$

and

$$\left(\overline{\vartheta}^{\lambda}\ (\mathrm{e})\right)_\alpha = \left\{u \in U : \vee_{a \in u\vartheta(e)} \lambda(a) \geq \alpha\right\} \text{ for all } e \in A.$$

Similarly, for $\delta \in \mathcal{F}(U)$, it follows that

$$\left({}^{\delta}\underline{\vartheta}(e)\right)_\alpha = \left\{u \in U : {}^{\delta}\underline{\vartheta}(e)(u) \geq \alpha\right\} = \left\{u \in U : \wedge_{a \in \vartheta(e)u} \delta(a) \geq \alpha\right\}$$

and

$$\left({}^{\delta}\overline{\vartheta}(e)\right)_\alpha = \left\{u \in U : \vee_{a \in \vartheta(e)u} \delta(a) \geq \alpha\right\} \text{ for all } e \in A$$

regarding F-sets.

Lemma 3. Let (F, A) be a SRR on a non-empty universe U, $\lambda \in \mathcal{F}\,(U)$ and $0 \lneqq \alpha \leq 1$. Then, the following assertions hold w.r.t the A-sets:

1) $\underline{\vartheta}^{(\lambda_\alpha)}(e) = \left(\underline{\vartheta}^{\lambda}(e)\right)_\alpha$ for all $e \in A$,
2) $\overline{\vartheta}^{(\lambda_\alpha)}(e) = \left(\overline{\vartheta}^{\lambda}(e)\right)_\alpha$ for all $e \in A$.

Proof.

1) Consider $\lambda \in \mathcal{F}\,(U)$ and $0 \lneqq \alpha \leq 1$. For the crisp set λ_α, we have

$$\underline{\vartheta}^{(\lambda_\alpha)}(e) = \{u \in U : u\vartheta(e) \subseteq \lambda_\alpha\} = \{u \in U : \lambda(a) \geq \alpha \; for \; all \; a \in u\vartheta(e)\}$$
$$= \left\{u \in U : \wedge_{a \in u\vartheta(e)} \lambda(a) \geq \alpha\right\} = \left(\underline{\vartheta}^{\lambda}(e)\right)_\alpha \text{ for all } e \in A.$$

2) It can be verified in the similar way as (1).

Lemma 4. Let (ϑ, A) be a SRR on a non-empty universe U, $\delta \in \mathcal{F}\,(U)$ and $0 \lneqq \alpha \leq 1$. Then, the following assertions hold regarding F-sets:

1) ${}^{(\delta_\alpha)}\underline{\vartheta}(e) = \left({}^{\delta}\underline{\vartheta}(e)\right)_\alpha$ for all $e \in A$,
2) ${}^{(\delta_\alpha)}\overline{\vartheta}(e) = \left({}^{\delta}\overline{\vartheta}(e)\right)_\alpha$ for all $e \in A$.

Proof. Similar to the proof of Lemma 3.

Now, we define the degrees of accuracy and roughness for membership functions of a FS in a non-empty finite universe.

Definition 22. Let (U, W, ϑ) be a generalized approximation space. The degree of accuracy for the membership of $\lambda \in \mathcal{F}\,(U)$, regarding parameters α, β such that $0 \lneqq \beta \leq \alpha \leq 1$ and regarding A-sets, is denoted and postulated as:

$$\gamma^{\vartheta}_{(\alpha,\beta)}(\lambda)(e_i) = \left|\underline{\vartheta}^{(\lambda_\alpha)}(e_i)\right| / \left|\overline{\vartheta}^{(\lambda_\beta)}(e_i)\right| \text{ for all } e_i \in A. \tag{6.2}$$

Likewise, the degree of accuracy for the membership of $\delta \in \mathcal{F}\,(U)$, regarding parameters α, β such that $0 \lneqq \beta \leq \alpha \leq 1$ and regarding F-sets, is denoted and portrayed as:

$${}^{\vartheta}_{(\alpha,\beta)}\gamma(\delta)(e_i) = \left|{}^{(\delta_\alpha)}\underline{\vartheta}(e_i)\right| / \left|{}^{(\delta_\beta)}\overline{\vartheta}(e_i)\right| \text{ for all } e_i \in A. \tag{6.3}$$

The degree of roughness for the membership of $\lambda \in \mathcal{F}\,(U)$, rergarding parameters α, β with $0 \lneqq \beta \leq \alpha \leq 1$ and regarding A-sets, is denoted and postulated as:

$$(\lambda)(e_i) = 1 - \gamma^{\vartheta}_{(\alpha,\beta)}(\lambda)(e_i) \text{ for all } e_i \in A. \tag{6.4}$$

In the same way, the degree of roughness for the membership of $\delta \in \mathcal{F}\,(U)$, regarding parameters α, β such that $0 \lneqq \beta \leq \alpha \leq 1$ and regarding F-sets, is denoted and described as:

$${}^{\vartheta}_{(\alpha,\beta)}\rho(\delta)(e_i) = 1 - {}^{\vartheta}_{(\alpha,\beta)}\gamma(\delta)(e_i) \text{ for all } e_i \in A. \tag{6.5}$$

Note that, in the case of SER, the concept of the F-sets and A-sets coincide. Further, $\underline{\vartheta}^{(\lambda_\alpha)}(e)$ or $\overline{\vartheta}^{(\lambda_\beta)}(e)$ comprise the objects of U having α or β as the least degree of definite or possible fulfilment in λ for all $e_i \in A$. Equivalently, $\underline{\vartheta}^{(\lambda_\alpha)}(e_i)$ or $\overline{\vartheta}^{(\lambda_\beta)}(e_i)$ can be interpreted as the union of the soft equivalence classes of U having a degree of fulfilment of at least α or β in the lower or upper fuzzy approximation of λ regarding A-sets. Therefore, the parameters α and β serve as the thresholds of sure and possible fulfilment of the objects of α or β in λ, respectively. Hence, $\gamma^{\vartheta}_{(\alpha,\beta)}(\lambda)(e_i)$ may be interpreted as the degree to which the membership function of λ is accurate, constrained to the threshold parameters α and β. In other words, $\gamma^{\vartheta}_{(\alpha,\beta)}(\lambda)(e_i)$ describes how accurate the membership function of the FS is regarding A-sets.

Example 3. Let $U = \{t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8, t_9, t_{10}, t_{11}\}$ be a collection of trees of different types and $A = \{e_1, e_2, e_3, e_4\}$ be a set of parameters such that e_1 stands for the attribute Height, e_2 stands for Age, e_3 stands for Fruitibility, and e_4 stands for the Thickness. Define a SER $\vartheta: A \to P(U \times U)$ for each $e \in A$. The corresponding soft equivalence class for each of the SERs is obtained as follows:

For $\vartheta(e_1)$, the soft equivalence classes $t\vartheta(e_1)$ are: $\{t_1, t_{10}\}, \{t_2, t_4, t_6, t_7\}, \{t_3, t_5, t_8, t_9,\}, \{t_{11}\}$.
For $\vartheta(e_2)$, the soft equivalence classes $t\vartheta(e_2)$ are: $\{t_1\}, \{t_2, t_{11}\}, \{t_4, t_7\}, \{t_3, t_5, t_8, t_9,\}, \{t_6, t_{10}\}$.
For $\vartheta(e_3)$, the soft equivalence classes $t\vartheta(e_3)$ are: $\{t_1\}, \{t_2\}, \{t_3, t_4, t_5, t_7, t_8, t_9, t_{10}\}, \{t_6\}, \{t_{11}\}$.
For $\vartheta(e_4)$, the soft equivalence classes $t\vartheta(e_4)$ are: $\{t_{10}\}, \{t_6\}, \{t_1, t_2, t_3, t_4, t_5, t_7, t_8, t_9\}, \{t_{11}\}$.

Define $\lambda : U \to [0,1]$ by

$$\lambda(t_1) = 0.9, \lambda(t_2) = 0.6, \lambda(t_3) = 0.3, \lambda(t_4) = 0,$$

$$\lambda(t_5) = 0.2, \lambda(t_6) = 0.4, \lambda(t_7) = 0.6, \lambda(t_8) = 0.8,$$

$$\lambda(t_9) = 1, \lambda(t_{10}) = 0, \lambda(t_{11}) = 1.$$

Take $\alpha = 0.7$ and $\beta = 0.6$. Then $\alpha -$level cuts $\lambda_{0.6}$ and $\lambda_{0.7}$ are calculated as:

$$\lambda_{0.6} = \{t_1, t_2, t_7, t_8, t_9, t_{11}\},$$

$$\lambda_{0.7} = \{t_1, t_8, t_9, t_{11}\}.$$

Now,

$$\underline{\vartheta}^{(\lambda_{0.7})}(e_1) = \{t_{11}\}, \underline{\vartheta}^{(\lambda_{0.7})}(e_2) = \{t_1\},$$

$$\underline{\vartheta}^{(\lambda_{0.7})}(e_3) = \{t_1, t_{11}\}, \underline{\vartheta}^{(\lambda_{0.7})}(e_4) = \{t_{11}\}.$$

And,

$$\overline{\vartheta}^{(\lambda_{0.6})}(e_1) = \{t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8, t_9, t_{10}, t_{11}\},$$

$$\overline{\vartheta}^{(\lambda_{0.6})}(e_2) = \{t_1, t_2, t_3, t_4, t_5, t_7, t_8, t_9, t_{11}\},$$

$$\overline{\vartheta}^{(\lambda_{0.6})}(e_3) = \{t_1, t_2, t_3, t_4, t_5, t_7, t_8, t_9, t_{10}, t_{11}\},$$

$$\overline{\vartheta}^{(\lambda_{0.6})}(e_4) = \{t_1, t_2, t_3, t_4, t_5, t_7, t_8, t_9, t_{11}\}.$$

The degree of accuracy for the membership of λ is calculated as follows:

$$\gamma^{\vartheta}_{(\alpha,\beta)}(\lambda)(e_1) = \left|\underline{\vartheta}^{(\lambda_{0.7})}(e_1)\right| / \left|\overline{\vartheta}^{(\lambda_{0.6})}(e_1)\right| = 1/11 = 0.091,$$

$$\gamma^{\vartheta}_{(\alpha,\beta)}(\lambda)(e_2) = \left|\underline{\vartheta}^{(\lambda_{0.7})}(e_2)\right| / \left|\overline{\vartheta}^{(\lambda_{0.6})}(e_2)\right| = 1/9 = 0.111,$$

$$\gamma^{\vartheta}_{(\alpha,\beta)}(\lambda)(e_3) = \left|\underline{\vartheta}^{(\lambda_{0.7})}(e_3)\right| / \left|\overline{\vartheta}^{(\lambda_{0.6})}(e_3)\right| = 1/5 = 0.200,$$

$$\gamma^{\vartheta}_{(\alpha,\beta)}(\lambda)(e_4) = \left|\underline{\vartheta}^{(\lambda_{0.7})}(e_4)\right| / \left|\overline{\vartheta}^{(\lambda_{0.6})}(e_4)\right| = 1/9 = 0.111.$$

Hence, $\gamma^{\vartheta}_{(\alpha,\beta)}(\lambda)(e_i)$ displays the degree to which the membership function of λ is accurately constrained to the parameters α and β for $i = 1,2,3,4$ regarding A-sets. Similarly, we can show it in the case of F-sets.

Theorem 22. Let (U, W, ϑ) be a generalized approximation space, $\lambda \in \mathcal{F}(U)$ and $0 \gneqq \beta \leq \alpha \leq 1$. Then, $0 \leq \gamma^{\vartheta}_{(\alpha,\beta)}(\lambda)(e) \leq 1$ for all $e \in A$ regarding A-sets.

Proof. For a FS $\lambda \in \mathcal{F}(U)$ and the parameters α, β such that $0 \gneqq \beta \leq \alpha \leq 1$. By using Lemma 6.3, $\lambda_\alpha(e) \subseteq \lambda_\beta(e)$. Now, according to Theorem 1, $\underline{\vartheta}^{\lambda_\alpha}(e) \leq \overline{\vartheta}^{\lambda_\alpha}(e) \leq \overline{\vartheta}^{\lambda_\beta}(e)$. So $\left|\underline{\vartheta}^{\lambda_\alpha}(\mathrm{e})\right| \leq \left|\overline{\vartheta}^{\lambda_\beta}(e)\right|$, therefore the ratio $\left|\underline{\vartheta}^{\lambda_\alpha}(\mathrm{e}) / \overline{\vartheta}^{\lambda_\beta}(e)\right|$ fluctuates between 0 and 1, which yields certainly $0 \leq \gamma^{\vartheta}_{(\alpha,\beta)}(\lambda)(e) \leq 1$ for all $e \in A$.

Corollary 3. Let (U, W, ϑ) be a generalized approximation space, $\lambda \in \mathcal{F}(U)$ and $0 \gneqq \beta \leq \alpha \leq 1$. Then, $0 \leq \rho^{\vartheta}_{(\alpha,\beta)}(\lambda)(e) \leq 1$ for all $e \in A$ regarding A-sets.

Proof. It is a direct consequence of Definition 22 and Theorem 22.

Theorem 23. Let (U, W, ϑ) be a generalized approximation space, $\lambda \in \mathcal{F}(U)$ and $0 \gneqq \beta \leq \alpha \leq 1$ for all $e \in A$ regarding A-sets.

1) If α stands fixed, then $\gamma^{\vartheta}_{(\alpha,\beta)}(\lambda)(e)$ increase with the increase in β.
2) If β stands fixed, then $\gamma^{\vartheta}_{(\alpha,\beta)}(\lambda)(e)$ decrease with the increase in α.

Proof.

1) Let α stand fixed and let $0 \gneqq \beta_1 \leq \beta_2 \leq 1$. Using Lemma 6.2, we have $\lambda_{\beta_2} \leq \lambda_{\beta_1}$. By Theorem 1, $\overline{\vartheta}^{\lambda_{\beta_2}}(e) \leq \overline{\vartheta}^{\lambda_{\beta_1}}(e)$ or $\left|\overline{\vartheta}^{\lambda_{\beta_2}}(e)\right| \leq \left|\overline{\vartheta}^{\lambda_{\beta_1}}(e)\right|$. This implies that $\left|\underline{\vartheta}^{\lambda_\alpha}(e)\right| / \left|\overline{\vartheta}^{\lambda_{\beta_1}}(e)\right| \leq \left|\underline{\vartheta}^{\lambda_\alpha}(e)\right| / \left|\overline{\vartheta}^{\lambda_{\beta_2}}(e)\right|$. That is $\gamma^{\vartheta}_{(\alpha,\beta_1)}(\lambda)(e) \leq \gamma^{\vartheta}_{(\alpha,\beta_2)}(\lambda)(e)$. This shows that $\gamma^{\vartheta}_{(\alpha,\beta)}(\lambda)(e)$ increase with the increase in β for all $e \in A$.

(2) Similar to the proof of (1).

Corollary 4. Let (U, W, ϑ) be a generalized approximation space, $\lambda \in \mathcal{F}(U)$ and $0 \gneqq \beta \leq \alpha \leq 1$ regarding A-sets.

1) If α stands fixed, then $\rho^{\vartheta}_{(\alpha,\beta)}(\lambda)(e)$ decrease with the increase in β.
2) If β stands fixed, then $\rho^{\vartheta}_{(\alpha,\beta)}(\lambda)(e)$ increase with the increase in α for all $e \in A$.

Proof. Direct consequence of Definition 22 and Theorem 23.

Theorem 24. Let (U, W, ϑ) be a generalized approximation space, $\lambda, \mu \in \mathcal{F}(U)$ and $0 \gneqq \beta \leq \alpha \leq 1$. Then, $\lambda \leq \mu$ implies the following assertions for all $e \in A$ and regarding A-sets.

1) $\gamma^{\vartheta}_{(\alpha,\beta)}(\lambda)(e) \leq \gamma^{\vartheta}_{(\alpha,\beta)}(\mu)(e)$, whenever $\overline{\vartheta}^{(\lambda_\beta)}(e) = \overline{\vartheta}^{(\mu_\beta)}(e)$;
2) $\gamma^{\vartheta}_{(\alpha,\beta)}(\lambda)(e) \geq \gamma^{\vartheta}_{(\alpha,\beta)}(\mu)(e)$, whenever $\underline{\vartheta}^{(\lambda_\alpha)}(e) = \underline{\vartheta}^{(\mu_\alpha)}(e)$.

Proof.

1) Presume that $0 \gneqq \beta \leq \alpha \leq 1$ and $\lambda, \mu \in \mathcal{F}(U)$ with $\lambda \leq \mu$. By Theorem 1, $\underline{\vartheta}^{(\lambda_\alpha)}(e) \leq \underline{\vartheta}^{(\mu_\alpha)}(\mathrm{e})$ or $\left|\underline{\vartheta}^{(\lambda_\alpha)}(\mathrm{e})\right| \leq \left|\underline{\vartheta}^{(\mu_\alpha)}(\mathrm{e})\right|$. This implies that $\left|\underline{\vartheta}^{(\lambda_\alpha)}(\mathrm{e})\right| / \left|\overline{\vartheta}^{(\lambda_\beta)}(e)\right| \leq \left|\underline{\vartheta}^{(\mu_\alpha)}(\mathrm{e})\right| / \left|\overline{\vartheta}^{(\mu_\beta)}(e)\right|$. Hence, $\gamma^{\vartheta}_{(\alpha,\beta)}(\lambda)(e) \leq \gamma^{\vartheta}_{(\alpha,\beta)}(\mu)(e)$.
2) Identical to the proof of (1).

Corollary 5. Let (U, W, ϑ) be a generalized approximation space, $\lambda, \mu \in \mathcal{F}(U)$ and $0 \gneqq \beta \leq \alpha \leq 1$. Then, $\lambda \leq \mu$ implies the following assertions for all $e \in A$ and regarding A-sets.

1) $\rho^{\vartheta}_{(\alpha,\beta)}(\lambda)(e) \geq \rho^{\vartheta}_{(\alpha,\beta)}(\mu)(e)$, whenever $\overline{\vartheta}^{(\lambda_\beta)}(e) = \overline{\vartheta}^{(\mu_\beta)}(e)$;
2) $\rho^{\vartheta}_{(\alpha,\beta)}(\lambda)(e) \leq \rho^{\vartheta}_{(\alpha,\beta)}(\mu)(e)$, whenever $\underline{\vartheta}^{(\lambda_\alpha)}(e) = \underline{\vartheta}^{(\mu_\alpha)}(e)$.

Proof. Immediately follows from Definition 22 and Theorem 23.

Theorem 25. Let (U, W, ϑ) be a generalized approximation space, $\lambda \in \mathcal{F}(U)$ and $0 \gneq \beta \leq \alpha \leq 1$. If (σ, A) is a SER on U such that $F(e) \subseteq \sigma(e)$. Then, $\gamma^{\vartheta}_{(\alpha,\beta)}(\lambda)(e) \geq \gamma^{\sigma}_{(\alpha,\beta)}(\lambda)(e)$ for all $e \in A$ regarding A-sets.

Proof. Let $\lambda \in \mathcal{F}(U)$ and let (ϑ, A) and (σ, A) be two SER on U such that $\vartheta(e) \subseteq \sigma(e)$. By Theorem 3.4, $\underline{\vartheta}^{\lambda}$ (e) $\geq \underline{\sigma}^{\lambda}$ (e) and $\overline{\vartheta}^{\lambda}(e) \leq \overline{\sigma}^{\lambda}(e)$. Using Lemma 1, $\underline{\vartheta}^{(\lambda_\alpha)}$ (e) $\geq \underline{\sigma}^{(\lambda_\alpha)}$ (e) and $\vartheta^{(\lambda_\beta)}$ (e) $\leq \overline{\sigma}^{(\lambda_\beta)}$ (e). By Lemma 3, $\left|\underline{\vartheta}^{(\lambda_\alpha)}\text{ (e)}\right| = \left|\underline{\vartheta}^{(\lambda)_\alpha}\text{ (e)}\right| \geq \left|\underline{\sigma}^{(\lambda)_\alpha}\text{ (e)}\right| = \left|\underline{\sigma}^{(\lambda_\alpha)}\text{ (e)}\right|$ and $\left|\overline{\vartheta}^{(\lambda_\beta)}\text{ (e)}\right| = \left|\overline{\vartheta}^{(\lambda)_\beta}\text{ (e)}\right| \geq \left|\overline{\sigma}^{(\lambda)_\beta}\text{ (e)}\right| = \left|\overline{\sigma}^{(\lambda_\beta)}\text{ (e)}\right|$.

Rearranging and dividing the above two equations, we get

$\left|\underline{\vartheta}^{(\lambda_\alpha)}\text{ (e)}\right| / \left|\overline{\vartheta}^{(\lambda_\beta)}\text{ (e)}\right| \geq \left|\underline{\sigma}^{(\lambda_\alpha)}\text{ (e)}\right| / \left|\overline{\sigma}^{(\lambda_\beta)}\text{ (e)}\right|$. Hence, $\gamma^{\vartheta}_{(\alpha,\beta)}(\lambda)(e) \geq \gamma^{\sigma}_{(\alpha,\beta)}(\lambda)(e)$ for all $e \in A$.

Corollary 6. Let (U, W, ϑ) be a generalized approximation space, $\lambda \in \mathcal{F}(U)$ and $0 \gneq \beta \leq \alpha \leq 1$. If (σ, A) is a SER on U such that $\vartheta(e) \subseteq \sigma(e)$. Then, $\rho^{\vartheta}_{(\alpha,\beta)}(\lambda)(e) \geq \rho^{\sigma}_{(\alpha,\beta)}(\lambda)(e)$ for all $e \in A$ regarding A-sets.

Proof. Direct consequence of Definition 22 and Theorem 25.

Theorem 26. Let (U, W, ϑ) be a generalized approximation space, $\lambda, \mu \in \mathcal{F}(U)$ and $0 \gneq \beta \leq \alpha \leq 1$. Then, $\lambda \simeq_F \mu$ implies the following assertions for all $e \in A$ regarding A-sets.

1) $\gamma^{\vartheta}_{(\alpha,\beta)}(\lambda \cap \mu)(e) \geq \gamma^{\vartheta}_{(\alpha,\beta)}(\lambda)(e)$;
2) $\gamma^{\vartheta}_{(\alpha,\beta)}(\lambda \cap \mu)(e) \geq \gamma^{\vartheta}_{(\alpha,\beta)}(\mu)(e)$.

Proof.

1) Let $\lambda, \mu \in \mathcal{F}(U)$ and $0 \gneq \beta \leq \alpha \leq 1$ such that $\lambda \simeq_F \mu$. From Definition 19, $\underline{F}^{\lambda}$ (e) = $\underline{\vartheta}^{\mu}$ (e). Now by Theorem 17, $\underline{\vartheta}^{\lambda \cap \mu}$ (e) = $\underline{\vartheta}^{\lambda}$ (e). Thus, $\underline{\vartheta}^{(\lambda \cap \mu)_\alpha}$ (e) = $\underline{\vartheta}^{(\lambda)_\alpha}$ (e). Therefore, $\left|\underline{\vartheta}^{(\lambda \cap \mu)_\alpha}\text{ (e)}\right| = \left|\underline{\vartheta}^{(\lambda)_\alpha}\text{ (e)}\right|$. On the other hand, $\lambda \cap \mu \leq \lambda$, which implies $\overline{\vartheta}^{(\lambda \cap \mu)}$ (e) $\leq \overline{\vartheta}^{(\lambda)}$ (e) or $\overline{\vartheta}^{(\lambda \cap \mu)_\beta}$ (e) $\leq \overline{\vartheta}^{(\lambda)_\beta}$ (e). Therefore, $\left|\overline{\vartheta}^{(\lambda \cap \mu)_\beta}\text{ (e)}\right| \leq \left|\overline{\vartheta}^{(\lambda)_\beta}\text{ (e)}\right|$. Therefore, by re-setting, we get $\left|\underline{\vartheta}^{(\lambda \cap \mu)_\alpha}\text{ (e)}\right| / \left|\overline{\vartheta}^{(\lambda \cap \mu)_\beta}\text{ (e)}\right| \geq \left|\underline{\vartheta}^{(\lambda)_\alpha}\text{ (e)}\right| / \left|\overline{\vartheta}^{(\lambda)_\beta}\text{ (e)}\right|$. Hence, $\gamma^{\vartheta}_{(\alpha,\beta)}(\lambda \cap \mu)(e) \geq \gamma^{\vartheta}_{(\alpha,\beta)}(\lambda)(e)$ for all $e \in A$.
2) This can be proved in the same manner as (1).

Corollary 7. Let (U, W, ϑ) be a generalized approximation space, $\lambda, \mu \in \mathcal{F}(U)$ and $0 \gneq \beta \leq \alpha \leq 1$. Then, $\lambda \simeq_F \mu$ implies the following assertions for all $e \in A$ regarding A-sets.

1) $\rho^{\vartheta}_{(\alpha,\beta)}(\lambda \cap \mu)(e) \geq \rho^{\vartheta}_{(\alpha,\beta)}(\lambda)(e)$;
2) $\rho^{\vartheta}_{(\alpha,\beta)}(\lambda \cap \mu)(e) \geq \rho^{\vartheta}_{(\alpha,\beta)}(\mu)(e)$.

Proof. Direct consequence of Definition 22 and Theorem 26.

Theorem 27. Let (U, W, ϑ) be a generalized approximation space, $\lambda, \mu \in \mathcal{F}(U)$ and $0 \gneq \beta \leq \alpha \leq 1$. Then, $\lambda \eqsim_F \mu$ implies the following assertions for all $e \in A$ regarding A-sets.

1) $\gamma^{\vartheta}_{(\alpha,\beta)}(\lambda \cup \mu)(e) \geq \gamma^{\vartheta}_{(\alpha,\beta)}(\lambda)(e)$;

2) $\gamma^{\vartheta}_{(\alpha,\beta)}(\lambda \cup \mu)(e) \geq \gamma^{\vartheta}_{(\alpha,\beta)}(\mu)(e)$.

Proof. Same the proof of Theorem 26.

Corollary 8. Let (U, W, ϑ) be a generalized approximation space, $\lambda, \mu \in \mathcal{F}(U)$ and $0 \lneqq \beta \leq \alpha \leq 1$. Then, $\lambda \eqsim_F \mu$ implies the following assertions for all $e \in A$ regarding A-sets.

1) $\rho^{\vartheta}_{(\alpha,\beta)}(\lambda \cup \mu)(e) \geq \rho^{\vartheta}_{(\alpha,\beta)}(\lambda)(e)$;

2) $\rho^{\vartheta}_{(\alpha,\beta)}(\lambda \cup \mu)(e) \geq \rho^{\vartheta}_{(\alpha,\beta)}(\mu)(e)$.

Proof. Direct consequence of Definition 22 and Theorem 27.

Theorem 28. Let (U, W, ϑ) be a generalized approximation space, $\lambda, \mu \in \mathcal{F}(U)$ and $0 \lneqq \beta \leq \alpha \leq 1$. Then, $\lambda \approx_F \mu$ implies that $\gamma^{\vartheta}_{(\alpha,\beta)}(\lambda)(e) = \gamma^{\vartheta}_{(\alpha,\beta)}(\mu)(e)$ for all $e \in A$ regarding A-sets.

Proof. Let $0 \lneqq \beta \leq \alpha \leq 1$ and $\lambda, \mu \in \mathcal{F}(U)$ such that $\lambda \approx_F \mu$. By Definition 19, $\underline{\vartheta}^{\lambda}(e) = \underline{\vartheta}^{\mu}(e)$ and $\overline{\vartheta}^{\lambda}(e) = \overline{\vartheta}^{\mu}(e)$. By Lemma 3, $\underline{\vartheta}^{(\lambda_\alpha)}(e) = \underline{\vartheta}^{(\mu_\alpha)}(e)$ and $\overline{\vartheta}^{(\lambda_\beta)}(e) = \overline{\vartheta}^{(\mu_\beta)}(e)$. That is, $\left|\underline{\vartheta}^{(\lambda_\alpha)}(e)\right| = \left|\underline{\vartheta}^{(\mu_\alpha)}(e)\right|$ and $\left|\overline{\vartheta}^{(\lambda_\beta)}(e)\right| = \left|\overline{\vartheta}^{(\mu_\beta)}(e)\right|$. This yields $\left|\underline{\vartheta}^{(\lambda_\alpha)}(e)\right| / \left|\overline{\vartheta}^{(\lambda_\beta)}(e)\right| = \left|\underline{\vartheta}^{(\mu_\alpha)}(e)\right| / \left|\overline{\vartheta}^{(\mu_\beta)}(e)\right|$. Hence, $\gamma^{\vartheta}_{(\alpha,\beta)}(\lambda)(e) = \gamma^{\vartheta}_{(\alpha,\beta)}(\mu)(e)$ for all $e \in A$.

Corollary 9. Let (U, W, ϑ) be a generalized approximation space, $\lambda, \mu \in \mathcal{F}(U)$ and $0 \lneqq \beta \leq \alpha \leq 1$. Then, $\lambda \approx_F \mu$ implies that $\rho^{\vartheta}_{(\alpha,\beta)}(\lambda)(e) = \rho^{\vartheta}_{(\alpha,\beta)}(\mu)(e)$ for all $e \in A$ regarding A-sets.

Proof. The proof follows immediately from Definition 22 and Theorem 28.

Note: In the context of SER, the notions of F-sets and A-sets are equivalent. Consequently, all the preceding results remain valid when applied to F-sets.

7. Accuracy measures

Definition 23. Let (ϑ, A) be an SBR over U and let λ be an FS in U. The upper and lower approximations $\overline{\vartheta}^{\lambda}$ and $\underline{\vartheta}^{\lambda}$ regarding $u\vartheta(e_i)$ are defined by:

$$\overline{\vartheta}^{\lambda}(e_i)\big(u\vartheta(e_i)\big) = max_{v \in u\vartheta(e_i)}\{\lambda(v)\}\ . \tag{7.1}$$

And

$$\underline{\vartheta}^{\lambda}(e_i)\big(u\vartheta(e_i)\big) = min_{v \in u\vartheta(e_i)}\{\lambda(v)\}. \tag{7.2}$$

These two FSSs, λ^* and λ_*, are generated by these lower and upper approximations in U postulated as:

$$\lambda^*(i)(u) = \overline{\vartheta}^{\lambda}(e_i)\big(u\vartheta(e_i)\big), \tag{7.3}$$

and

$$\lambda_*(i)(u) = \underline{\vartheta}^{\lambda}(e_i)\big(u\vartheta(e_i)\big). \tag{7.4}$$

For any $u \in U$, $\lambda^*(i)(u)$ and $\lambda_*(i)(u)$ can be seen as the degree to which u possibly (resp. definitely) belongs to the FS λ.

Let U be a universal set and ϑ be an SBR. For λ a normal FS in U, assume that the range of the membership function λ $\mathrm{rng}(\lambda) = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$, where $\alpha_i \gneqq \alpha_{i+1} \gneqq 0$, for $i = 1,2,3, \dots, n-1$ and $\alpha_1 = 1$. The mass assignment of λ denoted by m_λ satisfies $m_\lambda(\phi)(e_i) = (1 - \alpha_1)(e_i)$,

$m_\lambda(\lambda_i)(e_i) = (\alpha_i - \alpha_{i+1})(e_i)$ for $i = 1,2,3, \dots, n$ with $\alpha_{n+1} = 0$ by convention, where $\lambda_i = \{x \in U: \lambda(x) \geq \alpha_i\}$ for $1 \leq i \leq n$.

It has been proposed that the mass assignment of a fuzzy concept provides probability-based semantics for the fuzzy concept's membership function. The mass assignment theory has been utilized in several areas, including word computing and decision tree induction.

Now, we define the roughness measure of λ regarding SBRs and regarding A-sets as follows:

$$\hat{\rho}_{\vartheta}^{\lambda}(e_i) = \sum_{i=1}^{n} m_\lambda(\lambda_i)(e_i)\left(1 - \left|\underline{\vartheta}^{\lambda_i}(e_i)\right| / \left|\overline{\vartheta}^{\lambda_i}(e_i)\right|\right). \tag{7.5}$$

The roughness measure is not a number but a vector. Corresponding to each parameter of a SER, we have the corresponding component of the roughness measure vector.

For the sake of illustration, now consider an example as follows:

Example 4. Let $U = \{1,2,3,4,5,6,7,8,9,10,11\}$ and $A = \{e_1, e_2, e_3, e_4, e_5\}$ be a set of parameters. Define a SER $\vartheta: A \to P(U \times U)$ for each $e \in A$. The corresponding soft equivalence class for each of the SERs is obtained as follows:

For $\vartheta(e_1)$, the soft equivalence classes $u\vartheta(e_1)$ are: $\{1,10\}, \{2,4,6,7\}, \{3,5,8,9\}, \{11\}$.
For $\vartheta(e_2)$, the soft equivalence classes $u\vartheta(e_2)$ are: $\{1\}, \{2,11\}, \{4,7\}, \{3,5,8,9\}, \{6,10\}$.
For $\vartheta(e_3)$, the soft equivalence classes $u\vartheta(e_3)$ are: $\{1\}, \{2\}, \{3,4,5,7,8,9,10\}, \{6\}, \{11\}$.
For $\vartheta(e_4)$, the soft equivalence classes $u\vartheta(e_4)$ are: $\{2\}, \{3,4,5,7,8,9,11\}, \{1,6,10\}$.
For $\vartheta(e_5)$, the soft equivalence classes $u\vartheta(e_5)$ are: $\{10\}, \{6\}, \{1,2,3,4,5,7,8,9\}, \{11\}$.

Consider a linguistic value "$small$" whose membership function is defined by Table 7. The approximations of the FS "$small$" are given in Table 8. The mass assignment for "$small$" corresponding to each soft equivalence class is presented in Table 9. The lower and upper approximations corresponding to each soft equivalence class are given in Table 10. The order of the corresponding lower and upper approximations are displayed in Table 11.

Table 7. The membership function of "$small$".

u	1	2	3	4	5	6	7	8	9	10	11
$small(u)$	1	0.8	0.8	0.6	0.4	0.2	0	0	0	0	0

Table 8. The approximations of the FS $small$.

	$small_*$	$small^*$
Corresponding to $i = 1$	$\{0,0,0,0\}$	$\{1,0.8,0.8,0\}$
Corresponding to $i = 2$	$\{1,0,0,0,0\}$	$\{1,0.8,0.6,0.8,0.2\}$
Corresponding to $i = 3$	$\{1,0.8,0,0.2,0\}$	$\{1,0.8,0.8,0.2,0\}$
Corresponding to $i = 4$	$\{0.8,0,0\}$	$\{0.8,0.8,1\}$
Corresponding to $i = 5$	$\{0.2,0,0,0\}$	$\{0.2,0,1,0\}$

Table 9. Mass assignment corresponding to each soft equivalence class.

	$rn_g(small)$	$Small_\alpha$	m_{small} $(Small_\alpha)(e_i)$
For $i = 1$	$\{1, 0.8, 0.6, 0.4\}$	$\begin{pmatrix} \{1\}, \\ \{1,2,3\}, \\ \{1,2,3,4\}, \\ \{1,2,3,4,5\} \end{pmatrix}$	0.2
For $i = 2$	$\{1, 0.8, 0.6, 0.4, 0.2\}$	$\begin{pmatrix} \{1\}, \\ \{1,2,3\}, \\ \{1,2,3,4\}, \\ \{1,2,3,4,5\}, \\ \{1,2,3,4,5,6\} \end{pmatrix}$	0.2
For $i = 3$	$\{1, 0.8, 0.6, 0.4, 0.2\}$	$\begin{pmatrix} \{1\}, \\ \{1,2,3\}, \\ \{1,2,3,4\}, \\ \{1,2,3,4,5\}, \\ \{1,2,3,4,5,6\} \end{pmatrix}$	0.2
For $i = 4$	$\{1, 0.8, 0.6\}$	$\begin{pmatrix} \{1\}, \\ \{1,2,3\}, \\ \{1,2,3,4,\} \end{pmatrix}$	0.2
For $i = 5$	$\{1, 0.8, 0.6, 0.4\}$	$\begin{pmatrix} \{1\}, \\ \{1,2,3\}, \\ \{1,2,3,4\}, \\ \{1,2,3,4,5\} \end{pmatrix}$	0.2

Table 10. Lower and upper approximations corresponding to each soft equivalence class w.r.t. A-set.

	$\underline{\vartheta}^{small_\alpha}(e_i)$	$\overline{\vartheta}^{small_\alpha}(e_i)$
Corresponding to $i = 1$	$\{\phi, \phi, \phi, \phi\}$	$\begin{Bmatrix} \{1,10\}, \\ \{3,5,8,9\}, \\ \{3,5,8,9\}, \\ \phi \end{Bmatrix}$
Corresponding to $i = 2$	$\{\{1\}, \phi, \phi, \phi, \phi\}$	$\begin{Bmatrix} \{1\}, \\ \{3,5,8,9\}, \\ \{4,7\}, \\ \{3,5,8,9\}, \\ \{6,10\} \end{Bmatrix}$
Corresponding to $i = 3$	$\{\{1\}, \{2,11\}, \phi, \{6\}, \phi\}$	$\begin{Bmatrix} \{1\}, \\ \{3,4,5,7,8,9,10\}, \\ \{3,4,5,7,8,9,10\}, \\ \{6\}, \\ \phi \end{Bmatrix}$
Corresponding to $i = 4$	$\{\{2\}, \phi, \phi\}$	$\begin{Bmatrix} \{3,4,5,7,8,9,11\}, \\ \{3,4,5,7,8,9,11\}, \\ \{1,6,10\} \end{Bmatrix}$
Corresponding to $i = 5$	$\{\{6\}, \phi, \phi, \phi\}$	$\begin{Bmatrix} \{6\}, \\ \phi, \\ \{1,2,3,4,5,7,8,9\}, \\ \phi \end{Bmatrix}$

Table 11. Order of corresponding lower and upper approximations regarding A-sets.

	$\lvert\underline{\vartheta}^{small_\alpha}(e_i)\rvert$	$\lvert\overline{\vartheta}^{small_\alpha}(e_i)\rvert$
Corresponding to $i = 1$	{0,0,0,0}	{2,4,4,0}
Corresponding to $i = 2$	{1,0,0,0,0}	{1,4,2,4,2}
Corresponding to $i = 3$	{1,2,0,1,0}	{1,7,7,1,0}
Corresponding to $i = 4$	{1,0,0}	{7,7,3}
Corresponding to $i = 5$	{1,0,0,0}	{1,0,8,0}

Now, the measure of roughness yields

$$\hat{\rho}_{\vartheta}^{small}(e_i) = \sum_{\alpha \in rn_g\,(small)} m_{small}(small_\alpha)(e_i)\left(1 - \left|\underline{\vartheta}^{small_\alpha}(e_i)\right| / \left|\overline{\vartheta}^{small_\alpha}(e_i)\right|\right).$$

For $i = 1$:

$$\hat{\rho}_{\vartheta}^{small}(e_1) = \sum_{\alpha \in rn_g\,(small)} m_{small}(small_\alpha)(e_1)\left(1 - \left|\underline{\vartheta}^{small_\alpha}(e_1)\right| / \left|\overline{\vartheta}^{small_\alpha}(e_1)\right|\right) = 0.8.$$

For $i = 2$:

$$\hat{\rho}_{\vartheta}^{small}(e_2) = \sum_{\alpha \in rn_g\,(small)} m_{small}(small_\alpha)(e_2)\left(1 - \left|\underline{\vartheta}^{small_\alpha}(e_2)\right| / \left|\overline{\vartheta}^{small_\alpha}(e_2)\right|\right) = 0.8.$$

For $i = 3$:

$$\hat{\rho}_{\vartheta}^{small}(e_3) = \sum_{\alpha \in rn_g\,(small)} m_{small}(small_\alpha)(e_3)\left(1 - \left|\underline{\vartheta}^{small_\alpha}(e_3)\right| / \left|\overline{\vartheta}^{small_\alpha}(e_3)\right|\right) = 0.5.$$

For $i = 4$:

$$\hat{\rho}_{\vartheta}^{small}(e_4) = \sum_{\alpha \in rn_g\,(small)} m_{small}(small_\alpha)(e_4)\left(1 - \left|\underline{\vartheta}^{small_\alpha}(e_4)\right| / \left|\overline{\vartheta}^{small_\alpha}(e_4)\right|\right) = 0.5.$$

For $i = 5$:

$$\hat{\rho}_{\vartheta}^{small}(e_5) = \sum_{\alpha \in rn_g\,(small)} m_{small}(small_\alpha)(e_5)\left(1 - \left|\underline{\vartheta}^{small_\alpha}(e_5)\right| / \left|\overline{v}^{small_\alpha}(e_5)\right|\right) = 0.6.$$

Hence, corresponding to each parameter of a SER, the roughness measure vector is [0.8 0.8 0.5 0.5 0.6].

Note: Since the concepts of F-sets and A-sets match in the case of a SER, the same process may be used for the lower and upper approximations about the F-sets.

8. A decision-making scheme

SS theory and its many extensions have been used to address DM issues since Molodtsov introduced it (Maji et al. [5]). Some of its restrictions, including human perception and vision systems, are fundamentally humanistic and subjective. As Feng et al. [61] pointed out, there is not a single, consistent standard for judging decision alternatives.

As a result, every method now in use for making decisions based on SSs and the theory that extends from them will certainly have some benefits and some drawbacks. Every methodology currently in use for DM that is based on SSs and its extensions theory has successfully resolved a variety of decision problems. Roy and Maji [62] provided an FSS theory-based DM process. In Feng et al. [61], the authors created a new modified decision strategy based on FSS theory after carefully examining the shortcomings of the Roy and Maji decision method. Decision-makers still need to select the thresholds early even though the Roy and Maji approach's shortcomings have been addressed by the Feng et al. [61] technique. The results will then depend on the threshold.

In this work, we suggest a novel method for formulating decisions using $SBRs$ and fuzzy soft rough set theory. This method does not require any further information that could be supplied by decision-makers or in any other way; it will only make use of the data information provided by the problem of DM. As a result, it can prevent the impact of personal information on the results of decisions. Because of the influence of the subjective elements by various experts, the results might be more objective and prevent contradictory results for similar decision problems.

The current methods for managing decision-making issues with FSSs primarily concentrate on the object's score o_i as determined by the comparison table and the membership degree's choice value c_i (Roy and Maji [53]) regarding the parameter set for the specified item in the universe U. Selecting the universe U object with the highest choice value c_i, or maximum score, would be the most appropriate course for proceeding in this case.

Since the upper and rougher approximations are the two that are closest to the universe's approximated set, as a consequence, using the fuzzy soft upper and lower approximations of FS λ, we derive the two closest values, $\underline{\vartheta}^{\lambda}(e_i)(x_i)$ and $\overline{\vartheta}^{\lambda}(e_i)(x_i)$ regarding A-sets to the decision alternative $x_i \in U$ of the universe U. Thus, we redefine, regarding A-sets, the choice value γ_i for the decision alternative x_i on the universe U as follows:

$$\gamma_i = \sum_{i=1}^{n} \underline{\vartheta}^{\lambda}(e_i)(x_i) + \sum_{i=1}^{n} \overline{\vartheta}^{\lambda}(e_i)(x_i), x_i \in U. \tag{8.1}$$

In the end, the item $x_i \in U$ in the universe U with the largest choice value γ_i is considered the best choice for the assumed problem of DM, and the object $x_i \in U$ in the universe U with the least choice value γ_i is measured as the worst alternative. In general, if more than one item $x_i \in U$ has the identical maximum (minimum) choice value γ_i, select the random choice as the best (worst) alternative for the specified DM problem. In what follows, we propose two decision-making algorithms within the recommended framework to identify optimal choices. Algorithm 1 employs an A-sets method, while Algorithm 2 utilizes an F-sets approach.

Algorithm 1: Decision-making using A-sets.

Input: Two non-empty universes and a set of attributes
Output: Optimal choice

1) Determine the approximate values of the upper FSS $\overline{\boldsymbol{\vartheta}}^{\boldsymbol{\lambda}}$ and lower FSS $\underline{\boldsymbol{\vartheta}}^{\boldsymbol{\lambda}}$ for a given FS λ in relation to the A-sets.
2) For each $\boldsymbol{i}$ with regard to the A-sets, calculate the sum of the lower approximation $\sum_{\boldsymbol{i=1}}^{\boldsymbol{n}} \underline{\boldsymbol{\vartheta}}^{\boldsymbol{\lambda}}(\boldsymbol{e_i})(\boldsymbol{x_i})$ and the upper approximation $\sum_{\boldsymbol{i=1}}^{\boldsymbol{n}} \overline{\boldsymbol{\vartheta}}^{\boldsymbol{\lambda}}(\boldsymbol{e_i})(\boldsymbol{x_i})$.
3) Determine the choice value with regard to the A-sets using the following formula:
$$\boldsymbol{\gamma_i} = \sum_{\boldsymbol{i=1}}^{\boldsymbol{n}} \underline{\boldsymbol{\vartheta}}^{\boldsymbol{\lambda}}(\boldsymbol{e_i})(\boldsymbol{x_i}) + \sum_{\boldsymbol{i=1}}^{\boldsymbol{n}} \overline{\boldsymbol{\vartheta}}^{\boldsymbol{\lambda}}(\boldsymbol{e_i})(\boldsymbol{x_i}), \boldsymbol{x_i} \in \boldsymbol{U}$$
4) $\boldsymbol{x_k} \in \boldsymbol{U}$ is the optimal choice if $\boldsymbol{\gamma_k} = \boldsymbol{max_i}\, \boldsymbol{\gamma_i},\ \boldsymbol{i} = \boldsymbol{1}, \boldsymbol{2}, \dots, |\boldsymbol{U}|$.
5) $\boldsymbol{x_k} \in \boldsymbol{U}$ is the worst choice if $\boldsymbol{\gamma_k} = \boldsymbol{min_i}\, \boldsymbol{\gamma_i},\ \boldsymbol{i} = \boldsymbol{1}, \boldsymbol{2}, \dots, |\boldsymbol{U}|$.
6) Any one of $\boldsymbol{x_k}$ can be selected if $\boldsymbol{k}$ has multiple values.

Algorithm 2: Decision-making using F-sets.

Input: Two non-empty universes and a set of attributes
Output: Optimal choice

1) Determine the approximate values of the upper FSS ${}^{\boldsymbol{\delta}}\overline{\boldsymbol{\vartheta}}$ and lower FSS ${}^{\boldsymbol{\delta}}\underline{\boldsymbol{\vartheta}}$ for a given FS $\boldsymbol{\delta}$ in relation to the F-sets.
2) For each $\boldsymbol{i}$ with regard to the F-sets, calculate the sum of the lower approximation $\sum_{\boldsymbol{i=1}}^{\boldsymbol{n}} {}^{\boldsymbol{\delta}}\underline{\boldsymbol{\vartheta}}(\boldsymbol{e_i})(\boldsymbol{x_i})$ and the upper approximation $\sum_{\boldsymbol{i=1}}^{\boldsymbol{n}} {}^{\boldsymbol{\delta}}\overline{\boldsymbol{\vartheta}}(\boldsymbol{e_i})(\boldsymbol{x_i})$.
3) Determine the choice value regarding the F-sets according to the formula given as:
$$\boldsymbol{\gamma'_i} = \sum_{\boldsymbol{i=1}}^{\boldsymbol{n}} {}^{\boldsymbol{\delta}}\underline{\boldsymbol{\vartheta}}(\boldsymbol{e_i})(\boldsymbol{x_i}) + \sum_{\boldsymbol{i=1}}^{\boldsymbol{n}} {}^{\boldsymbol{\delta}}\overline{\boldsymbol{\vartheta}}(\boldsymbol{e_i})(\boldsymbol{x_i}), \boldsymbol{x_i} \in \boldsymbol{U}$$
4) $\boldsymbol{x_k} \in \boldsymbol{U}$ is the optimal choice if $\boldsymbol{\gamma'_k} = \boldsymbol{max_i}\, \boldsymbol{\gamma'_i}, \boldsymbol{i} = \boldsymbol{1}, \boldsymbol{2}, \dots, |\boldsymbol{U}|$
5) $\boldsymbol{x_k} \in \boldsymbol{U}$ is the poorest choice if $\boldsymbol{\gamma'_k} = \boldsymbol{min_i}\, \boldsymbol{\gamma'_i}, \boldsymbol{i} = \boldsymbol{1}, \boldsymbol{2}, \dots, |\boldsymbol{U}|$.
6) Any one of $\boldsymbol{x_k}$ can be selected if $\boldsymbol{k}$ has multiple values.

Figure 1 displays a graphic portrayal of Algorithms 1 and 2.

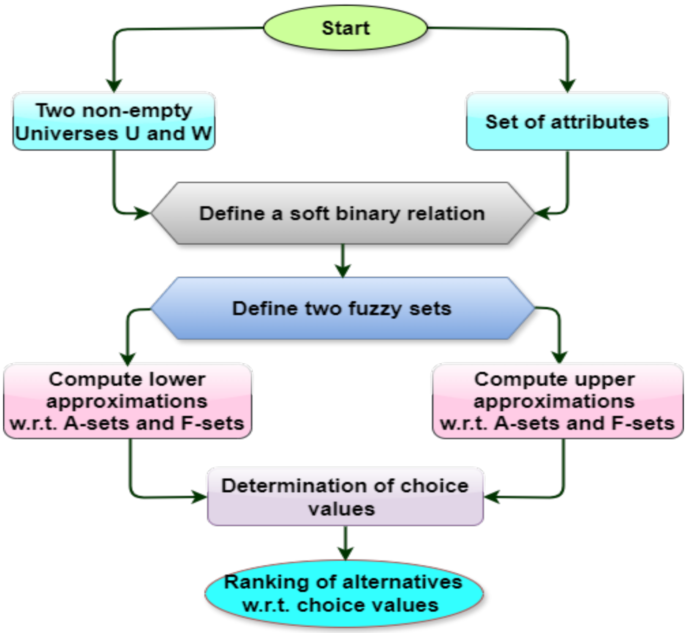


Figure 1. Flowchart of the proposed algorithms.

8.1. A practical illustration

In this subsection, we use the example of choosing a bike to demonstrate the steps of the DM process.

Example 5. Reviewing the SBR rovided in Example 1, where someone wishes to choose a bike out of six bike models and four bike colors.

Define $\lambda: W \to [0,1]$, which is $Mr. X's$ preferred color, such that $\lambda(c_1) = 0.3, \lambda(c_2) = 0.1, \lambda(c_3) = 0, \lambda(c_4) = 0.5$ and define $\delta: U \to [0,1]$, which indicates the color preference given by $Mr. X$ such that

$$\delta(d_1) = 1, \delta(d_2) = 0.7, \delta(d_3) = 0.5,$$

$$\delta(d_4) = 0.1, \delta(d_5) = 0, \delta(d_6) = 0.4.$$

After using the previously mentioned algorithm, examine the subsequent table.

Table 12. The decision algorithm's outcomes in relation to the A-sets.

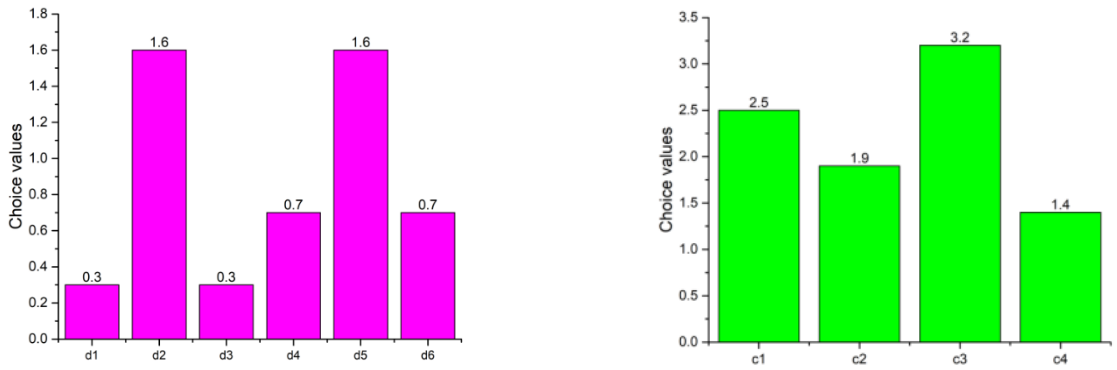
	$\underline{\vartheta}^{\lambda}(e_1)$	$\underline{\vartheta}^{\lambda}(e_2)$	$\underline{\vartheta}^{\lambda}(e_3)$	$\overline{\vartheta}^{\lambda}(e_1)$	$\overline{\vartheta}^{\lambda}(e_2)$	$\overline{\vartheta}^{\lambda}(e_3)$	Choice value γ_i
d_1	0	0	0	0.3	0	0	0.3
d_2	0.1	0	0.5	0.5	0	0.5	1.6
d_3	0	0	0	0	0	0.3	0.3
d_4	0	0.3	0	0.1	0.3	0	0.7
d_5	0	0.3	0	0.5	0.3	0.5	1.6
d_6	0.3	0	0	0.3	0.1	0	0.7

Table 13. The decision algorithm's outcomes in relation to the F-sets.

	${}^{\delta}\underline{\vartheta}(e_1)$	${}^{\delta}\underline{\vartheta}(e_2)$	${}^{\delta}\underline{\vartheta}(e_3)$	${}^{\delta}\overline{\vartheta}(e_1)$	${}^{\delta}\overline{\vartheta}(e_2)$	${}^{\delta}\overline{\vartheta}(e_3)$	Choice value γ_i'
c_1	0.4	0	0.5	1	0.1	0.5	2.5
c_2	0.1	0.4	0	1	0.4	0	1.9
c_3	0	0.7	0	1	1	0.5	3.2
c_4	0	0	0	0.7	0	0.7	1.4

Here, the chosen value $\gamma_i = \sum_{i=1}^{3} \underline{\vartheta}^{\lambda}(e_i)(x_i) + \sum_{i=1}^{3} \overline{\vartheta}^{\lambda}(e_i)(x_i)$ is calculated regarding A-sets and the choice value $\gamma_i' = \sum_{i=1}^{3} {}^{\delta}\underline{\vartheta}(e_i)(x_i) + \sum_{i=1}^{3} {}^{\delta}\overline{\vartheta}(e_i)(x_i)$ is calculated regarding F-sets.

Clearly, the maximum chosen value is $\gamma_k = 1.6 = \gamma_2 = \gamma_5$, scored by the models d_2 and d_5, and the conclusion is in support of selecting the model d_2 or d_5. Moreover, the models d_1 and d_3 are disregarded. Hence, $Mr.X$ will choose the bike of model d_2 or d_5 for his personal use, and he will not select the bikes of models d_1 and d_3 regarding A-sets. Similarly, the highest possible decision value is $\gamma_k' = 3.2 = \gamma_3'$, scored by the bike of color c_3, and the conclusion is in favor of selecting the bike of color c_3. Moreover, the bike of colour c_4 is completely disregarded. Hence, $Mr.X$ will choose the bike of color c_3 for his personal use, and he will not select the bike of color c_4 regarding F-sets. Furthermore, the graphical depiction of the ranking outcomes of the proposed approach is exhibited in Figure 2.



(a). Ranking w.r.t. A-sets.

(b). Ranking w.r.t. F-sets.

Figure 2. Ranking results.

9. Comparative analysis and discussion

The literature presents a diverse range of methodologies for addressing DM situations, each with its strengths and restrictions. The efficacy of any given method is largely dependent on the specific features of the problem under consideration. In this segment, we conduct an in-depth theoretical comparative analysis of the planned method against several widely adopted DM techniques within the given context of FSs, RSs, and SSs. Additionally, we discuss the advantages of the proposed approach to existing methods.

Handling ambiguous information is much improved by utilizing FS in combination with SS and RS. Molodtsov [3] introduced SS theory, and Maji et al. [5, 6] used it for decision analysis. Chen et al. [7] implemented parameterization reduction of SS and improved the SS-based decision-making scheme presented in [6]. The authors of [54] provided a uni-int DM technique by utilizing enhanced SS operations. The important thing to note about this approach is its fundamental limitation. For its methodology, this technique produces an empty set of optimal alternatives. Furthermore, every prior DM system only paid attention to crisp SS.

Depending on the DM situation, the findings of SSs entail the evaluation of every decision attribute. Moreover, there is typically no standard criteria for assessing decision attributes ([61]). Consequently, earlier DM techniques had shortcomings. DM was studied by Roy and Maji [62] within the context of FSSs. The strategy used by Roy and Maji [62] had certain drawbacks, which Feng et al. [61] addressed.

There are noteworthy differences between our method and previous research. The choice value of the membership grade for the attribute set for the provided alternatives in the universe and the score of alternatives from the comparison table are the primary foci of the current investigations of decision-making under the SS technique introduced by Maji et al. [5,6], Feng et al. [61], Roy and Maji [62], and Gogoi et al. [63]. The best option is then determined by selecting the universe's alternative with the highest choice value or score.

The present article introduces DM methods employing SBRs, which are very suitable for addressing uncertainty due to their parameterized collection of binary relations. These schemes rely on the FRS methodology. Expert feedback is incorporated into the planned investigation, and further information is not required. Additionally, we may use different parameters in SBRs according to the type of problem we are looking at. For this reason, our suggested method works better for solving ambiguous problems.

9.1. Numerical experimentation

In this subsection, we compare our proposed approach to existing schemes described in [30, 39, 62] within the framework of FSSs. The resultant ranking outcomes of this analysis are presented in Tables 14 and 15. Furthermore, as observed in Tables 20 and 21, the optimal alternative derived from the final ranking reveals minimal fluctuation concerning A-sets and F-sets. This difference is a common phenomenon in decision analysis, attributed to the dynamic nature of the DM setting.

Table 14. Ranking outcomes using various methods under the A-sets.

Methods	Choice values						Ranking result
	γ_1	γ_2	γ_3	γ_4	γ_5	γ_6	
Mehmood et al. [39]	0.8	1.3	1.3	1.3	0.9	1.3	$d_2 \approx d_3 \approx d_4 \approx d_6 \succcurlyeq d_5 \succcurlyeq d_1$
Roy and Maji [62]	-11	8	-12	1	13	1	$d_5 \succcurlyeq d_2 \succcurlyeq d_4 \approx d_6 \succcurlyeq d_1 \succcurlyeq d_3$
Zhan and Zhu [30]	1	4	1	3	4	3	$d_5 \approx d_2 \succcurlyeq d_4 \approx d_6 \succcurlyeq d_1 \succcurlyeq d_3$
Proposed scheme	0.3	1.6	0.3	0.7	1.6	0.7	$d_5 \approx d_2 \succcurlyeq d_4 \approx d_6 \succcurlyeq d_1 \succcurlyeq d_3$

Table 15. Ranking results using different methods under the F-sets.

Methods	Choice values				Ranking order
	γ_1'	γ_2'	γ_3'	γ_4'	
Mehmood et al. [39]	1	1	1.1	0.7	$c_3 \succcurlyeq c_1 \approx c_2 \succcurlyeq c_4$
Roy and Maji [62]	5	2	1	-8	$c_1 \succcurlyeq c_2 \succcurlyeq c_3 \succcurlyeq c_4$
Zhan and Zhu [30]	5	4	4	2	$c_1 \succcurlyeq c_2 \succcurlyeq c_3 \approx c_4$
Proposed scheme	2.5	1.9	3.2	1.4	$c_3 \succcurlyeq c_1 \succcurlyeq c_2 \succcurlyeq c_4$

9.2. Advantages of the proposed work

In summary, there are several advantages of our suggested method over current methods, which may be summed up as follows:

(1) One of the key advantages of the FRS variant based on SBRs is its capacity to represent ambiguity and uncertainty. FSs may handle a wider range of data types and more properly represent the uncertainty that commonly arises in real-world applications by merging with RSs and SBRs.

(2) Numerous academics investigated several DM methods under FSs and SSs, such as those introduced by Maji et al. [5, 6], Ali et al. [44], Çağman and Enginoğlu [54], Feng et al. [61], Roy and Maji [62], and Gogoi et al. [63]. Nevertheless, these systems' roughness was not investigated.

(3) A few studies on the roughness of FSs and SSs have been conducted by Molodtsov [3], Maji et al. [5, 6], Feng et al. [61], and Jiang et al. [64]. However, these methods cannot function effectively within the framework of dual universes.

10. Conclusion and future work

FS, RS, and SS theories serve as efficient mathematical frameworks for addressing uncertainty. By integrating these theories, various hybrid models have been established to manage the intrinsic uncertainty and vagueness involved in real-world dilemmas. This study introduces an FRS variant using an SBR over two universes. To this end, we described the roughness of an FS through an SBR w.r.t. the A-sets and F-sets. In this way, we attained two FSSs w.r.t. the A-sets and F-sets. Several substantial features of the devised approach have been systematically scrutinized with several concrete illustrations. Also, two types of fuzzy topologies are constructed via SRRs. In addition, numerous similarity relations linked with SBRs are also investigated. Based on the idea of FS mass assignment via SBRs, a novel accuracy and roughness measure was given in this study. Likewise, we offered two DM algorithms regarding the A-sets and F-sets. To illustrate these algorithms, we discussed a DM technique within the framework of the designed approach to resolve DM problems by evaluating the drawbacks and benefits of previous research. Finally, an applied example was adopted to confirm the soundness of the decision processes.

Admittedly, the proposed approach has certain limitations due to its current modelling capabilities, which are insufficient to effectively address the bipolarity inherent in inconsistent data. To overcome these limitations, future work will focus on the following domains:

- Extending the proposed framework to more general mathematical structures, including bipolar FSs, picture FSs, spherical FSs, t-spherical FSs, and bipolar SSs.
- Additionally, the framework will be utilized to explore the multi-granulation roughness of an FS based on SBRs across two universes.

- Further, the axiomatization of the proposed methodology represents an intriguing area for future research.
- We will also delve deeper into additional facets of the proposed framework under the context of a covering-based RS model.
- Another avenue involves investigating the potential integration of this framework with other methodologies to enhance the accuracy of the outcomes. Ultimately, these advanced techniques will be applied to real-world problems involving large-scale data sets, showcasing their practical applicability and effectiveness.

Author contributions

All authors of this article have been contributed equally. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

We declare that we have no conflict of interest in this paper.

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