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Cardinality rough neighborhoods via ideals with medical applications

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Abstract

In practical life, researchers aim to appropriately frame societal problems and challenges to address and find effective solutions. One efficient method for managing complex real-world data is rough set theory. Utilizing rough approximation operators, it identifies both confirmed and possible data obtainable through subsets. Earlier studies have introduced several rough approximation models inspired by neighborhood systems, which aim to enhance accuracy and satisfy the axioms of traditional approximation spaces as initially proposed by Pawlak. In this work, we put forward novel paradigms of rough sets depending on the cardinality rough neighborhoods and Ideals. These models are a suitable approach to cope with a wide range of examples including issues related to cardinal numbers, which are frequently encountered in contexts such as social media engagement, visitor counts at exhibitions, and the evaluation of applicants based on the number of their qualities. We amply investigate the master features of these paradigms and elucidate the interrelations between them as well as their connection with previous ones. Then, we tackle these paradigms from a topological view as an alternative instrument for describing the boundary regions and calculating the accuracy of data. Moreover, we examine our models' efficiency in dealing with dengue disease for some patients and conclude that the proposed rough-set paradigms ameliorate the properties of the previous approximation spaces. Ultimately, we demonstrate their pros in terms of expanding the confirmed knowledge obtained from subsets of data and keeping the main characteristics

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of original paradigms by Pawlak that were violated by forgoing models, as well as list the deficiencies of the present paradigms.

Keywords $\mathbb{E}_\sigma$-neighborhood · Rough set · Ideal · Lower and upper approximations · Accuracy criteria.

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1 Introduction

1.1 Literature review

Applications of rough sets theory to knowledge discovery involve collecting empirical data and building classification models from the data; see, Pawlak (2000), Pawlak (1991). In this theory, each subset is associated with two crisp sets (named lower and upper approximations) derived from an equivalence relation. To broaden the applications of rough set theory, many researchers have replaced the equivalence relation with various other types of relations; see, Abo-Tabl (2011), Dai et al. (2018), Qin et al. (2008), Slowinski and Vanderpooten (2000), Zhang et al. (2009). This led to replacing equivalence classes with different forms of neighborhoods, which represent blocks or granular computing to describe information systems. Some of these neighborhoods are right and left neighborhoods (Yao 1998, 1996), union and intersection neighborhoods (Allam et al. 2006, 2005)[17], minimal and maximal neighborhoods (Al-shami 2023; Dai and Xu 2012), equal neighborhoods (Atef et al. 2020; Mareay 2016), containment and subset neighborhoods (Al-shami 2021a; Al-shami and Ciucci 2022), cardinality neighborhoods (Al-shami et al. 2024a, c), overlapping containment neighborhoods (Al-shami and Mhemdi 2024), etcetera. Even though some of the new granular computing inspired by neighborhoods violates some properties of the standard model of Pawlak, the researchers proved their beneficial to cope with real scenarios in medicine, economics, and social issues and assist decision-makers in making accurate decisions. That is, they offer an extended framework free of restrictive conditions concerning the type of binary relations. It is worth noting that rough sets theory has proven its effectiveness as important tool for describing information content through a variety of frameworks and applications in multitudinous domains (see Abdelaziz et al. 2022; Akama et al. 2018; Hosny et al. 2022; Kryszkiewicz 1998; Mareay 2024).

The interconnection of topological and rough set theory was first explored by Wiweger (1989), who examined the topological aspects of rough sets. This led to a fusion of rough set and topological theories, becoming a central focus of numerous studies (Abo-Tabl 2013; Al-shami 2022, 2021b; Lashin et al. 2005; Salama 2010; Wu and Liu 2020)[57]. This interaction also involved generalizations of topology, such as supra topology (Al-shami and Alshammari 2023), infra topology (Al-shami and Mhemdi 2023), minimal structures (El-Sharkasy 2021)[7], nano-topology (Kaur et al. 2024), and bitopology (Salama 2020). For a comprehensive overview of the contributions investigating the interrelations between rough set theory and topology, we refer the readers to Singh and Tiwari (2020), Zhang et al. (2016). The symmetry between interior and closure topological operators with lower and upper approximations, respectively, allows us to use abstract tools to describe knowledge obtained from information systems and supply us with practical meanings for abstract concepts.

Ideals in a topological space, defined as a nonempty collection $\mathcal{I}$ of subsets of a universe that is closed under finite union and subsets, were first considered by Kuratowski (1966). Kandil et al. (2013) applied the concept of ideals with $\mathcal{D}_{\langle r\rangle}$-neighborhoods to generalize Pawlak's approximations, demonstrating that their results reduce the boundary region compared to the methods of Allam et al. (2006), and Yao (1998). The significance of this methodology of studying rough set theory is to enlarge the lower approximations of subsets, which refer to the confirmed knowledge extracted from the given data. In precise words, it makes its counterparts of generalized rough set models a special case induced when the ideal consists of only the empty set. For this reason, some interesting papers studied rough set theory described by ideals, such as (Al-shami et al. 2021; Al-shami and Hosny 2024; Güler et al. 2022; Hosny 2020; Hosny et al. 2022; Mustafa et al. 2023). Generating several generalized approximation spaces utilizing ideals with some maximal neighborhoods was the purpose of the articles of Al-shami and Hosny (2022); Hosny and Al-shami (2022)[32]. Al-shami et al. (2024b) proved the independence of some rough models generated by neighborhoods and ideals in terms of the size of approximation operators and the value of accuracy measures. Recently, Al-shami and Hosny have presented a novel technique (Al-shami et al. 2024c), free from an equivalence relation requirement, for addressing situations that focus on the cardinality number of $\mathcal{D}_\sigma$-neighborhoods, such as those encountered in social media or in categorizing applicants based on the number of their qualities.

1.2 Gap of research

It has been defined several sorts of neighborhood systems with different purposes such as rescinding the condition of an equivalence relation, increasing the accuracy measures, preserving the properties of Pawlak's lower and upper approximations, ect. However, these types of neighborhood systems do not pay attention to the cardinality numbers of neighborhoods, which is an important factors for some practical situations and an alternative tool to describe the relations between neighborhoods when the other types fail in analysis data of information systems. Examples include issues related to cardinal numbers, which are frequently encountered in contexts such as social media engagement, visitor counts at exhibitions, and the evaluation of applicants based on the number of their qualities. To illustrate this subject, consider the relation $\mathfrak{L} = \{(\mathsf{p}_1, \mathsf{p}_1), (\mathsf{p}_1, \mathsf{p}_4), (\mathsf{p}_3, \mathsf{p}_1),(\mathsf{p}_3, \mathsf{p}_5), (\mathsf{p}_3, \mathsf{p}_8), (\mathsf{p}_4, \mathsf{p}_1),(\mathsf{p}_4, \mathsf{p}_3), (\mathsf{p}_4, \mathsf{p}_8), (\mathsf{p}_5, \mathsf{p}_5), (\mathsf{p}_6, \mathsf{p}_6), (\mathsf{p}_7, \mathsf{p}_2), (\mathsf{p}_7, \mathsf{p}_6),(\mathsf{p}_8, \mathsf{p}_2)\}$ on the set $\mathbb{X} = \{\mathsf{p}_j : j\in \{1, 2, 3, 4, 5, 6, 7, 8\}\}$. Since $\mathcal{D}_r(\mathsf{p}) = \{\mathsf{e} \in \mathbb{X} : \mathsf{p}\ \mathfrak{L}\ \mathsf{e}\}$, it follows that $\mathcal{D}_r(\mathsf{p}_1) = \{\mathsf{p}_1, \mathsf{p}_4\}$, $\mathcal{D}_r(\mathsf{p}_2) = \emptyset$, $\mathcal{D}_r(\mathsf{p}_3) = \{\mathsf{p}_1, \mathsf{p}_5, \mathsf{p}_8\}$, $\mathcal{D}_r(\mathsf{p}_4) = \{\mathsf{p}_1, \mathsf{p}_3, \mathsf{p}_8\}$, $\mathcal{D}_r(\mathsf{p}_5) = \{\mathsf{p}_5\}$, $\mathcal{D}_r(\mathsf{p}_6) = \{\mathsf{p}_6\}$, $\mathcal{D}_r(\mathsf{p}_7) = \{\mathsf{p}_2, \mathsf{p}_6\}$, and $\mathcal{D}_r(\mathsf{p}_8) = \{\mathsf{p}_2\}$. Now, we have the following remarks which represent drawbacks of the previous systems of neighborhoods:

(i) The rough set models initiated by granules of $\mathcal{D}_r$-neighborhoods (Yao 1996) and ξ_r-neighborhoods [17] violate the property states that the lower approximation of a set is contained in it and a set is contained in its upper approximation. In addition, we should update the formula of accuracy measures in these models to avoid undefined cases obtained when the upper approximation is empty, or illogical cases obtained when the upper approximation of a set is a proper subset of its lower approximation.

(ii) The granules of $\mathbb{C}_r$-neighborhoods (Al-shami 2021a) and $\mathbb{S}_r$-neighborhoods (Al-shami and Ciucci 2022) produce the trivial case of most elements of $\mathbb{X}$; that is, $\mathbb{C}_r(\mathsf{p}_j) = \{\mathsf{p}_2, \mathsf{p}_j\}$ for each $j = 1, 4, 5, 6, 8$ and $\mathbb{S}_r(\mathsf{p}_j) = \{\mathsf{p}_j\}$ for each $j = 1, 3, 4, 7$, which decreases our reliability of the made decision depending on these systems of neighborhoods.

(iii) The granules of $\sharp_r$-neighborhoods (Atef et al. 2020; Mareay 2016) are a singleton set for each element; that is, $\sharp_r(\mathsf{p}_j) = \{\mathsf{p}_j\}$ for each j, which impedes extraction knowledge from the information systems under consideration and return us to the crisp case.

On the other hand, it can be noted that $\mathbb{E}_r(\mathsf{p}_1) = \mathbb{E}_r(\mathsf{p}_7) = \{\mathsf{p}_1, \mathsf{p}_7\}$, $\mathbb{E}_r(\mathsf{p}_2) = \{\mathsf{p}_2\}$, $\mathbb{E}_r(\mathsf{p}_3) = \mathbb{E}_r(\mathsf{p}_4) = \{\mathsf{p}_3, \mathsf{p}_4\}$, and $\mathbb{E}_r(\mathsf{p}_5) = \mathbb{E}_r(\mathsf{p}_6) = \mathbb{E}_r(\mathsf{p}_8) = \{\mathsf{p}_5, \mathsf{p}_6, \mathsf{p}_8\}$ provide a non trivial description allowing us to see the variety among the elements of sample under study $\mathbb{X}$. Moreover, suppose the comparison between some objects (i.e., candidates, students, applicants) is conducted according to the number of skills they have. In that case, the cardinality neighborhoods (Al-shami et al. 2024c) will be the best instrument to exemplify this situation logically.

1.3 Manuscript's design

We layout this paper in the following way. In the next section, we invoke the previous kind of rough neighborhoods and the main concepts related to them aiming to clarify why we need this study. Then, we divide Sect. 3 into two subsections including two fresh approximation spaces inspired by cardinality neighborhoods and ideals. In Sect. 3.1, we introduce a novel type of approximation space and scrutinize its main properties. To avoid failures of the preceding model regarding illogical characteristics and undefined cases, we update the previous model and reveal the advantages of the recent one in Sect. 3.2. Investigation of the proposed rough set models from a topological view and looking at their relationships with previous models and their counterparts introduced in Al-shami et al. (2024c) is the goal of Sect. 4. In Sect. 5, we prove the efficiency of the given models in dealing with a medical situation concerning dengue disease and point out how our technique helps improve decision-making and how we utilize a topological approach inspired by this technique to identify the most significant attributes or symptoms for making decisions. In the end, we discuss the advantages and disadvantages of the present models and epitomize the main contributions of this work with a plan for future work in Sects. 6 and 7, respectively.

2 Preliminaries

We dedicated this section to recalling the main definitions and results, while also elucidating the benefits of hybridizing ideals with cardinal rough neighborhoods to maximize accuracy.

2.1 Traditional approximation space

Definition 1 (Pawlak 1991, 1982) Let $\mathbb{X}$ be a nonempty finite set, known as a universe. A subclass $\mathfrak{L}$ of $\mathbb{X} \times \mathbb{X}$ is called a binary relation $\mathfrak{L}$ on $\mathbb{X}$. We write $\mathsf{a}\mathfrak{L}\mathsf{x}$ to refer to (a, x) is an element of $\mathfrak{L}$. We name a relation $\mathfrak{L}$ on $\mathbb{X}$ an equivalence if it is reflexive (i.e $\mathsf{a}\mathfrak{L}\mathsf{a}$ for any $\mathsf{a} \in \mathbb{X}$), symmetric (i.e. $\mathsf{a}\mathfrak{L}\mathsf{x} \Longleftrightarrow \mathsf{x}\mathfrak{L}\mathsf{a}$), and transitive (i.e $\mathsf{a}\mathfrak{L}\mathsf{y}$ when $\mathsf{a}\mathfrak{L}\mathsf{x}$ and $\mathsf{x}\mathfrak{L}\mathsf{y}$. Moreover, a relation satisfies $\mathsf{a}\mathfrak{L}\mathsf{x}$ or $\mathsf{x}\mathfrak{L}\mathsf{a}$ for all $\mathsf{a}, \mathsf{x} \in \mathbb{X}$ is named a comparable relation.

Definition 2 (Pawlak 1991, 1982) If $\mathfrak{L}$ is an equivalence relation on $\mathbb{X}$ and $\mathbb{X}/\mathfrak{L}$ denotes the set of all equivalence classes generated by $\mathfrak{L}$. Then, the lower and upper approximations of $W \subseteq \mathbb{X}$ are respectively given by:

$$\underline{\mathfrak{L}}(W) = \cup\{S \in \mathbb{X}/\mathfrak{L} \mid S \subseteq W\}.$$

$$\overline{\mathfrak{L}}(W) = \cup\{S \in \mathbb{X}/\mathfrak{L} \mid S \cap W \neq \emptyset\},$$

The term of traditional approximation space is given for the pair $(\mathbb{X}, \mathfrak{L})$ where $\mathfrak{L}$ is an equivalence. The relation between the lower and upper approximations is the criteria for characterizing a subset as rough or exact. That is, a subset W of $(\mathbb{X}, \mathfrak{L})$ is named exact (or definable) providing that the equality between $\overline{\mathfrak{L}}(W)$ and $\underline{\mathfrak{L}}(W)$ exists. Otherwise, a subset is named rough.

The main characteristics and properties of the traditional approximation space are demonstrated in the following proposition.

Proposition 1 *(Pawlak 1991, 1982) Let* $(\mathbb{X}, \mathfrak{L})$ *represent a traditional approximation space and let* $S, W \subseteq \mathbb{X}$*. Then, the subsequent results hold true:*

$(L1)$ $\underline{\mathfrak{L}}(S) \subseteq S$ $\quad$ $(U1)$ $S \subseteq \overline{\mathfrak{L}}(S)$

$(L2)$ $\underline{\mathfrak{L}}(\emptyset) = \emptyset$ $\quad$ $(U2)$ $\overline{\mathfrak{L}}(\emptyset) = \emptyset$

$(L3)$ $\underline{\mathfrak{L}}(\mathbb{X}) = \mathbb{X}$ $\quad$ $(U3)$ $\overline{\mathfrak{L}}(\mathbb{X}) = \mathbb{X}$

$(L4)$ $If\, S \subseteq W,\ then\ \underline{\mathfrak{L}}(S) \subseteq \underline{\mathfrak{L}}(W)$ $\quad$ $(U4)$ $If\, S \subseteq W,\ then\ \overline{\mathfrak{L}}(S) \subseteq \overline{\mathfrak{L}}(W)$

$(L5)$ $\underline{\mathfrak{L}}(S \cap W) = \underline{\mathfrak{L}}(S) \cap \underline{\mathfrak{L}}(W)$ $\quad$ $(U5)$ $\overline{\mathfrak{L}}(S \cap W) \subseteq \overline{\mathfrak{L}}(S) \cap \overline{\mathfrak{L}}(W)$

$(L6)$ $\underline{\mathfrak{L}}(S) \cup \underline{\mathfrak{L}}(W) \subseteq \underline{\mathfrak{L}}(S \cup W)$ $\quad$ $(U6)$ $\overline{\mathfrak{L}}(S \cup W) = \overline{\mathfrak{L}}(S) \cup \overline{\mathfrak{L}}(W)$

$(L7)$ $\underline{\mathfrak{L}}(S^c) = (\overline{\mathfrak{L}}(S))^c$ $\quad$ $(U7)$ $\overline{\mathfrak{L}}(S^c) = (\underline{\mathfrak{L}}(S))^c$

$(L8)$ $\underline{\mathfrak{L}}(\underline{\mathfrak{L}}(S)) = \underline{\mathfrak{L}}(S)$ $\quad$ $(U8)$ $\overline{\mathfrak{L}}(\overline{\mathfrak{L}}(S)) = \overline{\mathfrak{L}}(S)$

$(L9)$ $\underline{\mathfrak{L}}((\underline{\mathfrak{L}}(S))^c) = (\overline{\mathfrak{L}}(S))^c$ $\quad$ $(U9)$ $\overline{\mathfrak{L}}((\overline{\mathfrak{L}}(S))^c) = (\overline{\mathfrak{L}}(S))^c$

$(L10)$ $\underline{\mathfrak{L}}(W) = W, \forall W \in \mathbb{X}/\mathfrak{L}$ $\quad$ $(U10)$ $\overline{\mathfrak{L}}(W) = W, \forall W \in \mathbb{X}/\mathfrak{L}$

The traditional approximation space (Pawlak 1991, 1982) has been generalized using various methodologies, with research focusing on the extent to which the validity of the properties outlined in the aforementioned proposition is preserved within these methodologies. Unfortunately, some properties have been found to be completely or partially lost. It is worth noting that retaining as many of these properties as possible is considered a desirable attribute for the proposed methodologies.

The next two criteria were introduced to numerically describe a rough set.

Definition 3 (Pawlak 1991, 1982) Let $(\mathbb{X}, \mathfrak{L})$ represent a traditional approximation space and let $W \subseteq \mathbb{X}$. The criteria of **A**-accuracy and **R**-roughness of W are respectively computed by:

$$\mathbf{A}(W) = \frac{\mid \underline{\mathfrak{L}}(W) \mid}{\mid \overline{\mathfrak{L}}(W) \mid}, \mid \overline{\mathfrak{L}}(W) \mid \neq 0.$$
$$\mathbf{R}(W) = 1 - \mathbf{A}(W).$$

In many positions, the equivalence relations are not attainable. As a result, the traditional approach has been extended by utilizing relations that are weaker than equivalence relations.

2.2 Kinds of σ-neighborhood space

Definition 4 (Abo-Tabl 2011; Allam et al. 2006; Salama and Abd El-Monsef 2011; Yao 1998, 1996) The σ-neighborhoods of $\mathsf{v} \in \mathbb{X}$, symbolized by $\mathcal{D}_\sigma(\mathsf{v})$, are identified under an arbitrary relation $\mathfrak{L}$ on $\mathbb{X}$ for each $\sigma \in \{r, l, \langle r\rangle, \langle l\rangle, i, u, \langle i\rangle, \langle u\rangle\}$ as following:

(i) $\mathcal{D}_r(\mathsf{v}) = \{\mathsf{e} \in \mathbb{X} : \mathsf{v}\, \mathfrak{L}\, \mathsf{e}\}$.
(ii) $\mathcal{D}_l(\mathsf{v}) = \{\mathsf{e} \in \mathbb{X} : \mathsf{e}\, \mathfrak{L}\, \mathsf{v}\}$.
(iii)

$$\mathcal{D}_{\langle r\rangle}(\mathsf{v}) = \begin{cases} \bigcap\limits_{\mathsf{v}\in\mathcal{D}_r(\mathsf{e})} \mathcal{D}_r(\mathsf{e}) & : \ \exists\, \mathcal{D}_r(\mathsf{e})\ involving\ \mathsf{v} \\ \emptyset & : \ Elsewise \end{cases}$$

(iv)

$$\mathcal{D}_{\langle l\rangle}(\mathsf{v}) = \begin{cases} \bigcap\limits_{\mathsf{v}\in\mathcal{D}_l(\mathsf{e})} \mathcal{D}_l(\mathsf{e}) & : \ \exists\, \mathcal{D}_l(\mathsf{e})\ involving\ \mathsf{v} \\ \emptyset & : \ Elsewise \end{cases}$$

(v) $\mathcal{D}_i(\mathsf{v}) = \mathcal{D}_r(\mathsf{v}) \bigcap \mathcal{D}_l(\mathsf{v})$.
(vi) $\mathcal{D}_u(\mathsf{v}) = \mathcal{D}_r(\mathsf{v}) \bigcup \mathcal{D}_l(\mathsf{v})$.
(vii) $\mathcal{D}_{\langle i\rangle}(\mathsf{v}) = \mathcal{D}_{\langle r\rangle}(\mathsf{v}) \bigcap \mathcal{D}_{\langle l\rangle}(\mathsf{v})$.
$(viii)$ $\mathcal{D}_{\langle u\rangle}(\mathsf{v}) = \mathcal{D}_{\langle r\rangle}(\mathsf{v}) \bigcup \mathcal{D}_{\langle l\rangle}(\mathsf{v})$.

From this point onward, unless stated otherwise, σ will be assumed to be an element of the set $\{r, l, \langle r\rangle, \langle l\rangle, i, u, \langle i\rangle, \langle u\rangle\}$.

Definition 5 (Salama and Abd El-Monsef 2011) Consider ζ_σ denote a mapping from $\mathbb{X}$ to its power set $2^{\mathbb{X}}$ that associates every element v from $\mathbb{X}$ with its σ-neighborhood in $2^{\mathbb{X}}$. Consequently, the triple $(\mathbb{X}, \mathfrak{L}, \zeta_\sigma)$ is termed a σ-neighborhood space, abbreviated as σ-NS.

The previously mentioned sorts of neighborhoods were employed to develop new categories of lower and upper approximations, along with their measures of accuracy (or roughness). To improve the properties of approximations and optimize accuracy measures, extensive comparisons were made among these sorts of neighborhoods.

Definition 6 (Abo-Tabl 2011; Allam et al. 2006; Salama and Abd El-Monsef 2011; Yao 1998, 1996) The lower and upper approximations of a subset W in relative to $\mathcal{D}_\sigma$-neighborhoods are respectively computed by:

$$\mathcal{H}_{\mathcal{D}_\sigma}(W) = \{\mathsf{v} \in \mathbb{X} : \mathcal{D}_\sigma(\mathsf{v}) \subseteq W\},$$
$$\mathcal{H}^{\mathcal{D}_\sigma}(W) = \{\mathsf{v} \in \mathbb{X} : \mathcal{D}_\sigma(\mathsf{v}) \cap W \neq \emptyset\}.$$

Definition 7 (Abo-Tabl 2011; Allam et al. 2006; Salama and Abd El-Monsef 2011; Yao 1998, 1996) The $\mathbf{A}_{\mathcal{D}_\sigma}$-accuracy and $\mathcal{R}_{\mathcal{D}_\sigma}$-roughness measures of a nonempty set W in relative to $\mathcal{D}_\sigma$-neighborhoods are respectively computed by

$$\mathbf{A}_{\mathcal{D}_\sigma}(W) = \frac{|\ \mathcal{H}_{\mathcal{D}_\sigma}(W) \cap W\ |}{|\ \mathcal{H}^{\mathcal{D}_\sigma}(W) \cup W\ |},\ and$$
$$\mathcal{R}_{\mathcal{D}_\sigma}(W) = 1 - \mathbf{A}_{\mathcal{D}_\sigma}(W).$$

Definition 8 (Pawlak 1991, 1982) Consider two relations $\mathfrak{L}_1$ and $\mathfrak{L}_2$ on $\mathbb{X}$ such that $\mathfrak{L}_1 \subseteq \mathfrak{L}_2$. We say that the approximation space induced by $\mathcal{D}$-neighborhoods satisfies the property of monotonicity if $\mathbf{A}_{\mathcal{D}\sigma_1}(W) \geq \mathbf{A}_{\mathcal{D}\sigma_2}(W)$.

Definition 9 (Al-shami 2021a) The σ-containment neighborhoods of $\mathsf{v} \in \mathbb{X}$, symbolized by $\mathbb{C}_\sigma(\mathsf{v})$, are identified under an arbitrary relation $\mathfrak{L}$ on $\mathbb{X}$ for each σ as following:

(i) $\mathbb{C}_r(\mathsf{v}) = \{\mathsf{e} \in \mathbb{X} : \mathcal{D}_r(\mathsf{e}) \subseteq \mathcal{D}_r(\mathsf{v})\}$.

(ii) $\mathbb{C}_l(\mathsf{v}) = \{\mathsf{e} \in \mathbb{X} : \mathcal{D}_l(\mathsf{e}) \subseteq \mathcal{D}_l(\mathsf{v})\}$.
(iii) $\mathbb{C}_i(\mathsf{v}) = \mathbb{C}_r(\mathsf{v}) \cap \mathbb{C}_l(\mathsf{v})$.
(iv) $\mathbb{C}_u(\mathsf{v}) = \mathbb{C}_r(\mathsf{v}) \cup \mathbb{C}_l(\mathsf{v})$.
(v) $\mathbb{C}_{\langle r\rangle}(\mathsf{v}) = \{\mathsf{e} \in \mathbb{X} : \mathcal{D}_{\langle r\rangle}(\mathsf{e}) \subseteq \mathcal{D}_{\langle r\rangle}(\mathsf{v})\}$.
(vi) $\mathbb{C}_{\langle l\rangle}(\mathsf{v}) = \{\mathsf{e} \in \mathbb{X} : \mathcal{D}_{\langle l\rangle}(\mathsf{e}) \subseteq \mathcal{D}_{\langle l\rangle}(\mathsf{v})\}$.
(vii) $\mathbb{C}_{\langle i\rangle}(\mathsf{v}) = \mathbb{C}_{\langle r\rangle}(\mathsf{v}) \cap \mathbb{C}_{\langle l\rangle}(\mathsf{v})$.
(viii) $\mathbb{C}_{\langle u\rangle}(\mathsf{v}) = \mathbb{C}_{\langle r\rangle}(\mathsf{v}) \cup \mathbb{C}_{\langle l\rangle}(\mathsf{v})$.

Definition 10 (Al-shami and Ciucci 2022) The σ-subset neighborhoods of $\mathsf{v} \in \mathbb{X}$, symbolized by $\mathbb{S}_\sigma(\mathsf{v})$, are identified under an arbitrary relation $\mathfrak{L}$ on $\mathbb{X}$ for each σ as following:

(i) $\mathbb{S}_r(\mathsf{v}) = \{\mathsf{e} \in \mathbb{X} : \mathcal{D}_r(\mathsf{v}) \subseteq \mathcal{D}_r(\mathsf{e})\}$.
(ii) $\mathbb{S}_l(\mathsf{v}) = \{\mathsf{e} \in \mathbb{X} : \mathcal{D}_l(\mathsf{v}) \subseteq \mathcal{D}_l(\mathsf{e})\}$.
(iii) $\mathbb{S}_i(\mathsf{v}) = \mathbb{S}_r(\mathsf{v}) \cap \mathbb{S}_l(\mathsf{v})$.
(iv) $\mathbb{S}_u(\mathsf{v}) = \mathbb{S}_r(\mathsf{v}) \cup \mathbb{S}_l(\mathsf{v})$.
(v) $\mathbb{S}_{\langle r\rangle}(\mathsf{v}) = \{\mathsf{e} \in \mathbb{X} : \mathcal{D}_{\langle r\rangle}(\mathsf{v}) \subseteq \mathcal{D}_{\langle r\rangle}(\mathsf{e})\}$.
(vi) $\mathbb{S}_{\langle l\rangle}(\mathsf{v}) = \{\mathsf{e} \in \mathbb{X} : \mathcal{D}_{\langle l\rangle}(\mathsf{v}) \subseteq \mathcal{D}_{\langle l\rangle}(\mathsf{e})\}$.
(vii) $\mathbb{S}_{\langle i\rangle}(\mathsf{v}) = \mathbb{S}_{\langle r\rangle}(\mathsf{v}) \cap \mathbb{S}_{\langle l\rangle}(\mathsf{v})$.
(viii) $\mathbb{S}_{\langle u\rangle}(\mathsf{v}) = \mathbb{S}_{\langle r\rangle}(\mathsf{v}) \cup \mathbb{S}_{\langle l\rangle}(\mathsf{v})$.

Definition 11 (Atef et al. 2022; Mareay 2016) The σ-equality neighborhoods of $\mathsf{v} \in \mathbb{X}$, symbolized by $\sharp_\sigma(\mathsf{v})$, are identified under an arbitrary relation $\mathfrak{L}$ on $\mathbb{X}$ for each σ as following:

(i) $\sharp_r(\mathsf{v}) = \{\mathsf{e} \in \mathbb{X} : \mathcal{D}_r(\mathsf{v}) = \mathcal{D}_r(\mathsf{e})\}$.
(ii) $\sharp_l(\mathsf{v}) = \{\mathsf{e} \in \mathbb{X} : \mathcal{D}_l(\mathsf{v}) = \mathcal{D}_l(\mathsf{e})\}$.
(iii) $\sharp_i(\mathsf{v}) = \sharp_r(\mathsf{v}) \cap \sharp_l(\mathsf{v})$.
(iv) $\sharp_u(\mathsf{v}) = \sharp_r(\mathsf{v}) \cup \sharp_l(\mathsf{v})$.
(v) $\sharp_{\langle r\rangle}(\mathsf{v}) = \{\mathsf{e} \in \mathbb{X} : \mathcal{D}_{\langle r\rangle}(\mathsf{v}) = \mathcal{D}_{\langle r\rangle}(\mathsf{e})\}$.
(vi) $\sharp_{\langle l\rangle}(\mathsf{v}) = \{\mathsf{e} \in \mathbb{X} : \mathcal{D}_{\langle l\rangle}(\mathsf{v}) = \mathcal{D}_{\langle l\rangle}(\mathsf{e})\}$.
(vii) $\sharp_{\langle i\rangle}(\mathsf{v}) = \sharp_{\langle r\rangle}(\mathsf{v}) \cap \sharp_{\langle l\rangle}(\mathsf{v})$.
$(viii)$ $\sharp_{\langle u\rangle}(\mathsf{v}) = \sharp_{\langle r\rangle}(\mathsf{v}) \cup \sharp_{\langle l\rangle}(\mathsf{v})$.

2.3 Cardinality σ-neighborhood systems

According to any binary relation, this section is consecrated to introduce the notion of cardinality neighborhoods. The study of cardinality neighborhoods targets to handle certain scenarios that are affected by the number of members belonging to $\mathcal{D}_\sigma$-Neighborhoods. Their main properties will be explored and the conditions under which some of them are identical will be determined. To support the gained results and relationships, illustrative examples are included.

For each σ, $|\mathcal{D}_\sigma(.)|$ denotes the cardinality of $\mathcal{D}_\sigma(.)$.

Definition 12 (Al-shami et al. 2024c) The σ-cardinality neighborhoods of $\mathsf{v} \in \mathbb{X}$, symbolized by $\mathbb{E}_\sigma(\mathsf{v})$, are identified under an arbitrary relation $\mathfrak{L}$ on $\mathbb{X}$ for each σ as following:

(i) $\mathbb{E}_r(\mathsf{v}) = \{\mathsf{e} \in \mathbb{X} : |\mathcal{D}_r(\mathsf{v})| = |\mathcal{D}_r(\mathsf{e})|\}$.
(ii) $\mathbb{E}_l(\mathsf{v}) = \{\mathsf{e} \in \mathbb{X} : |\mathcal{D}_l(\mathsf{v})| = |\mathcal{D}_l(\mathsf{e})|\}$.
(iii) $\mathbb{E}_i(\mathsf{v}) = \mathbb{E}_r(\mathsf{v}) \cap \mathbb{E}_l(\mathsf{v})$.
(iv) $\mathbb{E}_u(\mathsf{v}) = \mathbb{E}_r(\mathsf{v}) \cup \mathbb{E}_l(\mathsf{v})$.
(v) $\mathbb{E}_{\langle r\rangle}(\mathsf{v}) = \{\mathsf{e} \in \mathbb{X} : |\mathcal{D}_{\langle r\rangle}(\mathsf{v})| = |\mathcal{D}_{\langle r\rangle}(\mathsf{e})|\}$.
(vi) $\mathbb{E}_{\langle l\rangle}(\mathsf{v}) = \{\mathsf{e} \in \mathbb{X} : |\mathcal{D}_{\langle l\rangle}(\mathsf{v})| = |\mathcal{D}_{\langle l\rangle}(\mathsf{e})|\}$.

(vii) $\mathbb{E}_{\langle i\rangle}(\mathsf{v}) = \mathbb{E}_{\langle r\rangle}(\mathsf{v}) \cap \mathbb{E}_{\langle l\rangle}(\mathsf{v})$.
(viii) $\mathbb{E}_{\langle u\rangle}(\mathsf{v}) = \mathbb{E}_{\langle r\rangle}(\mathsf{v}) \cup \mathbb{E}_{\langle l\rangle}(\mathsf{v})$.

Proposition 2 *(Al-shami et al. 2024c)*

(i) $\mathbb{E}_i \subseteq \mathbb{E}_r \cap \mathbb{E}_l \subseteq \mathbb{E}_r \cup \mathbb{E}_l \subseteq \mathbb{E}_u$, *and* $\mathbb{E}_{\langle i\rangle} \subseteq \mathbb{E}_{\langle r\rangle} \cap \mathbb{E}_{\langle l\rangle} \subseteq \mathbb{E}_{\langle r\rangle} \cup \mathbb{E}_{\langle l\rangle} \subseteq \mathbb{E}_{\langle u\rangle}$.
(iii) If $\mathfrak{L}$ is a symmetric relation, then all $\mathbb{E}_\sigma$ are equal.

Proposition 3 *(Al-shami et al. 2024c)*

(i) $\mathsf{v} \in \mathbb{E}_i(\mathsf{a})$ *iff* $|\mathcal{D}_r(\mathsf{v})| = |\mathcal{D}_r(\mathsf{a})|$ *and* $|\mathcal{D}_l(\mathsf{v})| = |\mathcal{D}_l(\mathsf{a})|$.
(ii) $\mathsf{v} \in \mathbb{E}_u(\mathsf{a})$ *iff* $|\mathcal{D}_r(\mathsf{v})| = |\mathcal{D}_r(\mathsf{a})|$ *or* $|\mathcal{D}_l(\mathsf{v})| = |\mathcal{D}_l(\mathsf{a})|$.
(iii) $\mathsf{v} \in \mathbb{E}_{\langle i\rangle}(\mathsf{a})$ *iff* $|\mathcal{D}_{\langle r\rangle}(\mathsf{v})| = |\mathcal{D}_{\langle r\rangle}(\mathsf{a})|$ *and* $|\mathcal{D}_{\langle l\rangle}(\mathsf{v})| = |\mathcal{D}_{\langle l\rangle}(\mathsf{a})|$.
(iv) $\mathsf{v} \in \mathbb{E}_{\langle u\rangle}(\mathsf{a})$ *iff* $|\mathcal{D}_{\langle r\rangle}(\mathsf{v})| = |\mathcal{D}_{\langle r\rangle}(\mathsf{a})|$ *or* $|\mathcal{D}_{\langle l\rangle}(\mathsf{v})| = |\mathcal{D}_{\langle l\rangle}(\mathsf{a})|$.

Corollary 1 *(Al-shami et al. 2024c) If $\mathfrak{L}$ is a symmetric relation, then:*

(i) $\mathbb{E}_i(\mathsf{a}) = \{\mathsf{e} \in \mathbb{X} : |\mathcal{D}_i(\mathsf{a})| = |\mathcal{D}_i(\mathsf{e})|\}$.
(ii) $\mathbb{E}_{\langle i\rangle}(\mathsf{a}) = \{\mathsf{e} \in \mathbb{X} : |\mathcal{D}_{\langle i\rangle}(\mathsf{a})| = |\mathcal{D}_{\langle i\rangle}(\mathsf{e})|\}$.
(iii) $\mathbb{E}_u(\mathsf{a}) = \{\mathsf{e} \in \mathbb{X} : |\mathcal{D}_u(\mathsf{a})| = |\mathcal{D}_u(\mathsf{e})|\}$.
(iv) $\mathbb{E}_{\langle u\rangle}(\mathsf{a}) = \{\mathsf{e} \in \mathbb{X} : |\mathcal{D}_{\langle u\rangle}(\mathsf{a})| = |\mathcal{D}_{\langle u\rangle}(\mathsf{e})|\}$.

Proposition 4 *(Al-shami et al. 2024c) Consider $(\mathbb{X}, \mathfrak{L}, \zeta_\sigma)$ as a σ-NS. If $\mathsf{v} \in \mathbb{X}$, then $\mathbb{E}_\sigma(\mathsf{v}) \neq \emptyset$ for each σ.*

Proposition 5 *(Al-shami et al. 2024c) Consider $(\mathbb{X}, \mathfrak{L}, \zeta_\sigma)$ as a σ-NS and $\mathsf{v} \in \mathbb{X}$. Then, $\mathsf{v} \in \mathbb{E}_\sigma(\mathsf{a})$ iff $\mathsf{a} \in \mathbb{E}_\sigma(\mathsf{v})$, for each σ.*

Proposition 6 *(Al-shami et al. 2024c) Consider $(\mathbb{X}, \mathfrak{L}, \zeta_\sigma)$ as a σ-NS. If $\mathsf{v} \in \mathbb{E}_\sigma(\mathsf{a})$, $\mathsf{a} \in \mathbb{E}_\sigma(\mathsf{x})$, then $\mathsf{v} \in \mathbb{E}_\sigma(\mathsf{x})$, in the cases of $\sigma \in \{r, \langle r\rangle, l, \langle l\rangle, i, \langle i\rangle\}$.*

Corollary 2 *(Al-shami et al. 2024c) Consider $(\mathbb{X}, \mathfrak{L}, \zeta_\sigma)$ as a σ-NS and $\mathsf{v} \in \mathbb{X}$. Then, $\mathsf{v} \in \mathbb{E}_\sigma(\mathsf{a})$ iff $\mathbb{E}_\sigma(\mathsf{v}) = \mathbb{E}_\sigma(\mathsf{a})$, in the cases of $\sigma \in \{r, \langle r\rangle, l, \langle l\rangle, i, \langle i\rangle\}$.*

Corollary 3 *(Al-shami et al. 2024c) In the cases of $\sigma \in \{r, \langle r\rangle, l, \langle l\rangle, i, \langle i\rangle\}$, the relation $\mathfrak{L}$ defined by $\mathsf{a}\mathfrak{L}\mathsf{v} \Longleftrightarrow \mathsf{a} \in \mathbb{E}_\sigma(\mathsf{v})$ is an equivalence. In other words, the cardinality neighborhoods of these cases form a partition for $\mathbb{X}$.*

Corollary 4 *(Al-shami et al. 2024c) The cardinality neighborhoods form a partition for $\mathbb{X}$ for every σ under a symmetric relation*

Proposition 7 *(Al-shami et al. 2024c) $\mathbb{E}_\sigma = \mathbb{E}_{\langle}\sigma\rangle$ for $\sigma \in \{r, l, i, u\}$, if $\mathfrak{L}$ is a preorder (i.e., reflexive, transitive) relation on $\mathbb{X}$.*

Proposition 8 *(Al-shami et al. 2024c) Consider $(\mathbb{X}, \mathfrak{L}, \zeta_\sigma)$ as a σ-NS. If $\mathsf{v} \in \mathbb{X}$, then $\sharp_\sigma(\mathsf{v}) \subseteq \mathbb{E}_\sigma(\mathsf{v})$, for each σ.*

Definition 13 (Al-shami et al. 2024c) Consider $(\mathbb{X}, \mathfrak{L}, \zeta_\sigma)$ as a σ-NS. Based on cardinality neighborhoods, the $\mathbb{E}_\sigma$-lower approximation $\mathcal{H}_{\mathbb{E}_\sigma}(W)$, and $\mathbb{E}_\sigma$-upper approximation $\mathcal{H}^{\mathbb{E}_\sigma}(W)$ of a set W, assigned as:

$$\mathcal{H}_{\mathbb{E}_\sigma}(W) = \{\mathsf{a} \in \mathbb{X} : \mathbb{E}_\sigma(\mathsf{a}) \subseteq W\}, \textit{ and}$$
$$\mathcal{H}^{\mathbb{E}_\sigma}(W) = \{\mathsf{a} \in \mathbb{X} : \mathbb{E}_\sigma(\mathsf{a}) \cap W \neq \emptyset\}$$

Definition 14 (Al-shami et al. 2024c) The $\mathbb{E}_\sigma$-boundary, $\mathbb{E}_\sigma$-positive, and $\mathbb{E}_\sigma$-negative regions of a subset W within a σ-NS $(\mathbb{X}, \mathfrak{L}, \zeta_\sigma)$ are identified respectively as:

$$\begin{aligned}
\mathbb{B}_{\mathbb{E}_\sigma}(W) &= \mathcal{H}^{\mathbb{E}_\sigma}(W) \setminus \mathcal{H}_{\mathbb{E}_\sigma}(W) \\
\mathbb{P}_{\mathbb{E}_\sigma}(W) &= \mathcal{H}_{\mathbb{E}_\sigma}(W), \\
\mathbb{N}_{\mathbb{E}_\sigma}(W) &= \mathbb{X} \setminus \mathcal{H}^{\mathbb{E}_\sigma}(W)
\end{aligned}$$

Definition 15 (Al-shami et al. 2024c) The $\mathbb{E}_\sigma$-accuracy and $\mathbb{E}_\sigma$-roughness criteria of $W \neq \emptyset$ of a σ-NS $(\mathbb{X}, \mathfrak{L}, \zeta_\sigma)$ are respectively endowed by:

$$\begin{aligned}
\mathbb{A}_{\mathbb{E}_\sigma}(W) &= \frac{\mid \mathcal{H}_{\mathbb{E}_\sigma}(W) \mid}{\mid \mathcal{H}^{\mathbb{E}_\sigma}(W) \mid}, \mid \mathcal{H}^{\mathbb{E}_\sigma}(W) \mid \neq 0. \\
\mathbb{R}_{\mathbb{E}_\sigma}(W) &= 1 - \mathbb{A}_{\mathbb{E}_\sigma}(W).
\end{aligned}$$

Theorem 1 *(Al-shami et al. 2024c) Consider $(\mathbb{X}, \mathfrak{L}, \zeta_\sigma)$ as a σ-NS. Based on cardinality neighborhoods, the family $\Omega_{\mathbb{E}_\sigma} = \{W \subseteq \mathbb{X}: \forall\, \mathsf{v} \in W, \mathbb{E}_\sigma(\mathsf{v}) \subseteq W\}$ constitutes a topology on $\mathbb{X}$, for each σ,*

Lemma 1 *(Al-shami et al. 2024c) Let $(\mathbb{X}, \mathfrak{L}, \zeta_\sigma)$ be a σ-NS and $\mathsf{v} \in \mathbb{X}$. If $\sigma \in \{r, \langle r\rangle, l, \langle l\rangle, i, \langle i\rangle\}$, then $\mathbb{E}_\sigma(\mathsf{v})$ is $\Omega_{\mathbb{E}\sigma}$-open set.*

Definition 16 A non-empty class $\mathcal{I} \subseteq 2^{\mathbb{X}}$ is defined as an ideal on $\mathbb{X}$ providing that it is closed under subset and finite union.

3 Novel rough-set paradigms generated by ideals and cardinality neighborhoods

This section is devoted to define and study novel rough approximation spaces directly generated from cardinality neighborhoods and ideals, which are used to specify new regions and accuracy and roughness criteria of any set.

3.1 First category of rough-set paradigms

This part is allocated to display new paradigms of rough sets induced by the notions of ideals and cardinality neighborhoods. We show that these paradigms aggrandize the lower approximation and minify the upper approximation compared with the preceding models of rough sets. On the other hand, we debate the deficiencies of the current models.

Definition 17 Consider $(\mathbb{X}, \mathfrak{L}, \zeta_\sigma)$ as a σ-NS and $\mathcal{I}$ is an ideal on $\mathbb{X}$. Regarding to ideals and cardinality neighborhoods, the duo $({}^{\mathcal{I}}\widetilde{\mathcal{H}}_{\mathbb{E}_\sigma}(W), {}^{\mathcal{I}}\widetilde{\mathcal{H}}^{\mathbb{E}_\sigma}(W))$ denotes the lower and upper approximations of a subset W, which are respectively computed by:

$$\begin{aligned}
{}^{\mathcal{I}}\widetilde{\mathcal{H}}_{\mathbb{E}_\sigma}(W) &= \{\mathsf{a} \in \mathbb{X} : \mathbb{E}_\sigma(\mathsf{a}) \setminus W \in \mathcal{I}\}, \\
{}^{\mathcal{I}}\widetilde{\mathcal{H}}^{\mathbb{E}_\sigma}(W) &= \{\mathsf{a} \in \mathbb{X} : \mathbb{E}_\sigma(\mathsf{a}) \cap W \notin \mathcal{I}\}
\end{aligned}$$

Remark 1 If $\mathcal{I} = \{\emptyset\}$ in Definition 17, the proposed approach identifies with the method outlined in Definition 4.1 of Al-shami et al. (2024c). Consequently, the current study can be regarded as a genuine generalization of the work presented in Al-shami et al. (2024c).

Next, we examine the features of ${}^{\mathcal{I}}\widetilde{\mathcal{H}}_{\mathbb{E}_\sigma}()$, ${}^{\mathcal{I}}\widetilde{\mathcal{H}}^{\mathbb{E}_\sigma}()$ for any given set, as detailed in the subsequent results.

Theorem 2 *Let $\mathcal{I}$ be an ideal on a σ-NS $(\mathbb{X}, \mathfrak{L}, \zeta_\sigma)$. If $S, W \subseteq \mathbb{X}$, then for each σ the next statements hold true.*

(i) ${}^{\mathcal{I}}\widetilde{\mathcal{H}}_{\mathbb{E}_\sigma}(\mathbb{X}) = \mathbb{X}$ *and* ${}^{\mathcal{I}}\widetilde{\mathcal{H}}^{\mathbb{E}_\sigma}(\emptyset) = \emptyset$.

(ii) If $S \subseteq W$, *then* ${}^{\mathcal{I}}\widetilde{\mathcal{H}}_{\mathbb{E}_\sigma}(S) \subseteq {}^{\mathcal{I}}\widetilde{\mathcal{H}}_{\mathbb{E}_\sigma}(W)$ *and* ${}^{\mathcal{I}}\widetilde{\mathcal{H}}^{\mathbb{E}_\sigma}(S) \subseteq {}^{\mathcal{I}}\widetilde{\mathcal{H}}^{\mathbb{E}_\sigma}(W)$.

(iii) ${}^{\mathcal{I}}\widetilde{\mathcal{H}}_{\mathbb{E}_\sigma}(S \cap W) = {}^{\mathcal{I}}\widetilde{\mathcal{H}}_{\mathbb{E}_\sigma}(S) \cap {}^{\mathcal{I}}\widetilde{\mathcal{H}}_{\mathbb{E}_\sigma}(W)$ *and* ${}^{\mathcal{I}}\widetilde{\mathcal{H}}^{\mathbb{E}_\sigma}(S \cup W) = {}^{\mathcal{I}}\widetilde{\mathcal{H}}^{\mathbb{E}_\sigma}(S) \cup {}^{\mathcal{I}}\widetilde{\mathcal{H}}^{\mathbb{E}_\sigma}(W)$.

(iv) ${}^{\mathcal{I}}\widetilde{\mathcal{H}}_{\mathbb{E}_\sigma}(W^c) = ({}^{\mathcal{I}}\widetilde{\mathcal{H}}^{\mathbb{E}_\sigma}(W))^c$ *and* ${}^{\mathcal{I}}\widetilde{\mathcal{H}}^{\mathbb{E}_\sigma}(W^c) = ({}^{\mathcal{I}}\widetilde{\mathcal{H}}_{\mathbb{E}_\sigma}(W))^c$.

(v) If $W^c \in \mathcal{I}$, *then* ${}^{\mathcal{I}}\widetilde{\mathcal{H}}_{\mathbb{E}_\sigma}(W) = \mathbb{X}$ *and* ${}^{\mathcal{I}}\widetilde{\mathcal{H}}^{\mathbb{E}_\sigma}(W^c) = \emptyset$.

(vi) ${}^{\mathcal{I}}\widetilde{\mathcal{H}}_{\mathbb{E}_\sigma}({}^{\mathcal{I}}\widetilde{\mathcal{H}}_{\mathbb{E}_\sigma}(W)) \supseteq {}^{\mathcal{I}}\widetilde{\mathcal{H}}_{\mathbb{E}_\sigma}(W)$ *and* ${}^{\mathcal{I}}\widetilde{\mathcal{H}}^{\mathbb{E}_\sigma}({}^{\mathcal{I}}\widetilde{\mathcal{H}}^{\mathbb{E}_\sigma}(W)) \subseteq {}^{\mathcal{I}}\widetilde{\mathcal{H}}^{\mathbb{E}_\sigma}(W)$, *for every* $\sigma \in \{r, l, i, \langle r\rangle, \langle l\rangle, \langle i\rangle\}$.

(vii) ${}^{\mathcal{I}}\widetilde{\mathcal{H}}_{\mathbb{E}_\sigma}(\mathbb{E}_\sigma(\mathsf{e})) \supseteq \mathbb{E}_\sigma(\mathsf{e})$, *for every* $\sigma \in \{r, l, i, \langle r\rangle, \langle l\rangle, \langle i\rangle\}$.

Proof (i) ${}^{\mathcal{I}}\widetilde{\mathcal{H}}_{\mathbb{E}_\sigma}(\mathbb{X}) = \{\mathsf{a} \in \mathbb{X} : \mathbb{E}_\sigma(\mathsf{a}) \backslash \mathbb{X} = \emptyset \in \mathcal{I}\} = \mathbb{X}$ and ${}^{\mathcal{I}}\widetilde{\mathcal{H}}^{\mathbb{E}_\sigma}(\emptyset) = \{\mathsf{a} \in \mathbb{X} : \mathbb{E}_\sigma(\mathsf{a}) \cap \emptyset \in \mathcal{I}\} = \emptyset$.

(ii) Obvious.

(iii) It follows from (ii) that ${}^{\mathcal{I}}\widetilde{\mathcal{H}}_{\mathbb{E}_\sigma}(S \cap W) \subseteq {}^{\mathcal{I}}\widetilde{\mathcal{H}}_{\mathbb{E}_\sigma}(S) \cap {}^{\mathcal{I}}\widetilde{\mathcal{H}}_{\mathbb{E}_\sigma}(W)$.

Conversely, let $\mathsf{a} \in {}^{\mathcal{I}}\widetilde{\mathcal{H}}_{\mathbb{E}_\sigma}(S) \cap {}^{\mathcal{I}}\widetilde{\mathcal{H}}_{\mathbb{E}_\sigma}(W)$. Then $\mathsf{a} \in {}^{\mathcal{I}}\widetilde{\mathcal{H}}_{\mathbb{E}_\sigma}(S)$ and $\mathsf{a} \in {}^{\mathcal{I}}\widetilde{H}_{\mathbb{E}_\sigma}(W)$ which means that ${}^{\mathcal{I}}\mathbb{E}_\sigma(\mathsf{a}) \backslash S \in \mathcal{I}$ and ${}^{\mathcal{I}}\mathbb{E}_\sigma(\mathsf{a}) \backslash W \in \mathcal{I}$. Therefore, ${}^{\mathcal{I}}\mathbb{E}_\sigma(\mathsf{a}) \backslash (S \cap W) \in \mathcal{I}$. Thus, $\mathsf{a} \in {}^{\mathcal{I}}\widetilde{\mathcal{H}}_{\mathbb{E}_\sigma}(S \cap W)$. Hence, ${}^{\mathcal{I}}\widetilde{\mathcal{H}}_{\mathbb{E}_\sigma}(S) \cap {}^{\mathcal{I}}\widetilde{\mathcal{H}}_{\mathbb{E}_\sigma}(W) \subseteq {}^{\mathcal{I}}\widetilde{\mathcal{H}}_{\mathbb{E}_\sigma}(S \cap W)$.

In the same way it can be proved, ${}^{\mathcal{I}}\widetilde{\mathcal{H}}^{\mathbb{E}_\sigma}(S \cup W) = {}^{\mathcal{I}}\widetilde{\mathcal{H}}^{\mathbb{E}_\sigma}(S) \cup {}^{\mathcal{I}}\widetilde{\mathcal{H}}^{\mathbb{E}_\sigma}(W)$.

(iv) $\mathsf{a} \in {}^{\mathcal{I}}\widetilde{\mathcal{H}}_{\mathbb{E}_\sigma}(W^c) \Longleftrightarrow \mathbb{E}_\sigma(\mathsf{a}) \backslash W^c \in \mathcal{I}$

$$\begin{aligned} &\Longleftrightarrow \mathbb{E}_\sigma(\mathsf{a}) \cap W \in \mathcal{I} \\ &\Longleftrightarrow \mathsf{a} \notin {}^{\mathcal{I}}\widetilde{\mathcal{H}}^{\mathbb{E}_\sigma}(W) \\ &\Longleftrightarrow \mathsf{a} \in ({}^{\mathcal{I}}\widetilde{\mathcal{H}}^{\mathbb{E}_\sigma}(W))^c. \end{aligned}$$

Similarly, it can be proven ${}^{\mathcal{I}}\widetilde{\mathcal{H}}^{\mathbb{E}_\sigma}(W^c) = ({}^{\mathcal{I}}\widetilde{\mathcal{H}}_{\mathbb{E}_\sigma}(W))^c$.

(v) Let $W^c \in \mathcal{I}$. Then for any $\mathsf{a} \in \mathbb{X}$, $\mathbb{E}_\sigma(\mathsf{a}) \backslash W = \mathbb{E}_\sigma(\mathsf{a}) \cap W^c \in \mathcal{I}$. Hence, ${}^{\mathcal{I}}\widetilde{\mathcal{H}}_{\mathbb{E}_\sigma}(W) = \mathbb{X}$. By using (iv), ${}^{\mathcal{I}}\widetilde{\mathcal{H}}^{\mathbb{E}_\sigma}(W^c) = \emptyset$.

(vi) Suppose $\sigma \in \{r, l, i, \langle r\rangle, \langle l\rangle, \langle i\rangle\}$. We will prove only ${}^{\mathcal{I}}\widetilde{\mathcal{H}}^{\mathbb{E}_\sigma}({}^{\mathcal{I}}\widetilde{\mathcal{H}}^{\mathbb{E}_\sigma}(W)) \subseteq {}^{\mathcal{I}}\widetilde{\mathcal{H}}^{\mathbb{E}_\sigma}(W)$. Let $\mathsf{a} \in {}^{\mathcal{I}}\widetilde{\mathcal{H}}^{\mathbb{E}_\sigma}({}^{\mathcal{I}}\widetilde{\mathcal{H}}^{\mathbb{E}_\sigma}(W))$, then $\mathbb{E}_\sigma(\mathsf{a}) \cap {}^{\mathcal{I}}\widetilde{\mathcal{H}}^{\mathbb{E}_\sigma}(W) \notin \mathcal{I}$. Hence, $\mathbb{E}_\sigma(\mathsf{a}) \cap {}^{\mathcal{I}}\widetilde{\mathcal{H}}^{\mathbb{E}_\sigma}(W) \neq \emptyset$ i.e there exists $y \in \mathbb{X}$ s.t. $y \in \mathbb{E}_\sigma(\mathsf{a})$, and $y \in {}^{\mathcal{I}}\widetilde{\mathcal{H}}^{\mathbb{E}_\sigma}(W)$. This leads to that $\mathbb{E}_\sigma(y) \cap W \notin \mathcal{I}$. Regarding to Corollary 2, $\mathbb{E}_\sigma(y) = \mathbb{E}_\sigma(\mathsf{a})$. Consequently, $\mathbb{E}_\sigma(\mathsf{a}) \cap W \notin \mathcal{I}$ and so $\mathsf{a} \in {}^{\mathcal{I}}\widetilde{\mathcal{H}}^{\mathbb{E}_\sigma}(W)$

(vii) Suppose $\sigma \in \{r, l, i, \langle r\rangle, \langle l\rangle, \langle i\rangle\}$. Let $\mathsf{a} \in \mathbb{E}_\sigma(\mathsf{e})$. According to Corollary 2, $\mathbb{E}_\sigma(\mathsf{e}) = \mathbb{E}_\sigma(\mathsf{a})$. Then, $\mathbb{E}_\sigma(\mathsf{a}) \setminus \mathbb{E}_\sigma(\mathsf{e}) = \emptyset \in \mathcal{I}$ and so $\mathsf{a} \in {}^{\mathcal{I}}\widetilde{\mathcal{H}}_{\mathbb{E}_\sigma}(\mathbb{E}_\sigma(\mathsf{e}))$. Consequently, $\mathbb{E}_\sigma(\mathsf{e}) \subseteq {}^{\mathcal{I}}\widetilde{\mathcal{H}}_{\mathbb{E}_\sigma}(\mathbb{E}_\sigma(\mathsf{e}))$.

□

In light of points (ii) from Theorem 2, the following corollary is evident.

Table 1 $\mathbb{E}_\sigma$-neighborhoods for members of $\mathbb{X}$

	a	x	y	v
$\mathbb{E}_r$	{a, y}	{x}	{a, y}	{v}
$\mathbb{E}_l$	{a}	{x}	{y, v}	{y, v}
$\mathbb{E}_i$	{a}	{x}	{y}	{v}
$\mathbb{E}_u$	{a, y}	{x}	{a, y, v}	{y, v}
$\mathbb{E}_{\langle r\rangle}$	{a}	{x, v}	{y}	{x, v}
$\mathbb{E}_{\langle l\rangle}$	{a}	{x, y}	{x, y}	{v}
$\mathbb{E}_{\langle i\rangle}$	{a}	{x}	{y}	{v}
$\mathbb{E}_{\langle u\rangle}$	{a}	{x, y, v}	{x, y}	{x, v}

Corollary 5 *Let $\mathcal{I}$ be an ideal on a σ-NS $(\mathbb{X}, \mathfrak{L}, \zeta_\sigma)$. If $S, W \subseteq \mathbb{X}$, then the following statements hold for each σ:*

(i) ${}^{\mathcal{I}}\widetilde{\mathcal{H}}_{\mathbb{E}_\sigma}(S) \cup {}^{\mathcal{I}}\widetilde{\mathcal{H}}_{\mathbb{E}_\sigma}(W) \subseteq {}^{\mathcal{I}}\widetilde{\mathcal{H}}_{\mathbb{E}_\sigma}(S \cup W)$.

(ii) ${}^{\mathcal{I}}\widetilde{\mathcal{H}}^{\mathbb{E}_\sigma}(S \cap W) \subseteq {}^{\mathcal{I}}\widetilde{\mathcal{H}}^{\mathbb{E}_\sigma}(S) \cap {}^{\mathcal{I}}\widetilde{\mathcal{H}}^{\mathbb{E}_\sigma}(W)$.

Proposition 9 *Let $\mathcal{I}$ be an ideal on a σ-NS $(\mathbb{X}, \mathfrak{L}, \zeta_\sigma)$. If $W \subseteq \mathbb{X}$, then*

(i) ${}^{\mathcal{I}}\widetilde{\mathcal{H}}_{\mathbb{E}u}(W) \subseteq {}^{\mathcal{I}}\widetilde{\mathcal{H}}_{\mathbb{E}r}(W) \cap {}^{\mathcal{I}}\widetilde{\mathcal{H}}_{\mathbb{E}l}(W) \subseteq {}^{\mathcal{I}}\widetilde{\mathcal{H}}_{\mathbb{E}r}(W) \cup {}^{\mathcal{I}}\widetilde{\mathcal{H}}_{\mathbb{E}l}(W) \subseteq {}^{\mathcal{I}}\widetilde{\mathcal{H}}_{\mathbb{E}i}(W)$.

(ii) ${}^{\mathcal{I}}\widetilde{\mathcal{H}}^{\mathbb{E}i}(W) \subseteq {}^{\mathcal{I}}\widetilde{\mathcal{H}}^{\mathbb{E}r}(W) \cap {}^{\mathcal{I}}\widetilde{\mathcal{H}}^{\mathbb{E}l}(W) \subseteq {}^{\mathcal{I}}\widetilde{\mathcal{H}}^{\mathbb{E}r}(W) \cup {}^{\mathcal{I}}\widetilde{\mathcal{H}}^{\mathbb{E}l}(W) \subseteq {}^{\mathcal{I}}\widetilde{\mathcal{H}}^{\mathbb{E}u}(W)$.

(iii) ${}^{\mathcal{I}}\widetilde{\mathcal{H}}_{\mathbb{E}\langle u\rangle}(W) \subseteq {}^{\mathcal{I}}\widetilde{\mathcal{H}}_{\mathbb{E}\langle r\rangle}(W) \cap {}^{\mathcal{I}}\widetilde{\mathcal{H}}_{\mathbb{E}\langle l\rangle}(W) \subseteq {}^{\mathcal{I}}\widetilde{\mathcal{H}}_{\mathbb{E}\langle r\rangle}(W) \cup {}^{\mathcal{I}}\widetilde{\mathcal{H}}_{\mathbb{E}\langle l\rangle}(W) \subseteq {}^{\mathcal{I}}\widetilde{\mathcal{H}}_{\mathbb{E}\langle i\rangle}(W)$.

(iv) ${}^{\mathcal{I}}\widetilde{\mathcal{H}}^{\mathbb{E}\langle i\rangle}(W) \subseteq {}^{\mathcal{I}}\widetilde{\mathcal{H}}^{\mathbb{E}\langle r\rangle}(W) \cap {}^{\mathcal{I}}\widetilde{\mathcal{H}}^{\mathbb{E}\langle l\rangle}(W) \subseteq {}^{\mathcal{I}}\widetilde{\mathcal{H}}^{\mathbb{E}\langle r\rangle}(W) \cup {}^{\mathcal{I}}\widetilde{\mathcal{H}}^{\mathbb{E}\langle l\rangle}(W) \subseteq {}^{\mathcal{I}}\widetilde{\mathcal{H}}^{\mathbb{E}\langle u\rangle}(W)$.

Proof Follows from (i) of Proposition 2. □

Proposition 10 *Let $\mathcal{I}$ be an ideal on a σ-NS $(\mathbb{X}, \mathfrak{L}, \zeta_\sigma)$ s.t. $\mathfrak{L}$ is a symmetric relation. Then, for every $W \subseteq \mathbb{X}$, all ${}^{\mathcal{I}}\widetilde{\mathcal{H}}_{\mathbb{E}\sigma}(W)$ (${}^{\mathcal{I}}\widetilde{\mathcal{H}}^{\mathbb{E}\sigma}(W)$) are equal.*

Proof Follows from (ii) of Proposition 2. □

We furnish the next example to show that:

1) the converse of items (ii), (vi), and (vii) of Theorem 2 fails in general,
2) the converse of Corollary 5 is not always true,
3) the subsets ${}^{\mathcal{I}}\widetilde{\mathcal{H}}_{\mathbb{E}_\sigma}(\mathbb{E}_\sigma(\mathsf{e}))$ and $\mathbb{E}_\sigma(\mathsf{e})$ are independent of each other in the cases of $\sigma \in \{u, \langle u\rangle\}$,
4) the converse of Proposition 9 need not be true, and
5) some properties of Pawlak's paradigm are violated in the current models.

Example 1 Consider $\mathfrak{L} = \{(\mathsf{a}, \mathsf{x}), (\mathsf{x}, \mathsf{x}), (\mathsf{x}, \mathsf{y}), (\mathsf{y}, \mathsf{v})\}$ is a binary relation on $\mathbb{X} = \{\mathsf{a}, \mathsf{x}, \mathsf{y}, \mathsf{v}\}$. Then, in Table 1, we compute the cardinality neighborhoods for all elements of $\mathbb{X}$.

If $\mathcal{I} = \{\emptyset, \{\mathsf{y}\}\}$, then for each σ, the ${}^{\mathcal{I}}\widetilde{\mathcal{H}}_{\mathbb{E}_\sigma}(W)$, ${}^{\mathcal{I}}\widetilde{\mathcal{H}}^{\mathbb{E}_\sigma}(W)$ are computed in Tables 2 and 3.

Now, it can be seen the following:

Table 2 The approximations for $\{r, l, i, u\}$

W	${}^{\mathcal{I}}\widetilde{\mathcal{H}}_{\mathbb{E}_r}(W)$	${}^{\mathcal{I}}\widetilde{\mathcal{H}}^{\mathbb{E}_r}(W)$	${}^{\mathcal{I}}\widetilde{\mathcal{H}}_{\mathbb{E}_l}(W)$	${}^{\mathcal{I}}\widetilde{\mathcal{H}}^{\mathbb{E}_l}(W)$	${}^{\mathcal{I}}\widetilde{\mathcal{H}}_{\mathbb{E}_i}(W)$	${}^{\mathcal{I}}\widetilde{\mathcal{H}}^{\mathbb{E}_i}(W)$	${}^{\mathcal{I}}\widetilde{\mathcal{H}}_{\mathbb{E}_u}(W)$	${}^{\mathcal{I}}\widetilde{\mathcal{H}}^{\mathbb{E}_u}(W)$
{a}	{a, y}	{a, y}	{a}	{a}	{a, y}	{a}	{a}	{a, y}
{x}	{x}	{x}	{x}	{x}	{x, y}	{x}	{x}	{x}
{y}	∅	∅	∅	∅	{y}	∅	∅	∅
{v}	{v}	{v}	{y, v}	{y, v}	{y, v}	{v}	{v}	{y, v}
{a, x}	{a, x, y}	{a, x, y}	{a, x}	{a, x}	{a, x, y}	{a, x}	{a, x}	{a, x, y}
{a, y}	{a, y}	{a, y}	{a}	{a}	{a, y}	{a}	{a}	{a, y}
{a, v}	{a, y, v}	{a, y, v}	{a, y, v}	{a, y, v}	{a, y, v}	{a, v}	{a, y, v}	{a, y, v}
{x, y}	{x}	{x}	{x}	{x}	{x, y}	{x}	{x}	{x}
{x, v}	{x, v}	{x, v}	{x, y, v}	{x, y, v}	{x, y, v}	{x, v}	{x, v}	{x, y, v}
{y, v}	{v}	{v}	{y, v}	{y, v}	{y, v}	{v}	{v}	{y, v}
{a, x, y}	{a, x, y}	{a, x, y}	{a, x}	{a, x}	{a, x, y}	{a, x}	{a, x}	{a, x, y}
{a, x, v}	$\mathbb{X}$	$\mathbb{X}$	$\mathbb{X}$	$\mathbb{X}$	$\mathbb{X}$	{a, x, v}	$\mathbb{X}$	$\mathbb{X}$
{a, y, v}	{a, y, v}	{a, y, v}	{a, y, v}	{a, y, v}	{a, y, v}	{a, v}	{a, y, v}	{a, y, v}
{x, y, v}	{x, v}	{x, v}	{x, y, v}	{x, y, v}	{x, y, v}	{x, v}	{x, v}	{x, y, v}
$\mathbb{X}$	$\mathbb{X}$	$\mathbb{X}$	$\mathbb{X}$	$\mathbb{X}$	$\mathbb{X}$	{a, x, v}	$\mathbb{X}$	$\mathbb{X}$
∅	∅	∅	∅	∅	{y}	∅	∅	∅

Table 3 The approximations for $\{\langle r\rangle, \langle l\rangle, \langle i\rangle, \langle u\rangle\}$

W	${}^{\mathcal{I}}\widetilde{\mathcal{H}}_{\mathbb{E}_{\langle r\rangle}}(W)$	${}^{\mathcal{I}}\widetilde{\mathcal{H}}^{\mathbb{E}_{\langle r\rangle}}(W)$	${}^{\mathcal{I}}\widetilde{\mathcal{H}}_{\mathbb{E}_{\langle l\rangle}}(W)$	$\widetilde{\mathcal{H}}^{\mathbb{E}_{\langle l\rangle}}(W)$	${}^{\mathcal{I}}\widetilde{\mathcal{H}}_{\mathbb{E}_{\langle i\rangle}}(W)$	${}^{\mathcal{I}}\widetilde{\mathcal{H}}^{\mathbb{E}_{\langle i\rangle}}(W)$	${}^{\mathcal{I}}\widetilde{\mathcal{H}}_{\mathbb{E}_{\langle u\rangle}}(W)$	${}^{\mathcal{I}}\widetilde{\mathcal{H}}^{\mathbb{E}_{\langle u\rangle}}(W)$
{a}	{a, y}	{a}	{a}	{a}	{a, y}	{a}	{a}	{a}
{x}	{y}	{x, v}	{x, y}	{x, y}	{x, y}	{x}	{y}	{x, y, v}
{y}	{y}	∅	∅	∅	{y}	∅	∅	∅
{v}	{y}	{x, v}	{v}	{v}	{y, v}	{v}	∅	{x, v}
{a, x}	{a, y}	{a, x, v}	{a, x, y}	{a, x, y}	{a, x, y}	{a, x}	{a, y}	$\mathbb{X}$
{a, y}	{a, y}	{a}	{a}	{a}	{a, y}	{a}	{a}	{a}
{a, v}	{a, y}	{a, x, v}	{a, v}	{a, v}	{a, y, v}	{a, v}	{a}	{a, x, v}
{x, y}	{y}	{x, v}	{x, y}	{x, y}	{x, y}	{x}	{y}	{x, y, v}
{x, v}	{x, y, v}	{x, v}	{x, y, v}	{x, y, v}	{x, y, v}	{x, v}	{x, y, v}	{x, y, v}
{y, v}	{y}	{x, v}	{v}	{v}	{y, v}	{v}	∅	{x, v}
{a, x, y}	{a, y}	{a, x, v}	{a, x, y}	{a, x, y}	{a, x, y}	{a, x}	{a, y}	$\mathbb{X}$
{a, x, v}	$\mathbb{X}$	{a, x, v}	$\mathbb{X}$	$\mathbb{X}$	$\mathbb{X}$	{a, x, v}	$\mathbb{X}$	$\mathbb{X}$
{a, y, v}	{a, y}	{a, x, v}	{a, v}	{a, v}	{a, y, v}	{a, v}	{a}	{a, x, v}
{x, y, v}	{x, y, v}	{x, v}	{x, y, v}	{x, y, v}	{x, y, v}	{x, v}	{x, y, v}	{x, y, v}
$\mathbb{X}$	$\mathbb{X}$	{a, x, v}	$\mathbb{X}$	$\mathbb{X}$	$\mathbb{X}$	{a, x, v}	$\mathbb{X}$	$\mathbb{X}$
∅	{y}	∅	∅	∅	{y}	∅	∅	∅

(i) ${}^{\mathcal{I}}\widetilde{\mathcal{H}}_{\mathbb{E}_\sigma}(\{\mathsf{y}\}) \subseteq {}^{\mathcal{I}}\widetilde{\mathcal{H}}_{\mathbb{E}_\sigma}(\{\mathsf{x}\})$ and ${}^{\mathcal{I}}\widetilde{\mathcal{H}}^{\mathbb{E}_\sigma}(\{\mathsf{y}\}) \subseteq {}^{\mathcal{I}}\widetilde{\mathcal{H}}^{\mathbb{E}_\sigma}(\{\mathsf{x}\})$ for each σ, whereas $\{x\}$ and $\{y\}$ are independent of each other with respect to inclusion relation.

(ii) ${}^{\mathcal{I}}\widetilde{\mathcal{H}}_{\mathbb{E}_{\langle r\rangle}}(\{\mathsf{x}\} \cup \{\mathsf{v}\}) = \{\mathsf{x}, \mathsf{y}, \mathsf{v}\} \nsubseteq {}^{\mathcal{I}}\widetilde{\mathcal{H}}_{\mathbb{E}_{\langle r\rangle}}(\{\mathsf{x}\}) \cup {}^{\mathcal{I}}\widetilde{\mathcal{H}}_{\mathbb{E}_{\langle r\rangle}}(\{\mathsf{v}\}) = \{\mathsf{y}\}$.

(iii) ${}^{\mathcal{I}}\widetilde{\mathcal{H}}^{\mathbb{E}_{\langle r\rangle}}(\{\mathsf{x}\}) \cap {}^{\mathcal{I}}\widetilde{\mathcal{H}}^{\mathbb{E}_{\langle r\rangle}}(\{\mathsf{v}\}) = \{\mathsf{x}, \mathsf{v}\} \nsubseteq {}^{\mathcal{I}}\widetilde{\mathcal{H}}^{\mathbb{E}_{\langle r\rangle}}(\{\mathsf{x}\} \cap \{\mathsf{v}\}) = \emptyset$.

The main advantages of the current models, compared to the preceding models introduced in Al-shami et al. (2024c), are to enlarge the lower approximation and downsize the upper approximation of subsets, which leads to minimize the boundary region. The next result proves this matter.

Theorem 3 *Let $\mathcal{I}$ be an ideal on a σ-NS $(\mathbb{X}, \mathfrak{L}, \zeta_\sigma)$ and let $S \subseteq \mathbb{X}$. We have the subsequent relations for each σ.*

(i) $\mathcal{H}_{\mathbb{E}_\sigma}(S) \subseteq {}^{\mathcal{I}}\widetilde{\mathcal{H}}_{\mathbb{E}_\sigma}(S)$,

(ii) ${}^{\mathcal{I}}\widetilde{\mathcal{H}}^{\mathbb{E}_\sigma}(S) \subseteq \mathcal{H}^{\mathbb{E}_\sigma}(S)$.

Proof Let $\mathsf{e} \in \mathcal{H}_{\mathbb{E}_\sigma}(S)$. Then, $\mathbb{E}_\sigma(\mathsf{e}) \subseteq S$. So $\mathbb{E}_\sigma(\mathsf{e}) \setminus S = \emptyset \in \mathcal{I}$. Now, we have $\mathsf{e} \in {}^{\mathcal{I}}\widetilde{\mathcal{H}}_{\mathbb{E}_\sigma}(S)$. Hence, $\mathcal{H}_{\mathbb{E}_\sigma}(S) \subseteq {}^{\mathcal{I}}\widetilde{\mathcal{H}}_{\mathbb{E}_\sigma}(S)$. One can prove the second statement following a similar argument. □

To clarify that the converse of the aforementioned theorem fails, we give the next example.

Example 2 In Example 1, take $S = \{\mathsf{a}\}$ and $W = \{\mathsf{y}\}$. Then, $\mathcal{H}_{\mathbb{E}_r}(S) = \emptyset$, whereas ${}^{\mathcal{I}}\widetilde{\mathcal{H}}_{\mathbb{E}_r}(S) = \{\mathsf{a}, \mathsf{y}\}$. Also, ${}^{\mathcal{I}}\widetilde{\mathcal{H}}^{\mathbb{E}_r}(W) = \emptyset$, whereas $\mathcal{H}^{\mathbb{E}_\sigma}(W) = \{\mathsf{a}, \mathsf{y}\}$.

In what follows, we demonstrate some failures of the models given herein.

Remark 2 Let $\mathcal{I}$ be an ideal on a σ-NS $(\mathbb{X}, \mathfrak{L}, \zeta_\sigma)$ and $S, W \subseteq \mathbb{X}$. The next statements demonstrate some drawbacks of the current rough set models.

(i) ${}^{\mathcal{I}}\widetilde{\mathcal{H}}_{\mathbb{E}_\sigma}(W) \nsubseteq W \nsubseteq {}^{\mathcal{I}}\widetilde{\mathcal{H}}^{\mathbb{E}_\sigma}(W)$.

(ii) ${}^{\mathcal{I}}\widetilde{\mathcal{H}}_{\mathbb{E}_\sigma}(\emptyset) \neq \emptyset$.

(iii) ${}^{\mathcal{I}}\widetilde{\mathcal{H}}^{\mathbb{E}_\sigma}(\mathbb{X}) \neq \mathbb{X}$.

(iv) ${}^{\mathcal{I}}\widetilde{\mathcal{H}}_{\mathbb{E}_\sigma}({}^{\mathcal{I}}\widetilde{\mathcal{H}}_{\mathbb{E}_\sigma}(W)) \neq {}^{\mathcal{I}}\widetilde{\mathcal{H}}_{\mathbb{E}_\sigma}(W)$ in the cases of $\sigma \in \{r, l, i, \langle r\rangle, \langle l\rangle, \langle i\rangle\}$.

(v) ${}^{\mathcal{I}}\widetilde{\mathcal{H}}_{\mathbb{E}_\sigma}({}^{\mathcal{I}}\widetilde{\mathcal{H}}_{\mathbb{E}_\sigma}(W)) \nsupseteq {}^{\mathcal{I}}\widetilde{\mathcal{H}}_{\mathbb{E}_\sigma}(W)$ in the cases of $\sigma \in \{u, \langle u\rangle\}$.

(vi) Let $\mathsf{e} \in \mathbb{X}$. Then ${}^{\mathcal{I}}\widetilde{\mathcal{H}}_{\mathbb{E}_\sigma}(\mathbb{E}_\sigma(\mathsf{e})) \nsubseteq \mathbb{E}_\sigma(\mathsf{e})$ in the cases of $\sigma \in \{r, l, i, \langle r\rangle, \langle l\rangle, \langle i\rangle\}$.

(vii) Let $\mathsf{e} \in \mathbb{X}$. Then ${}^{\mathcal{I}}\widetilde{\mathcal{H}}_{\mathbb{E}_\sigma}(\mathbb{E}_\sigma(\mathsf{e})) \nsupseteq \mathbb{E}_\sigma(\mathsf{e})$ in the cases of $\sigma \in \{u, \langle u\rangle\}$.

Example 3 illustrates property (i) of Remark 2.

Example 3 Continued in Example 1. Let $\mathcal{I} = \{\emptyset, \{\mathsf{y}\}\}$, and $\sigma = r$.

(i) If $W = \{\mathsf{a}, \mathsf{x}, \mathsf{v}\}$, then ${}^{\mathcal{I}}\widetilde{\mathcal{H}}_{\mathbb{E}_r}(W) = \mathbb{X} \nsubseteq W$,

(ii) If $W = \{\mathsf{x}, \mathsf{y}\}$, then $W \nsubseteq \{\mathsf{x}\} = {}^{\mathcal{I}}\widetilde{\mathcal{H}}^{\mathbb{E}_r}(W)$.

Example 4 illustrates properties (ii), (iii), (vi) of Remark 2.

Example 4 Continued in Example 1. Let $\mathcal{I} = \{\emptyset, \{\mathsf{y}\}\}$, and $\sigma = i$.

(i) ${}^{\mathcal{I}}\widetilde{\mathcal{H}}_{\mathbb{E}_i}(\emptyset) = \{\mathsf{y}\} \neq \emptyset$.

(ii) ${}^{\mathcal{I}}\widetilde{\mathcal{H}}^{\mathbb{E}_i}(\mathbb{X}) = \{\mathsf{a}, \mathsf{x}, \mathsf{v}\} \neq \mathbb{X}$.

(iii) ${}^{\mathcal{I}}\widetilde{\mathcal{H}}_{\mathbb{E}_i}(\{\mathsf{a}\}) = \{\mathsf{a}, \mathsf{y}\} \nsubseteq \{\mathsf{a}\}$, ${}^{\mathcal{I}}\widetilde{\mathcal{H}}_{\mathbb{E}_i}(\{\mathsf{x}\}) = \{\mathsf{x}, \mathsf{y}\} \nsubseteq \{\mathsf{x}\}$, ${}^{\mathcal{I}}\widetilde{\mathcal{H}}_{\mathbb{E}_i}(\{\mathsf{v}\}) = \{\mathsf{y}, \mathsf{v}\} \nsubseteq \{\mathsf{v}\}$. Hence, for each $\mathsf{e} \in \mathbb{X}$, ${}^{\mathcal{I}}\widetilde{\mathcal{H}}_{\mathbb{E}_i}(\mathbb{E}_i(\mathsf{e})) \nsubseteq \mathbb{E}_i(\mathsf{e})$.

Example 5 illustrates properties (v) of Remark 2.

Example 5 Continued in Example 1. Let $\mathcal{I} = \{\emptyset, \{\mathsf{a}\}\}$, and $\sigma = u$. If $W = \{\mathsf{x}, \mathsf{y}\}$, then ${}^{\mathcal{I}}\widetilde{\mathcal{H}}_{\mathbb{E}_u}(W) = \{\mathsf{a}, \mathsf{x}\} \nsubseteq \{\mathsf{x}\} = {}^{\mathcal{I}}\widetilde{\mathcal{H}}_{\mathbb{E}_u}({}^{\mathcal{I}}\widetilde{\mathcal{H}}_{\mathbb{E}_u}(W))$.

Example 6 illustrates properties (vii) of Remark 2.

Example 6 Continued in Example 1. Let $\mathcal{I} = \{\emptyset, \{\mathsf{y}\}\}$, and $\sigma = u$. Manifestly ${}^{\mathcal{I}}\widetilde{\mathcal{H}}_{\mathbb{E}_u}(\mathbb{E}_u(\mathsf{a})) = \{\mathsf{a}\} \nsupseteq \{\mathsf{a}, \mathsf{y}\} = \mathbb{E}_u(\mathsf{a})$, and ${}^{\mathcal{I}}\widetilde{\mathcal{H}}_{\mathbb{E}_u}(\mathbb{E}_u(\mathsf{v})) = \{\mathsf{v}\} \nsupseteq \{\mathsf{y}, \mathsf{v}\} = \mathbb{E}_u(\mathsf{v})$ Hence, for each $\mathsf{e} \in \mathbb{X}$, ${}^{\mathcal{I}}\widetilde{\mathcal{H}}_{\mathbb{E}_u}(\mathbb{E}_u(\mathsf{e})) \nsupseteq \mathbb{E}_u(\mathsf{e})$.

One can give the proof of the next result easily, so we omit the proof.

Proposition 11 *Let $\mathcal{I}, \mathcal{J}$ be ideals on a σ-NS $(\mathbb{X}, \mathfrak{L}, \zeta_\sigma)$, and $W \subseteq \mathbb{X}$. If $\mathcal{I} \subseteq \mathcal{J}$, then the following statements hold for each σ:*

(i) ${}^{\mathcal{I}}\widetilde{\mathcal{H}}_{\mathbb{E}_\sigma}(W) \subseteq {}^{\mathcal{J}}\widetilde{\mathcal{H}}_{\mathbb{E}_\sigma}(W)$,

(ii) ${}^{\mathcal{J}}\widetilde{\mathcal{H}}^{\mathbb{E}_\sigma}(W) \subseteq {}^{\mathcal{I}}\widetilde{\mathcal{H}}^{\mathbb{E}_\sigma}(W)$,

Remark 3 Continued in Example 1. Let $\mathcal{I} = \{\emptyset, \{\mathsf{y}\}\}$, $\mathcal{J} = \{\emptyset, \{\mathsf{x}\}, \{\mathsf{y}\}, \{\mathsf{x}, \mathsf{y}\}\}$.

(i) If $W = \{\mathsf{a}, \mathsf{v}\}$, then ${}^{\mathcal{J}}\widetilde{\mathcal{H}}_{\mathbb{E}_r}(W) = \mathbb{X} \nsubseteq \{\mathsf{a}, \mathsf{y}, \mathsf{v}\} = {}^{\mathcal{I}}\widetilde{\mathcal{H}}_{\mathbb{E}_r}(W)$,

(ii) If $W = \{\mathsf{x}, \mathsf{y}\}$, then ${}^{\mathcal{I}}\widetilde{\mathcal{H}}^{\mathbb{E}_r}(W) = \{\mathsf{x}\} \nsubseteq \emptyset = {}^{\mathcal{J}}\widetilde{\mathcal{H}}^{\mathbb{E}_r}(W)$,

3.2 Second category of rough-set paradigms

In the first category of rough-set paradigms, we face some shortcomings and undesirable properties such as

- The property says that ${}^{\mathcal{I}}\widetilde{\mathcal{H}}_{\mathbb{E}_\sigma}(W) \subseteq W \subseteq {}^{\mathcal{I}}\widetilde{\mathcal{H}}^{\mathbb{E}_\sigma}(W)$ does not hold for some subsets, which leads to illogical characterizations of those rough set models or suspicion of the knowledge extracted from them; especially, when ${}^{\mathcal{I}}\widetilde{\mathcal{H}}_{\mathbb{E}_\sigma}(\emptyset) \neq \emptyset$ and ${}^{\mathcal{I}}\widetilde{\mathcal{H}}^{\mathbb{E}_\sigma}(\mathbb{X}) \neq \mathbb{X}$.
- We cannot use the original formula of accuracy measure since it produces values greater than one or undefined case for some subsets, i.e. in example 1 we have $\frac{|{}^{\mathcal{I}}\widetilde{\mathcal{H}}_{\mathbb{E}_i}(\{\mathsf{a},\mathsf{x},\mathsf{y}\})|}{|{}^{\mathcal{I}}\widetilde{\mathcal{H}}^{\mathbb{E}_i}(\{\mathsf{a},\mathsf{x},\mathsf{y}\})|} = \frac{3}{2} > 1$, and $\frac{|{}^{\mathcal{I}}\widetilde{\mathcal{H}}_{\mathbb{E}_i}(\emptyset)|}{|{}^{\mathcal{I}}\widetilde{\mathcal{H}}^{\mathbb{E}_i}(\emptyset)|} = \frac{1}{0}$. Such cases are meaningless and useless for practical issues.

To fix these failures and keep the advantages of the first rough-set paradigm in connection with increasing lower approximation and maximizing upper approximation, we do this subsection of manuscript. Let us begin with the following definition.

Definition 18 Let $\mathcal{I}$ be an ideal on a σ-NS $(\mathbb{X}, \mathfrak{L}, \zeta_\sigma)$. Based on ideals and cardinality neighborhoods, the ${}^{\mathcal{I}}\mathbb{E}_\sigma$-lower approximation ${}^{\mathcal{I}}\mathcal{H}_{\mathbb{E}_\sigma}()$, and ${}^{\mathcal{I}}\mathbb{E}_\sigma$-upper approximation ${}^{\mathcal{I}}\mathcal{H}^{\mathbb{E}_\sigma}()$ of $W \subseteq \mathbb{X}$ are respectively computed by:

$$
\begin{aligned}
{}^{\mathcal{I}}\mathcal{H}_{\mathbb{E}_\sigma}(W) &= {}^{\mathcal{I}}\widetilde{\mathcal{H}}_{\mathbb{E}_\sigma}(W) \cap W, \\
{}^{\mathcal{I}}\mathcal{H}^{\mathbb{E}_\sigma}(W) &= {}^{\mathcal{I}}\widetilde{\mathcal{H}}^{\mathbb{E}_\sigma}(W) \cup W
\end{aligned}
$$

Definition 19 The ${}^{\mathcal{I}}\mathbb{E}_{\sigma}$-boundary, ${}^{\mathcal{I}}\mathbb{E}_{\sigma}$-positive, and ${}^{\mathcal{I}}\mathbb{E}_{\sigma}$-negative regions of a subset W within a σ-NS $(\mathbb{X}, \mathfrak{L}, \zeta_{\sigma})$ with ideal $\mathcal{I}$ on $\mathbb{X}$ are respectively given by

$$
\begin{aligned}
{}^{\mathcal{I}}\mathbb{B}_{\mathbb{E}_{\sigma}}(W) &= {}^{\mathcal{I}}\mathcal{H}^{\mathbb{E}_{\sigma}}(W) \setminus {}^{\mathcal{I}}\mathcal{H}_{\mathbb{E}_{\sigma}}(W)\\
{}^{\mathcal{I}}\mathbb{P}_{\mathbb{E}_{\sigma}}(W) &= {}^{\mathcal{I}}\mathcal{H}_{\mathbb{E}_{\sigma}}(W),\\
{}^{\mathcal{I}}\mathbb{N}_{\mathbb{E}_{\sigma}}(W) &= \mathbb{X} \setminus {}^{\mathcal{I}}\mathcal{H}^{\mathbb{E}_{\sigma}}(W).
\end{aligned}
$$

Numerically, rough sets can be described with respect to $\mathbb{E}_{\sigma}$-neighborhoods and ideals by next measures.

Definition 20 The ${}^{\mathcal{I}}\mathbb{E}_{\sigma}$-accuracy and ${}^{\mathcal{I}}\mathbb{E}_{\sigma}$-roughness criteria of $W \neq \emptyset$ of a σ-NS $(\mathbb{X}, \mathfrak{L}, \zeta_{\sigma})$ with ideal $\mathcal{I}$ on $\mathbb{X}$ are respectively given by

$$
\begin{aligned}
{}^{\mathcal{I}}\mathcal{A}_{\mathbb{E}_{\sigma}}(W) &= \frac{|{}^{\mathcal{I}}\mathcal{H}_{\mathbb{E}_{\sigma}}(W)|}{|{}^{\mathcal{I}}\mathcal{H}^{\mathbb{E}_{\sigma}}(W)|}, \ |{}^{\mathcal{I}}\mathcal{H}^{\mathbb{E}_{\sigma}}(W)| \neq 0.\\
{}^{\mathcal{I}}\mathcal{R}_{\mathbb{E}_{\sigma}}(W) &= 1 - {}^{\mathcal{I}}\mathcal{A}_{\mathbb{E}_{\sigma}}(W).
\end{aligned}
$$

In the following theorem, description of Pawlak's properties according to the ${}^{\mathcal{I}}\mathbb{E}_{\sigma}$-lower and ${}^{\mathcal{I}}\mathbb{E}^{\sigma}$-upper approximations will be examined.

Theorem 4 *Let $\mathcal{I}$ be an ideal on a σ-NS $(\mathbb{X}, \mathfrak{L}, \zeta_{\sigma})$ and let $S, W \subseteq \mathbb{X}$. Then, we have the subsequent properties.*

(i) ${}^{\mathcal{I}}\mathcal{H}_{\mathbb{E}_{\sigma}}(W) \subseteq W \subseteq {}^{\mathcal{I}}\mathcal{H}^{\mathbb{E}_{\sigma}}(W)$.

(ii) ${}^{\mathcal{I}}\mathcal{H}_{\mathbb{E}_{\sigma}}(\emptyset) = \emptyset$, *and* ${}^{\mathcal{I}}\mathcal{H}^{\mathbb{E}_{\sigma}}(\emptyset) = \emptyset$.

(iii) ${}^{\mathcal{I}}\mathcal{H}_{\mathbb{E}_{\sigma}}(\mathbb{X}) = \mathbb{X}$, *and* ${}^{\mathcal{I}}\mathcal{H}^{\mathbb{E}_{\sigma}}(\mathbb{X}) = \mathbb{X}$.

(iv) *If* $S \subseteq W$, *then* ${}^{\mathcal{I}}\widetilde{\mathcal{H}}_{\mathbb{E}_{\sigma}}(S) \subseteq {}^{\mathcal{I}}\widetilde{\mathcal{H}}_{\mathbb{E}_{\sigma}}(W)$ *and* ${}^{\mathcal{I}}\widetilde{\mathcal{H}}^{\mathbb{E}_{\sigma}}(S) \subseteq {}^{\mathcal{I}}\widetilde{\mathcal{H}}^{\mathbb{E}_{\sigma}}(W)$.

(v) ${}^{\mathcal{I}}\mathcal{H}_{\mathbb{E}_{\sigma}}({}^{\mathcal{I}}\mathcal{H}_{\mathbb{E}_{\sigma}}(W)) = {}^{\mathcal{I}}\mathcal{H}_{\mathbb{E}_{\sigma}}(W)$ *in the cases of* $\sigma \in \{r, l, i, \langle r\rangle, \langle l\rangle, \langle i\rangle\}$.

(vi) ${}^{\mathcal{I}}\mathcal{H}_{\mathbb{E}_{\sigma}}({}^{\mathcal{I}}\mathcal{H}_{\mathbb{E}_{\sigma}}(W)) \subseteq {}^{\mathcal{I}}\mathcal{H}_{\mathbb{E}_{\sigma}}(W)$ *in the cases of* $\sigma \in \{u, \langle u\rangle\}$.

(vii) *Let* $\mathsf{e} \in \mathbb{X}$. *Then* ${}^{\mathcal{I}}\mathcal{H}_{\mathbb{E}_{\sigma}}(\mathbb{E}_{\sigma}(\mathsf{e})) = \mathbb{E}_{\sigma}(\mathsf{e})$ *in the cases of* $\sigma \in \{r, l, i, \langle r\rangle, \langle l\rangle, \langle i\rangle\}$.

(viii) *Let* $\mathsf{e} \in \mathbb{X}$. *Then* ${}^{\mathcal{I}}\mathcal{H}_{\mathbb{E}_{\sigma}}(\mathbb{E}_{\sigma}(\mathsf{e})) \subseteq \mathbb{E}_{\sigma}(\mathsf{e})$ *in the cases of* $\sigma \in \{u, \langle u\rangle\}$.

(ix) $\mathcal{H}^{\mathbb{E}_{\sigma}}(\mathcal{H}^{\mathbb{E}_{\sigma}}(W)) = \mathcal{H}^{\mathbb{E}_{\sigma}}(W)$ *in the cases of* $\sigma \in \{r, \langle r\rangle, l, \langle l\rangle, i, \langle i\rangle\}$.

(x) $\mathcal{H}^{\mathbb{E}_{\sigma}}(\mathcal{H}^{\mathbb{E}_{\sigma}}(W)) \supseteq \mathcal{H}^{\mathbb{E}_{\sigma}}(W)$ *in the cases of* $\sigma \in \{u, \langle u\rangle\}$.

(xi) ${}^{\mathcal{I}}\mathcal{H}_{\mathbb{E}\sigma}(S) \cap {}^{\mathcal{I}}\mathcal{H}_{\mathbb{E}\sigma}(W) = {}^{\mathcal{I}}\mathcal{H}_{\mathbb{E}\sigma}(S \cap W)$ *in the cases of* σ.

(xii) ${}^{\mathcal{I}}\mathcal{H}^{\mathbb{E}\sigma}(S) \cup {}^{\mathcal{I}}\mathcal{H}^{\mathbb{E}\sigma}(W) = {}^{\mathcal{I}}\mathcal{H}^{\mathbb{E}\sigma}(S \cup W)$ *in the cases of* σ.

Proof According to Definition 18, the validation of (i), (ii), (iii), (iv), (xi), (xii) is facile.

(v) Suppose that $\sigma \in \{r, l, i, \langle r\rangle, \langle l\rangle, \langle i\rangle\}$. Using properties (i) and (iv) of the current theorem,

${}^{\mathcal{I}}\mathcal{H}_{\mathbb{E}_{\sigma}}({}^{\mathcal{I}}\mathcal{H}_{\mathbb{E}_{\sigma}}(W)) \subseteq {}^{\mathcal{I}}\mathcal{H}_{\mathbb{E}_{\sigma}}(W)$. The other direction is proved using (vi) of Theorem 2. Hence, ${}^{\mathcal{I}}\mathcal{H}_{\mathbb{E}_{\sigma}}({}^{\mathcal{I}}\mathcal{H}_{\mathbb{E}_{\sigma}}(W)) = {}^{\mathcal{I}}\mathcal{H}_{\mathbb{E}_{\sigma}}(W)$.

(vi) Suppose that $\sigma \in \{u, \langle u\rangle\}$. Using properties (i) and (iv) of the current theorem, ${}^{\mathcal{I}}\mathcal{H}_{\mathbb{E}_{\sigma}}({}^{\mathcal{I}}\mathcal{H}_{\mathbb{E}_{\sigma}}(W)) \subseteq {}^{\mathcal{I}}\mathcal{H}_{\mathbb{E}_{\sigma}}(W)$.

(vii) Suppose that $\sigma \in \{r, l, i, \langle r\rangle, \langle l\rangle, \langle i\rangle\}$. Using property (i) of the current theorem, then ${}^{\mathcal{I}}\mathcal{H}_{\mathbb{E}_\sigma}(\mathbb{E}_\sigma(\mathsf{e})) \subseteq \mathbb{E}_\sigma(\mathsf{e})$ for each $\mathsf{e} \in \mathbb{X}$. The other direction is proved using (vii) of Theorem 2. Hence, ${}^{\mathcal{I}}\mathcal{H}_{\mathbb{E}_\sigma}(\mathbb{E}_\sigma(\mathsf{e})) = \mathbb{E}_\sigma(\mathsf{e})$

$(viii)$ Suppose that $\sigma \in \{u, \langle u\rangle\}$. Using property (i) of the current theorem, then ${}^{\mathcal{I}}\mathcal{H}_{\mathbb{E}_\sigma}(\mathbb{E}_\sigma(\mathsf{e})) \subseteq \mathbb{E}_\sigma(\mathsf{e})$ for each $\mathsf{e} \in \mathbb{X}$.

(ix) Similar to the proof of property (v).

(x) Similar to the proof of property (vi).

□

Proposition 12 *Let $\mathcal{I}$ be an ideal on a σ-NS $(\mathbb{X}, \mathfrak{L}, \zeta_\sigma)$. If $W \subseteq \mathbb{X}$, then*

(i) ${}^{\mathcal{I}}\mathcal{H}_{\mathbb{E}u}(W) \subseteq {}^{\mathcal{I}}\mathcal{H}_{\mathbb{E}r}(W) \cap {}^{\mathcal{I}}\mathcal{H}_{\mathbb{E}l}(W) \subseteq {}^{\mathcal{I}}\mathcal{H}_{\mathbb{E}r}(W) \cup {}^{\mathcal{I}}\mathcal{H}_{\mathbb{E}l}(W) \subseteq {}^{\mathcal{I}}\mathcal{H}_{\mathbb{E}i}(W)$.

(ii) ${}^{\mathcal{I}}\mathcal{H}^{\mathbb{E}i}(W) \subseteq {}^{\mathcal{I}}\mathcal{H}^{\mathbb{E}r}(W) \cap {}^{\mathcal{I}}\mathcal{H}^{\mathbb{E}l}(W) \subseteq {}^{\mathcal{I}}\mathcal{H}^{\mathbb{E}r}(W) \cup {}^{\mathcal{I}}\mathcal{H}^{\mathbb{E}l}(W) \subseteq {}^{\mathcal{I}}\mathcal{H}^{\mathbb{E}u}(W)$.

(iii) ${}^{\mathcal{I}}\mathcal{H}_{\mathbb{E}\langle u\rangle}(W) \subseteq {}^{\mathcal{I}}\mathcal{H}_{\mathbb{E}\langle r\rangle}(W) \cap {}^{\mathcal{I}}\mathcal{H}_{\mathbb{E}\langle l\rangle}(W) \subseteq {}^{\mathcal{I}}\mathcal{H}_{\mathbb{E}\langle r\rangle}(W) \cup {}^{\mathcal{I}}\mathcal{H}_{\mathbb{E}\langle l\rangle}(W) \subseteq {}^{\mathcal{I}}\mathcal{H}_{\mathbb{E}\langle i\rangle}(W)$.

(iv) ${}^{\mathcal{I}}\mathcal{H}^{\mathbb{E}\langle i\rangle}(W) \subseteq {}^{\mathcal{I}}\mathcal{H}^{\mathbb{E}\langle r\rangle}(W) \cap {}^{\mathcal{I}}\mathcal{H}^{\mathbb{E}\langle l\rangle}(W) \subseteq {}^{\mathcal{I}}\mathcal{H}^{\mathbb{E}\langle r\rangle}(W) \cup {}^{\mathcal{I}}\mathcal{H}^{\mathbb{E}\langle l\rangle}(W) \subseteq {}^{\mathcal{I}}\mathcal{H}^{\mathbb{E}\langle u\rangle}(W)$.

Proof Follows from Proposition 9 and Definition 18. □

Corollary 6 *Let $\mathcal{I}$ be an ideal on a σ-NS $(\mathbb{X}, \mathfrak{L}, \zeta_\sigma)$. If $W \subseteq \mathbb{X}$, then*

(i) ${}^{\mathcal{I}}\mathbf{A}_{\mathbb{E}u}(W) \leq {}^{\mathcal{I}}\mathbf{A}_{\mathbb{E}r}(W) \leq {}^{\mathcal{I}}\mathbf{A}_{\mathbb{E}i}(W)$.

(ii) ${}^{\mathcal{I}}\mathbf{A}_{\mathbb{E}u}(W) \leq {}^{\mathcal{I}}\mathbf{A}_{\mathbb{E}l}(W) \leq {}^{\mathcal{I}}\mathbf{A}_{\mathbb{E}i}(W)$.

(iii) ${}^{\mathcal{I}}\mathbf{A}_{\mathbb{E}\langle u\rangle}(W) \leq {}^{\mathcal{I}}\mathbf{A}_{\mathbb{E}\langle r\rangle}(W) \leq {}^{\mathcal{I}}\mathbf{A}_{\mathbb{E}\langle i\rangle}(W)$.

(iv) ${}^{\mathcal{I}}\mathbf{A}_{\mathbb{E}\langle u\rangle}(W) \leq {}^{\mathcal{I}}\mathbf{A}_{\mathbb{E}\langle l\rangle}(W) \leq {}^{\mathcal{I}}\mathbf{A}_{\mathbb{E}\langle i\rangle}(W)$.

Proposition 13 *If W is a nonempty subset of $\mathbb{X}$, $0 \leq {}^{\mathcal{I}}\mathbf{A}_{\mathbb{E}_\sigma}(W) \leq 1$ for any σ.*

Proof Follows by the fact that ${}^{\mathcal{I}}\mathcal{H}_{\mathbb{E}_\sigma}(W) \subseteq W \subseteq {}^{\mathcal{I}}\mathcal{H}^{\mathbb{E}_\sigma}(W)$. □

Definition 21 We call a subset W ${}^{\mathcal{I}}\mathbb{E}\sigma$-exact if ${}^{\mathcal{I}}\mathbf{A}_{\mathbb{E}_\sigma}(W) = 1$. Otherwise, W is called ${}^{\mathcal{I}}\mathbb{E}\sigma$-rough.

The next example demonstrates that the converse of Corollary 6 fails in general.

Example 7 Continued in Example 1. If $\mathcal{I} = \{\emptyset, \{\mathsf{y}\}\}$, then for each σ, the accuracy criteria ${}^{\mathcal{I}}\mathcal{A}_{\mathbb{E}_\sigma}(W)$ are computed in the Tables 4 and 5.

The next theorem elucidates how the present models get better the operators of approximation compared with the models furnished in Al-shami et al. (2024c). They make a real shrink (or removal) for the boundary regions and an obvious increase in accuracy measures of subsets.

Theorem 5 *Let $\mathcal{I}$ be an ideal on a σ-NS $(\mathbb{X}, \mathfrak{L}, \zeta_\sigma)$. If $S \subseteq \mathbb{X}$, then for each σ the next statements hold true.*

(i) $\mathcal{H}_{\mathbb{E}_\sigma}(S) \subseteq {}^{\mathcal{I}}\mathcal{H}_{\mathbb{E}_\sigma}(S)$,

(ii) ${}^{\mathcal{I}}\mathcal{H}^{\mathbb{E}_\sigma}(S) \subseteq \mathcal{H}^{\mathbb{E}_\sigma}(S)$.

Table 4 The Accuracy criteria for $\{r, l, i, u\}$

W	${}^{\mathcal{I}}\mathcal{A}_{\mathbb{E}_r}(W)$	${}^{\mathcal{I}}\mathcal{A}_{\mathbb{E}_l}(W)$	${}^{\mathcal{I}}\mathcal{A}_{\mathbb{E}_i}(W)$	${}^{\mathcal{I}}\mathcal{A}_{\mathbb{E}_u}(W)$
{a}	$\frac{1}{2}$	1	1	$\frac{1}{2}$
{x}	1	1	1	1
{y}	0	0	1	0
{v}	1	$\frac{1}{2}$	1	$\frac{1}{2}$
{a, x}	$\frac{2}{3}$	1	1	$\frac{2}{3}$
{a, y}	1	$\frac{1}{2}$	1	$\frac{1}{2}$
{a, v}	$\frac{2}{3}$	$\frac{2}{3}$	1	$\frac{2}{3}$
{x, y}	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$
{x, v}	1	$\frac{2}{3}$	1	$\frac{2}{3}$
{y, v}	$\frac{1}{2}$	1	1	$\frac{1}{2}$
{a, x, y}	1	$\frac{2}{3}$	1	$\frac{2}{3}$
{a, x, v}	$\frac{3}{4}$	$\frac{3}{4}$	1	$\frac{3}{4}$
{a, y, v}	1	1	1	1
{x, y, v}	$\frac{2}{3}$	1	1	$\frac{2}{3}$
$\mathbb{X}$	1	1	1	1

Table 5 The Accuracy criteria for $\{\langle r\rangle, \langle l\rangle, \langle i\rangle, \langle u\rangle\}$

W	${}^{\mathcal{I}}\mathcal{A}_{\mathbb{E}_{\langle r\rangle}}(W)$	${}^{\mathcal{I}}\mathcal{A}_{\mathbb{E}_{\langle l\rangle}}(W)$	${}^{\mathcal{I}}\mathcal{A}_{\mathbb{E}_{\langle i\rangle}}(W)$	${}^{\mathcal{I}}\mathcal{A}_{\mathbb{E}_{\langle u\rangle}}(W)$
{a}	1	1	1	1
{x}	0	$\frac{1}{2}$	1	0
{y}	1	0	1	0
{v}	0	1	1	0
{a, x}	$\frac{1}{3}$	$\frac{2}{3}$	1	$\frac{1}{4}$
{a, y}	1	$\frac{1}{2}$	1	$\frac{1}{2}$
{a, v}	$\frac{1}{3}$	1	1	$\frac{1}{3}$
{x, y}	$\frac{1}{3}$	1	1	$\frac{1}{3}$
{x, v}	1	$\frac{2}{3}$	1	$\frac{2}{3}$
{y, v}	$\frac{1}{3}$	$\frac{1}{2}$	1	0
{a, x, y}	$\frac{1}{2}$	1	1	$\frac{1}{2}$
{a, x, v}	1	$\frac{3}{4}$	1	$\frac{3}{4}$
{a, y, v}	$\frac{1}{2}$	$\frac{2}{3}$	1	$\frac{1}{4}$
{x, y, v}	1	1	1	1
$\mathbb{X}$	1	1	1	1

Proof By using Theorem 3, then $\mathcal{H}_{\mathbb{E}_\sigma}(S) \subseteq {}^{\mathcal{I}}\widetilde{\mathcal{H}}_{\mathbb{E}_\sigma}(S)$. Since $\mathcal{H}_{\mathbb{E}_\sigma}(S) \subseteq W$, then $\mathcal{H}_{\mathbb{E}_\sigma}(S) \subseteq {}^{\mathcal{I}}\mathcal{H}_{\mathbb{E}_\sigma}(S)$. By following a similar argument, the second statement can be proven. □

Corollary 7 *Let $\mathcal{I}$ be an ideal on a σ-NS $(\mathbb{X}, \mathfrak{L}, \zeta_\sigma)$. If $S \subseteq \mathbb{X}$, then*

$$\mathbb{A}_{\mathbb{E}_\sigma}(S) \leq^{\mathcal{I}} \mathbf{A}_{\mathbb{E}_\sigma}(S),\ for each \sigma.$$

To clarify that the converse of the aforementioned theorem and corollary fails, we give the next example.

Example 8 Continued from Example 1. If $\mathcal{I} = \{\emptyset, \{\mathsf{x}\}\}$, then ${}^{\mathcal{I}}\mathcal{H}_{\mathbb{E}_\sigma}$, ${}^{\mathcal{I}}\mathcal{H}^{\mathbb{E}_\sigma}$, and ${}^{\mathcal{I}}\mathbf{A}_{\mathbb{E}_\sigma}$ of $W = \{\mathsf{a}, \mathsf{y}\}$ are computed for $\sigma \in \{l, u\}$ as follows:

$(i)\mathcal{H}_{\mathbb{E}_\sigma}(W) = \{\mathsf{a}\}$, $\mathcal{H}^{\mathbb{E}_\sigma}(W) = \{\mathsf{a}, \mathsf{y}, \mathsf{v}\}$, and $\mathbb{A}_{\mathbb{E}_\sigma}(W) = \frac{1}{3}$.
$(\widetilde{ii})^{\mathcal{I}}\mathcal{H}_{\mathbb{E}_\sigma}(W) = \{\mathsf{a}, \mathsf{y}\}$, ${}^{\mathcal{I}}\mathcal{H}^{\mathbb{E}_\sigma}(W) = \{\mathsf{a}, \mathsf{x}, \mathsf{y}\}$, and ${}^{\mathcal{I}}\mathbf{A}_{\mathbb{E}_\sigma}(W) = \frac{2}{3}$,

Proposition 14 *For each σ, suppose that $\mathcal{I}$ is an ideal on a σ-NS $(\mathbb{X}, \mathfrak{L}, \zeta_\sigma)$. If $W \subseteq \mathbb{X}$, then*

(i) ${}^{\mathcal{I}}\mathcal{H}_{\mathbb{E}_\sigma}(W) \subseteq {}^{\mathcal{I}}\mathcal{H}_{\sharp_\sigma}(W)$.

(ii) ${}^{\mathcal{I}}\mathcal{H}^{\sharp_\sigma}(W) \subseteq {}^{\mathcal{I}}\mathcal{H}^{\mathbb{E}_\sigma}(W)$

(iii) ${}^{\mathcal{I}}\mathcal{A}_{\mathbb{E}_\sigma}(W) \leq {}^{\mathcal{I}}\mathcal{A}_{\sharp_\sigma}(W)$.

Remark 4 Let $\mathcal{I}$, $\mathcal{J}$ be ideals on a σ-NS $(\mathbb{X}, \mathfrak{L}, \zeta_\sigma)$, and $W \subseteq \mathbb{X}$. If $\mathcal{I} \subseteq \mathcal{J}$, then ${}^{\mathcal{I}}\mathcal{A}_{\mathbb{E}_\sigma}(W) \leq {}^{\mathcal{J}}\mathcal{A}_{\mathbb{E}_\sigma}(W)$, for each σ.

Finally, we present Algorithm 1, which determines whether a set is ${}^{\mathcal{I}}\mathbb{E}\sigma$-rough or ${}^{\mathcal{I}}\mathbb{E}\sigma$-exact and calculates its measure of accuracy.

Input : The sample under study representing by $\mathbb{X}$ as the universe.
Output: Identify whether sets under considaration are ${}^{\mathcal{I}}\mathbb{E}\sigma$-rough or ${}^{\mathcal{I}}\mathbb{E}\sigma$-exact and compute their accuracy.

Provide an ideal $\mathcal{I}$ and a binary relation $\mathfrak{L}$ over $\mathbb{X}$ as given by the expert;
Select a type of σ;
for *all* $\alpha \in \mathbb{X}$ **do**
    Compute $\mathcal{D}_\sigma(\alpha)$
end
for *all* $\alpha \in \mathbb{X}$ **do**
    Compute $\mathbb{E}\sigma(\alpha)$
end
for *each subset* $S \neq \emptyset$ *of* $\mathbb{X}$ **do**
    Compute ${}^{\mathcal{I}}\widetilde{\mathcal{H}}_{\mathbb{E}\sigma}(S)$ (by the formula of Definition 17);
    Compute ${}^{\mathcal{I}}\mathcal{H}_{\mathbb{E}\sigma}(S) = {}^{\mathcal{I}}\widetilde{\mathcal{H}}_{\mathbb{E}\sigma}(S) \cap S$;
    Compute ${}^{\mathcal{I}}\widetilde{\mathcal{H}}^{\mathbb{E}\sigma}(S)$ (by the formula of Definition 17);
    Compute ${}^{\mathcal{I}}\mathcal{H}^{\mathbb{E}\sigma}(S) = {}^{\mathcal{I}}\widetilde{\mathcal{H}}^{\mathbb{E}\sigma}(S) \cup S$;
    if ${}^{\mathcal{I}}\mathcal{H}_{\mathbb{E}\sigma}(S) = {}^{\mathcal{I}}\mathcal{H}^{\mathbb{E}\sigma}(S)$ **then**
        a subset S is ${}^{\mathcal{I}}\mathbb{E}\sigma$-exact;
        Print ${}^{\mathcal{I}}\mathcal{A}_{\mathbb{E}_\sigma}(W) = 1$
    else
        a subset S is ${}^{\mathcal{I}}\mathbb{E}\sigma$-rough;
        Compute ${}^{\mathcal{I}}\mathcal{A}_{\mathbb{E}_\sigma}(W) = \frac{|{}^{\mathcal{I}}\mathcal{H}_{\mathbb{E}_\sigma}(W)|}{|{}^{\mathcal{I}}\mathcal{H}^{\mathbb{E}_\sigma}(W)|}$
    end
end

Algorithm 1: Determination of whether a subset is ${}^{\mathcal{I}}\mathbb{E}\sigma$-rough or ${}^{\mathcal{I}}\mathbb{E}\sigma$-exact, along with the computation of its accuracy measure.

4 Assorted topologies generated by ideals and cardinality neighborhoods

In this part, we utilize ideals and cardinal neighborhoods to originate various topologies that are finer than those previously generated by cardinal neighborhoods as described in Al-shami et al. (2024c), for any given relation.

Theorem 6 *Let $\mathcal{I}$ be an ideal on a σ-NS $(\mathbb{X}, \mathfrak{L}, \zeta_\sigma)$. For each σ, the family ${}^{\mathcal{I}}\Omega_{\mathbb{E}_\sigma} = \{W \subseteq \mathbb{X}$: $\forall\, \mathsf{v} \in W, \mathbb{E}_\sigma(\mathsf{v})\setminus W \in \mathcal{I}\}$ constitutes a topology on $\mathbb{X}$.*

Proof Firstly, suppose $W_\iota \in {}^{\mathcal{I}}\Omega_{\mathbb{E}_\sigma}$, for each $\iota \in \Delta$. Let $\mathsf{v} \in \cup_{\iota\in\Delta} W_\iota$, then there is $\iota_0 \in \Delta$ s.t. $\mathsf{v} \in W_{\iota_0}$ and $\mathbb{E}_\sigma(\mathsf{v})\setminus W_{\iota_0} \in \mathcal{I}$. Since $W_{\iota_0} \subseteq \cup_{\iota\in\Delta} W_\iota$. Therefore, $\mathbb{E}_\sigma(\mathsf{v}) \setminus (\cup_{\iota\in\Delta} W_\iota) \in \mathcal{I}$, this means that $\cup_{\iota\in\Delta} W_\iota \in {}^{\mathcal{I}}\Omega_{\mathbb{E}_\sigma}$.

Secondly, let W_1, W_2 be elements of ${}^{\mathcal{I}}\Omega_{\mathbb{E}_\sigma}$ and v belongs to the intersection of W_1 and W_2. Then $\mathbb{E}_\sigma(\mathsf{v})\setminus W_1 \in \mathcal{I}$ and $\mathbb{E}_\sigma(\mathsf{v})\setminus W_2 \in \mathcal{I}$. Hence, $\mathbb{E}_\sigma(\mathsf{v}) \setminus [W_1 \cap W_2] \in \mathcal{I}$. This means that $W_1 \cap W_2 \in {}^{\mathcal{I}}\Omega_{\mathbb{E}_\sigma}$.

Finally, it is evident that $\emptyset, \mathbb{X} \in {}^{\mathcal{I}}\Omega_{\mathbb{E}_\sigma}$, for each σ. Consequently, ${}^{\mathcal{I}}\Omega_{\mathbb{E}_\sigma}$ is a topology on $\mathbb{X}$. □

If $W \in {}^{\mathcal{I}}\Omega_{\mathbb{E}\sigma}$, then W is said to be ${}^{\mathcal{I}}\Omega_{\mathbb{E}_\sigma}$-open set while the complement of W is called a ${}^{\mathcal{I}}\perp_{\mathbb{E}\sigma}$-closed set, where ${}^{\mathcal{I}}\perp_{\mathbb{E}\sigma} = \{F : F^c \in {}^{\mathcal{I}}\Omega_{\mathbb{E}\sigma}\}$.

Proposition 15 *Let $\mathcal{I}$ be an ideal on a σ-NS $(\mathbb{X}, \mathfrak{L}, \zeta_\sigma)$. Then*

(i) For each σ, $\Omega_{\mathbb{E}_\sigma} \subseteq {}^{\mathcal{I}}\Omega_{\mathbb{E}_\sigma}$

(ii) For each σ, ${}^{\mathcal{I}}\Omega_{\mathbb{E}_\sigma} \subseteq {}^{\mathcal{I}}\Omega_{\sharp_\sigma}$.

(iii) If $\mathfrak{L}$ is preorder relation, then ${}^{\mathcal{I}}\Omega_{\mathbb{E}\langle\sigma\rangle} = {}^{\mathcal{I}}\Omega_{\mathbb{E}_\sigma}$, for $\sigma \in \{r, l, i, u\}$.

Proof (i): Directly from the fact that $\mathbb{E}_\sigma(\mathsf{v}) \subseteq S$ for each $\mathsf{e} \in S$ implies that $\mathbb{E}_\sigma(\mathsf{v})\setminus S \in \mathcal{I}$ for each $\mathsf{e} \in S$.
(ii): By Proposition 8, we have $\sharp_\sigma(\mathsf{v}) \subseteq \mathbb{E}_\sigma(\mathsf{v})$ for each σ; therefore, we find by the property of ideal that $\mathbb{E}_\sigma(\mathsf{v}) \setminus S \in \mathcal{I}$ implies that $\sharp_\sigma(\mathsf{v}) \setminus S \in \mathcal{I}$.
(iii): By Proposition 7. □

Example 9 Continuing from Example 1.

$\Omega_{\mathbb{E}r} = \{\emptyset, \mathbb{X}, \{x\}, \{v\}, \{x, v\}, \{a, y\}, \{a, x, y\}, \{a, y, v\}\}$.
$\Omega_{\mathbb{E}l} = \{\emptyset, \mathbb{X}, \{x\}, \{a\}, \{y, v\}, \{a, x\}, \{x, y, v\}, \{a, y, v\}\}$.
$\Omega_{\mathbb{E}i} = 2^{\mathbb{X}} = \perp_{\mathbb{E}i}$.
$\Omega_{\mathbb{E}u} = \{\emptyset, \mathbb{X}, \{x\}, \{a, y, v\}\}$.
$\Omega_{\mathbb{E}\langle r\rangle} = \{\emptyset, \mathbb{X}, \{a\}, \{y\}, \{x, v\}, \{a, y\}, \{a, x, v\}, \{x, y, v\}\}$.
$\Omega_{\mathbb{E}\langle l\rangle} = \{\emptyset, \mathbb{X}, \{a\}, \{v\}, \{a, v\}, \{x, y\}, \{a, x, y\}, \{x, y, v\}\}$.
$\Omega_{\mathbb{E}\langle i\rangle} = 2^{\mathbb{X}}$.
$\Omega_{\mathbb{E}\langle u\rangle} = \{\emptyset, \mathbb{X}, \{a\}, \{x, y, v\}\}$.
If $\mathcal{I} = \emptyset, \{y\}$, then
${}^{\mathcal{I}}\Omega_{\mathbb{E}_r} = \{\emptyset, \mathbb{X}, \{a\}, \{x\}, \{v\}, \{a, x\}, \{a, y\}, \{a, v\}, \{x, v\}, \{a, x, y\}, \{a, x, v\}, \{a, y, v\}\}$.
${}^{\mathcal{I}}\Omega_{\mathbb{E}_l} = \{\emptyset, \mathbb{X}, \{a\}, \{x\}, \{v\}, \{a, x\}, \{y, v\}, \{a, v\}, \{x, v\}, \{x, y, v\}, \{a, x, v\}, \{a, y, v\}\}$.
${}^{\mathcal{I}}\Omega_{\mathbb{E}_i} = 2^{\mathbb{X}}$.
${}^{\mathcal{I}}\Omega_{\mathbb{E}_u} = \{\emptyset, \mathbb{X}, \{a\}, \{x\}, \{v\}, \{a, x\}, \{a, v\}, \{x, v\}, \{a, x, v\}, \{a, y, v\}\}$.
${}^{\mathcal{I}}\Omega_{\mathbb{E}\langle r\rangle} = \{\emptyset, \mathbb{X}, \{a\}, \{y\}, \{a, y\}, \{x, v\}, \{a, x, v\}, \{x, y, v\}\}$.
${}^{\mathcal{I}}\Omega_{\mathbb{E}\langle l\rangle} = \{\emptyset, \mathbb{X}, \{a\}, \{x\}, \{v\}, \{a, x\}, \{a, v\}, \{x, y\}, \{x, v\}, \{a, x, y\}, \{a, x, v\}, \{x, y, v\}\}$.
${}^{\mathcal{I}}\Omega_{\mathbb{E}\langle i\rangle} = 2^{\mathbb{X}}$.
${}^{\mathcal{I}}\Omega_{\mathbb{E}\langle u\rangle} = \{\emptyset, \mathbb{X}, \{a\}, \{x, v\}, \{a, x, v\}, \{x, y, v\}\}$.

Lemma 2 *Let $\mathcal{I}, \mathcal{J}$ be ideals on a σ-NS $(\mathbb{X}, \mathfrak{L}, \zeta_\sigma)$ such that $\mathcal{I} \subseteq \mathcal{J}$. Then, for all σ, we have ${}^{\mathcal{I}}\Omega_{\mathbb{E}_\sigma} \subseteq {}^{\mathcal{J}}\Omega_{\mathbb{E}_\sigma}$.*

Proof Direct to prove.

The reverse implication of Lemma 2 does not necessarily hold, as demonstrated in the next example.

Example 10 Continuing from Example 1. Let $\mathcal{I} = \{\emptyset, \{y\}\}$, $\mathcal{J} = \{\emptyset, \{a\}, \{y\}, \{a, y\}\}$, and $\sigma = r$. Then, ${}^{\mathcal{J}}\Omega_{\mathbb{E}_r} = 2^{\mathbb{X}} \nsubseteq \{\emptyset, \mathbb{X}, \{a\}, \{x\}, \{v\}, \{a, x\}, \{a, y\}, \{a, v\}, \{x, v\}, \{a, x, y\}, \{a, x, v\}, \{a, y, v\}\} = {}^{\mathcal{I}}\Omega_{\mathbb{E}_r}$.

Theorem 7 *The subsequent relations between topologies are satisfied:*

(i) ${}^{\mathcal{I}}\Omega_{\mathbb{E}_u} \subseteq {}^{\mathcal{I}}\Omega_{\mathbb{E}_r} \cap {}^{\mathcal{I}}\Omega_{\mathbb{E}_l} \subseteq {}^{\mathcal{I}}\Omega_{\mathbb{E}_r} \cup {}^{\mathcal{I}}\Omega_{\mathbb{E}_l} \subseteq {}^{\mathcal{I}}\Omega_{\mathbb{E}_i}$.

(ii) ${}^{\mathcal{I}}\Omega_{\mathbb{E}_{\langle u\rangle}} \subseteq {}^{\mathcal{I}}\Omega_{\mathbb{E}_{\langle r\rangle}} \cap {}^{\mathcal{I}}\Omega_{\mathbb{E}_{\langle l\rangle}} \subseteq {}^{\mathcal{I}}\Omega_{\mathbb{E}_{\langle r\rangle}} \cup {}^{\mathcal{I}}\Omega_{\mathbb{E}_{\langle l\rangle}} \subseteq {}^{\mathcal{I}}\Omega_{\mathbb{E}_{\langle i\rangle}}$.

Proof These relations are warranted by the first item of Proposition 2. □

Example 9 displays that ${}^{\mathcal{I}}\Omega_{\mathbb{E}_i} \neq {}^{\mathcal{I}}\Omega_{\mathbb{E}_r}$, ${}^{\mathcal{I}}\Omega_{\mathbb{E}_i} \neq {}^{\mathcal{I}}\Omega_{\mathbb{E}_l}$, ${}^{\mathcal{I}}\Omega_{\mathbb{E}_i} \neq {}^{\mathcal{I}}\Omega_{\mathbb{E}_u}$ ${}^{\mathcal{I}}\Omega_{\mathbb{E}_u} \neq {}^{\mathcal{I}}\Omega_{\mathbb{E}_r}$, and ${}^{\mathcal{I}}\Omega_{\mathbb{E}_u} \neq {}^{\mathcal{I}}\Omega_{\mathbb{E}_l}$. ${}^{\mathcal{I}}\Omega_{\mathbb{E}\langle i\rangle} \neq {}^{\mathcal{I}}\Omega_{\mathbb{E}\langle r\rangle}$, ${}^{\mathcal{I}}\Omega_{\mathbb{E}\langle i\rangle} \neq {}^{\mathcal{I}}\Omega_{\mathbb{E}\langle l\rangle}$, ${}^{\mathcal{I}}\Omega_{\mathbb{E}\langle i\rangle} \neq {}^{\mathcal{I}}\Omega_{\mathbb{E}\langle u\rangle}$, ${}^{\mathcal{I}}\Omega_{\mathbb{E}\langle u\rangle} \neq {}^{\mathcal{I}}\Omega_{\mathbb{E}\langle r\rangle}$, and ${}^{\mathcal{I}}\Omega_{\mathbb{E}\langle u\rangle} \neq {}^{\mathcal{I}}\Omega_{\mathbb{E}\langle l\rangle}$.

In the next, various types of rough approximations will be constructed by utilizing their counterparts via topologies initiated by ideals and cardinal neighborhoods. Also, some of properties of these rough approximations will be discussed.

Definition 22 Let ${}^{\mathcal{I}}\Omega_{\mathbb{E}\sigma}$ represent a topology induced by ideals and cardinality neighborhoods. Then, for each σ, the lower and upper approximations of a set $W \subseteq \mathbb{X}$ are respectively given by:

$\underline{{}^{\mathcal{I}}\delta_\sigma}(W) = {}^{\mathcal{I}}int_{\mathbb{E}_\sigma}(W)$, $\overline{{}^{\mathcal{I}}\delta_\sigma}(W) = {}^{\mathcal{I}}cl_{\mathbb{E}_\sigma}(W)$, where ${}^{\mathcal{I}}int_{\mathbb{E}_\sigma}(W)$, ${}^{\mathcal{I}}cl_{\mathbb{E}_\sigma}(W)$ respectively represent the interior and closure of a set W with respect the topology ${}^{\mathcal{I}}\Omega_{\mathbb{E}\sigma}$. Additionally, the accuracy criteria of W is assigned as: ${}^{\mathcal{I}}\lambda_{\delta_\sigma}(W) = \frac{|\underline{{}^{\mathcal{I}}\delta_\sigma}(W)|}{|\overline{{}^{\mathcal{I}}\delta_\sigma}(W)|}$, $|\overline{{}^{\mathcal{I}}\delta_\sigma}(W)| \neq 0$.

It is evident that $0 \leq^{\mathcal{I}} \lambda_{\delta_\sigma} \leq 1$. If ${}^{\mathcal{I}}\lambda_{\delta_\sigma}(W) = 1$, then W is referred to as an ${}^{\mathcal{I}}\mathbb{E}_\sigma$-exact set. Elsewise, W is termed an ${}^{\mathcal{I}}\mathbb{E}_\sigma$-rough set.

Concerning to Definition 22, the following results can be proven using the topological characteristics of interior and closure operators. It is noteworthy that certain properties absent in the ${}^{\mathcal{I}}\widetilde{\mathcal{H}}_{\mathbb{E}_\sigma}$-, ${}^{\mathcal{I}}\widetilde{\mathcal{H}}^{\mathbb{E}_\sigma}$-approximations are still valid for the $\underline{{}^{\mathcal{I}}\delta_\sigma}$-, $\overline{{}^{\mathcal{I}}\delta_\sigma}$-approximations such as item (i) of Theorem 2.

Theorem 8 *For each σ, suppose that ${}^{\mathcal{I}}\Omega_{\mathbb{E}\sigma}$ is a topology generated by ideals and cardinality neighborhoods and let $S, W \subseteq \mathbb{X}$. Then, we have the next properties:*

(i) $\underline{{}^{\mathcal{I}}\delta_\sigma}(W) \subseteq W$.

(ii) $\underline{{}^{\mathcal{I}}\delta_\sigma}(\emptyset) = \emptyset$.

(iii) $\underline{{}^{\mathcal{I}}\delta_\sigma}(\mathbb{X}) = \mathbb{X}$.

(iv) *If $S \subseteq W$, then* $\underline{{}^{\mathcal{I}}\delta_\sigma}(S) \subseteq \underline{{}^{\mathcal{I}}\delta_\sigma}(W)$.

(v) $\underline{{}^{\mathcal{I}}\delta_\sigma}(S \cap W) = \underline{{}^{\mathcal{I}}\delta_\sigma}(S) \cap \underline{{}^{\mathcal{I}}\delta_\sigma}(W)$.

(vi) $\underline{^{\mathcal{I}}\delta_\sigma}(W^c) = (\overline{^{\mathcal{I}}\delta_\sigma}(W))^c$.

(vii) $\underline{^{\mathcal{I}}\delta_\sigma}(\underline{^{\mathcal{I}}\delta_\sigma}(W)) = \underline{^{\mathcal{I}}\delta_\sigma}(W)$ *for each* σ.

Proof These relations are valid due to the correspondence between interior topological and lower approximation operators. □

Corollary 8 *For each* σ*, suppose that* $^{\mathcal{I}}\Omega_{\mathbb{E}\sigma}$ *is a topology induced by ideals and cardinality neighborhoods. Then* $\underline{^{\mathcal{I}}\delta_\sigma}(S) \cup \underline{^{\mathcal{I}}\delta_\sigma}(W) \subseteq \underline{^{\mathcal{I}}\delta_\sigma}(S \cup W)$ *for any* $S, W \subseteq \mathbb{X}$.

Theorem 9 *For each* σ*, suppose that* $^{\mathcal{I}}\Omega_{\mathbb{E}\sigma}$ *is a topology generated by ideals and cardinality neighborhoods and let* $S, W \subseteq \mathbb{X}$*. Then, we have the next properties:*

(i) $W \subseteq \overline{^{\mathcal{I}}\delta_\sigma}(W)$.

(ii) $\overline{^{\mathcal{I}}\delta_\sigma}(\emptyset) = \emptyset$.

(iii) $\overline{^{\mathcal{I}}\delta_\sigma}(\mathbb{X}) = \mathbb{X}$.

(iv) *If* $S \subseteq W$*, then* $\overline{^{\mathcal{I}}\delta_\sigma}(S) \subseteq^{\mathcal{I}} \overline{\delta_\sigma}(W)$.

(v) $\overline{^{\mathcal{I}}\delta_\sigma}(S \cup W) = \overline{^{\mathcal{I}}\delta_\sigma}(S) \cup \overline{^{\mathcal{I}}\delta_\sigma}(W)$.

(vi) $\overline{^{\mathcal{I}}\delta_\sigma}(W^c) = (\underline{^{\mathcal{I}}\delta_\sigma}(W))^c$.

(vii) $\overline{^{\mathcal{I}}\delta_\sigma}(\overline{^{\mathcal{I}}\delta_\sigma}(W)) = \overline{^{\mathcal{I}}\delta_\sigma}(W)$ *for each* σ.

Proof These relations are valid due to the correspondence between closure topological and upper approximation operators. □

Corollary 9 *Let* $^{\mathcal{I}}\Omega_{\mathbb{E}\sigma}$ *be a topology induced by ideals and cardinality neighborhoods. Then* $\overline{^{\mathcal{I}}\delta_\sigma}(S \cap W) \subseteq \overline{^{\mathcal{I}}\delta_\sigma}(S) \cap \overline{^{\mathcal{I}}\delta_\sigma}(W)$ *for any* $S, W \subseteq \mathbb{X}$.

Proposition 16 *If* $\emptyset \neq W \subseteq \mathbb{X}$*, then* $0 \leq {^{\mathcal{I}}\lambda_{\delta_\sigma}}(W) \leq 1$ *and* $^{\mathcal{I}}\lambda_{\delta_\sigma}(\mathbb{X}) = 1$ *for any* σ.

Proposition 17 *The subsequent inclusion relations are valid for every subset* S *of a topological space* $(\mathbb{X}, ^{\mathcal{I}}\Omega_{\mathbb{E}\sigma})$*:*

(i) $\underline{^{\mathcal{I}}\delta_u}(S) \subseteq \underline{^{\mathcal{I}}\delta_r}(S) \cap \underline{^{\mathcal{I}}\delta_l}(S) \subseteq \underline{^{\mathcal{I}}\delta_r}(S) \cup \underline{^{\mathcal{I}}\delta_l}(S) \subseteq \underline{^{\mathcal{I}}\delta_i}(S)$.

(ii) $\overline{^{\mathcal{I}}\delta_i}(S) \subseteq \overline{^{\mathcal{I}}\delta_r}(S) \cap \overline{^{\mathcal{I}}\delta_l}(S) \subseteq \overline{^{\mathcal{I}}\delta_r}(S) \cup \overline{^{\mathcal{I}}\delta_l}(S) \subseteq \overline{^{\mathcal{I}}\delta_u}(S)$.

(iii) $\underline{^{\mathcal{I}}\delta_{\langle u\rangle}}(S) \subseteq \underline{^{\mathcal{I}}\delta_{\langle r\rangle}}(S) \cap \underline{^{\mathcal{I}}\delta_{\langle l\rangle}} \subseteq \underline{^{\mathcal{I}}\delta_{\langle r\rangle}}(S) \cup \underline{^{\mathcal{I}}\delta_{\langle l\rangle}}(S) \subseteq \underline{^{\mathcal{I}}\delta_{\langle i\rangle}}(S)$.

(iv) $\overline{^{\mathcal{I}}\delta_{\langle i\rangle}}(S) \subseteq \overline{^{\mathcal{I}}\delta_{\langle r\rangle}}(S) \cap \overline{^{\mathcal{I}}\delta_{\langle l\rangle}}(S) \subseteq \overline{^{\mathcal{I}}\delta_{\langle r\rangle}}(S) \cup \overline{^{\mathcal{I}}\delta_{\langle l\rangle}}(S) \subseteq \overline{^{\mathcal{I}}\delta_{\langle u\rangle}}(S)$.

Corollary 10 *The subsequent inequalities are valid for every nonempty subset* S *of a topological space* $(\mathbb{X}, ^{\mathcal{I}}\Omega_{\mathbb{E}\sigma})$*:*

(i) $^{\mathcal{I}}\lambda_{\delta_u}(S) \leq^{\mathcal{I}} \lambda_{\delta_r}(S) \leq^{\mathcal{I}} \lambda_{\delta_i}(S)$.

(ii) $^{\mathcal{I}}\lambda_{\delta_u}(S) \leq^{\mathcal{I}} \lambda_{\delta_l}(S) \leq^{\mathcal{I}} \lambda_{\delta_i}(S)$.

(iii) $^{\mathcal{I}}\lambda_{\delta\langle u\rangle}(S) \leq^{\mathcal{I}} \lambda_{\delta\langle r\rangle}(S) \leq^{\mathcal{I}} \lambda_{\delta\langle i\rangle}(S)$.
(iv) $^{\mathcal{I}}\lambda_{\delta\langle u\rangle}(S) \leq^{\mathcal{I}} \lambda_{\delta\langle l\rangle}(S) \leq^{\mathcal{I}} \lambda_{\delta\langle i\rangle}(S)$.

The approximations and accuracy criteria presented in this section, founded on topological spaces, will now be compared with the methods discussed in the previous section.

Table 6 Information system of sample of patients

Patients	Rashes	Fever	Headache	Vomiting	Fatigue	Decision
p_1	+	+	−	−	+	✓
p_2	−	+	+	+	+	✓
p_3	+	+	−	+	−	✓
p_4	−	−	+	+	−	✗
p_5	+	+	−	−	+	✗
p_6	+	+	−	+	+	✗
p_7	+	+	−	+	−	✓
p_8	+	−	−	−	−	✗

Proposition 18 *For each σ and $S \subseteq \mathbb{X}$, we have the following relations:*

(i) $\underline{{}^{\mathcal{I}}\delta_\sigma}(S) \subseteq {}^{\mathcal{I}}\mathcal{H}_{\mathbb{E}_\sigma}(S)$, *and*

(ii) $\overline{{}^{\mathcal{I}}\delta_\sigma}(S) \supseteq {}^{\mathcal{I}}\mathcal{H}^{\mathbb{E}_\sigma}(S)$

Proof To prove (i). Let $\mathsf{a} \in \underline{{}^{\mathcal{I}}\delta_\sigma}(S)$. Then we find a subset $V \in {}^{\mathcal{I}}\Omega_\sigma$ with $\mathsf{a} \in V \subseteq S$. It follows from the way of structuring topology, we obtain $\mathbb{E}_\sigma(\mathsf{a}) \setminus V \in \mathcal{I}$. Now, we get $\mathbb{E}_\sigma(\mathsf{a}) \setminus S \in \mathcal{I}$ since $V \subseteq S$. Hence, $\mathsf{a} \in {}^{\mathcal{I}}\widetilde{\mathcal{H}}_{\mathbb{E}_\sigma}(S)$. Since $\mathsf{a} \in S$, then $\mathsf{a} \in {}^{\mathcal{I}}\mathcal{H}_{\mathbb{E}_\sigma}(S)$ and so $\underline{{}^{\mathcal{I}}\delta_\sigma}(S) \subseteq {}^{\mathcal{I}}\mathcal{H}_{\mathbb{E}_\sigma}(S)$. By the same manner, one can prove (ii). □

The converse of proposition 18 need not to be true, refer to Table 2 and Example 9. Suppose that $\sigma = r$ and $W = \{\mathsf{a}, \mathsf{x}\}$. Then ${}^{\mathcal{I}}\mathcal{H}_{\mathbb{E}_\sigma}(S) = \{\mathsf{a}, \mathsf{x}, \mathsf{y}\}$, $\underline{{}^{\mathcal{I}}\delta_\sigma}(S) = \{\mathsf{a}, \mathsf{x}\}$. Hence, ${}^{\mathcal{I}}\mathcal{H}_{\mathbb{E}_\sigma}(S) \nsubseteq \underline{{}^{\mathcal{I}}\delta_\sigma}(S)$.

Corollary 11 *For each σ, consider $\mathcal{I}$ is an ideal on a σ-NS $(\mathbb{X}, \mathfrak{L}, \zeta_\sigma)$. If $S \subseteq \mathbb{X}$, then* ${}^{\mathcal{I}}\lambda_{\delta_\sigma}(S) \leq {}^{\mathcal{I}}\mathcal{A}_{\mathbb{E}_\sigma}(S)$.

5 An examination of the current rough-set paradigms for analyzing the diagnosis of dengue

In this practical section, we evaluate the effectiveness of the suggested models in handling dengue information systems for certain patients. We illustrate how the present method enhances decision-making and how we leverage a topological technique to identify the most critical symptoms for determining infection of dengue. Based on the following analysis, we conclude that the proposed rough-set paradigms outperform their counterparts of rough-set paradigms based on cardinality neighborhoods without the use of ideals. Additionally, we refer to the limitations associated with the method outlined in Sect. 3.1.

Table 6 presents data for a group of eight patients, denoted as $\mathbb{X} = \{p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8\}$, along with their corresponding symptoms, which are represented as conditional attributes: rashes, fever, headache, vomiting, and fatigue. The dengue diagnosis is considered the decision attribute. For each conditional attribute (symptom), we assign a value of "+" or "-" to indicate whether the symptom is present or absent in the patient. Similarly, the decision attribute is marked with either "✓" or "✗" to denote a positive or negative dengue report, respectively.

Presume that the system's expert suggested the next relation $\mathfrak{L}$ on the set of patients $\mathbb{X}$, to describe the connections between patients according to their symptoms:

$$\mathsf{p}_i \mathfrak{L} \mathsf{p}_j \iff \text{the common positive symptoms between } \mathsf{p}_i \text{ and } \mathsf{p}_j \text{ are more than two}$$

Then $\mathfrak{L} = \{(\mathsf{p}_1, \mathsf{p}_1\}), (\mathsf{p}_2, \mathsf{p}_2\}), (\mathsf{p}_3, \mathsf{p}_3\}), (\mathsf{p}_5, \mathsf{p}_5\}), (\mathsf{p}_6, \mathsf{p}_6\}), (\mathsf{p}_7, \mathsf{p}_7\}), (\mathsf{p}_1, \mathsf{p}_5\}), (\mathsf{p}_5, \mathsf{p}_1\}), (\mathsf{p}_1, \mathsf{p}_6\}), (\mathsf{p}_6, \mathsf{p}_1\}), (\mathsf{p}_2, \mathsf{p}_6\}), (\mathsf{p}_6, \mathsf{p}_2\}), (\mathsf{p}_3, \mathsf{p}_6\}), (\mathsf{p}_6, \mathsf{p}_3\}), (\mathsf{p}_3, \mathsf{p}_7\}), (\mathsf{p}_7, \mathsf{p}_3\}),$

$(\mathsf{p}_5, \mathsf{p}_6\}), (\mathsf{p}_6, \mathsf{p}_5\}), (\mathsf{p}_6, \mathsf{p}_7\}), (\mathsf{p}_7, \mathsf{p}_6\})\}$.

Remark that $\mathfrak{L}$ is a symmetric relation. In contrast, we have $(\mathsf{p}_4, \mathsf{p}_4\}) \notin \mathfrak{L}$ so $\mathfrak{L}$ is not a reflexive relation; also, we have $(\mathsf{p}_2, \mathsf{p}_6\}), (\mathsf{p}_6, \mathsf{p}_5\}) \in \mathfrak{L}$ but $(\mathsf{p}_2, \mathsf{p}_5\}) \notin \mathfrak{L}$ so $\mathfrak{L}$ is not a transitive relation. We begin processing the data, described by the given relation, by constructing the $\mathbb{E}_\sigma$-neighborhood systems. Due to the symmetry of the proposed relation, we infer that all $\mathbb{E}_\sigma$-neighborhoods are identical as established in Proposition 2. Table 7 presents the $\mathbb{E}_\sigma$-neighborhood for each patient.

Also, let $\mathcal{I} = \{\emptyset, \{\mathsf{p}_2\}, \{\mathsf{p}_5\}, \{\mathsf{p}_2, \mathsf{p}_5\}\}$ refer to the ideal given by the expert. In the frameworks of the present rough-set models and those given in Al-shami et al. (2024c), we calculate, in the following items, the approximations (lower and upper) and accuracy for $S = \{\mathsf{p}_1, \mathsf{p}_2, \mathsf{p}_3, \mathsf{p}_7\}$, which represents a set of patients with a positive report of dengue:

– Rough-set model presented in Al-shami et al. (2024c).

  (i) $\mathcal{H}_{\mathbb{E}_\sigma}(S) = \{\mathsf{p}_2\}$,

  (ii) $\mathcal{H}^{\mathbb{E}_\sigma}(S) = \{\mathsf{p}_1, \mathsf{p}_2, \mathsf{p}_3, \mathsf{p}_5, \mathsf{p}_7\}$,

  (iii) $\mathbb{B}_{\mathbb{E}_\sigma}(S) = \mathcal{H}^{\mathbb{E}_\sigma}(S) \setminus \mathcal{H}_{\mathbb{E}_\sigma}(S) = \{\mathsf{p}_1, \mathsf{p}_3, \mathsf{p}_5, \mathsf{p}_7\}$, and

  (iv) $\mathbb{A}_{\mathbb{E}_\sigma}(S) = \frac{|\mathcal{H}_{\mathbb{E}_\sigma}(S)|}{|\mathcal{H}^{\mathbb{E}_\sigma}(S)|} = \frac{1}{5}$.

– Our rough-set model presented in Sect. 3.1.

  (i) ${}^{\mathcal{I}}\widetilde{\mathcal{H}}_{\mathbb{E}_\sigma}(S) = \{\mathsf{p}_1, \mathsf{p}_2, \mathsf{p}_3, \mathsf{p}_5, \mathsf{p}_7\}$ and

  (ii) ${}^{\mathcal{I}}\widetilde{\mathcal{H}}^{\mathbb{E}_\sigma}(S) = \{\mathsf{p}_1, \mathsf{p}_3, \mathsf{p}_5, \mathsf{p}_7\}$.

– Our rough-set model presented in Sect. 3.2.

  (i) ${}^{\mathcal{I}}\mathcal{H}_{\mathbb{E}_\sigma}(S) = S$,

  (ii) ${}^{\mathcal{I}}\mathcal{H}^{\mathbb{E}_\sigma}(S) = \{\mathsf{p}_1, \mathsf{p}_2, \mathsf{p}_3, \mathsf{p}_5, \mathsf{p}_7\}$,

  (iii) ${}^{\mathcal{I}}\mathbb{B}_{\mathbb{E}_\sigma}(S) = {}^{\mathcal{I}}\mathcal{H}^{\mathbb{E}_\sigma}(S) \setminus {}^{\mathcal{I}}\mathcal{H}_{\mathbb{E}_\sigma}(S) = \{\mathsf{p}_5\}$, and

  (iv) ${}^{\mathcal{I}}\mathcal{A}_{\mathbb{E}_\sigma}(S) = \frac{|{}^{\mathcal{I}}\mathcal{H}_{\mathbb{E}_\sigma}(S)|}{|{}^{\mathcal{I}}\mathcal{H}^{\mathbb{E}_\sigma}(S)|} = \frac{4}{5}$.

– Topological models established in Sect. 4. To apply these models, we first initiate a topology according to Table 7 as follows: ${}^{\mathcal{I}}\Omega_{\mathbb{E}_\sigma} = \{\emptyset, \mathbb{X}, \{\mathsf{p}_2\}, \{\mathsf{p}_6\}, \{\mathsf{p}_2, \mathsf{p}_6\}, \{\mathsf{p}_4, \mathsf{p}_8\}, \{\mathsf{p}_2, \mathsf{p}_4, \mathsf{p}_8\}, \{\mathsf{p}_4, \mathsf{p}_6, \mathsf{p}_8\}, \{\mathsf{p}_2, \mathsf{p}_4, \mathsf{p}_6, \mathsf{p}_8\}, \{\mathsf{p}_1, \mathsf{p}_3, \mathsf{p}_7\}, \{\mathsf{p}_1, \mathsf{p}_2, \mathsf{p}_3, \mathsf{p}_7\}, \{\mathsf{p}_1, \mathsf{p}_3, \mathsf{p}_5, \mathsf{p}_7\}, \{\mathsf{p}_1, \mathsf{p}_2, \mathsf{p}_3, \mathsf{p}_5, \mathsf{p}_7\}, \{\mathsf{p}_1, \mathsf{p}_3, \mathsf{p}_5, \mathsf{p}_6, \mathsf{p}_7\}, \{\mathsf{p}_1, \mathsf{p}_2, \mathsf{p}_3, \mathsf{p}_5, \mathsf{p}_6, \mathsf{p}_7\}\{\mathsf{p}_1, \mathsf{p}_3, \mathsf{p}_6, \mathsf{p}_7\}, \{\mathsf{p}_1, \mathsf{p}_2, \mathsf{p}_3, \mathsf{p}_6, \mathsf{p}_7\}, \{\mathsf{p}_1, \mathsf{p}_3, \mathsf{p}_4, \mathsf{p}_7, \mathsf{p}_8\}, \{\mathsf{p}_1, \mathsf{p}_2, \mathsf{p}_3, \mathsf{p}_4, \mathsf{p}_7, \mathsf{p}_8\}, \{\mathsf{p}_1, \mathsf{p}_3, \mathsf{p}_4, \mathsf{p}_6, \mathsf{p}_7, \mathsf{p}_8\}, \{\mathsf{p}_1, \mathsf{p}_3, \mathsf{p}_4, \mathsf{p}_5, \mathsf{p}_6, \mathsf{p}_7, \mathsf{p}_8\}, \{\mathsf{p}_1, \mathsf{p}_2, \mathsf{p}_3, \mathsf{p}_4, \mathsf{p}_5, \mathsf{p}_7, \mathsf{p}_8\}, \{\mathsf{p}_1, \mathsf{p}_3, \mathsf{p}_4, \mathsf{p}_5, \mathsf{p}_7, \mathsf{p}_8\}\}$. Then, we calculate, in the following items, the approximations (lower and upper) and accuracy for S:

  (i) ${}^{\mathcal{I}}\underline{\delta_\sigma}(S) = {}^{\mathcal{I}}int_{\mathbb{E}_\sigma}(S) = S$,

Table 7 $\mathbb{E}_\sigma$ for each patient

	p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8
$\mathcal{D}_r()$	$\{p_1, p_5, p_6\}$	$\{p_2, p_6\}$	$\{p_3, p_6, p_7\}$	$\emptyset$	$\{p_1, p_5, p_6\}$	$\{p_1, p_2, p_3, p_5, p_6, p_7\}$	$\{p_3, p_6, p_7\}$	$\emptyset$
$\mathcal{D}_{\langle r \rangle}()$	$\{p_1, p_5, p_6\}$	$\{p_2, p_6\}$	$\{p_3, p_6, p_7\}$	$\emptyset$	$\{p_1, p_5, p_6\}$	$\{p_6\}$	$\{p_3, p_6, p_7\}$	$\emptyset$
$\mathbb{E}_r()$	$\{p_1, p_3, p_5, p_7\}$	$\{p_2\}$	$\{p_1, p_3, p_5, p_7\}$	$\{p_4, p_8\}$	$\{p_1, p_3, p_5, p_7\}$	$\{p_6\}$	$\{p_1, p_3, p_5, p_7\}$	$\{p_4, p_8\}$

(ii) $\overline{^{\mathcal{I}}\delta_{\sigma}}(S) = {}^{\mathcal{I}}cl_{\mathbb{E}_{\sigma}}(S) = \{\mathsf{p}_1, \mathsf{p}_2, \mathsf{p}_3, \mathsf{p}_5, \mathsf{p}_7\}$,

(iii) $^{\mathcal{I}}\mathbb{B}_{\mathbb{E}_{\sigma}}(S) = {}^{\mathcal{I}} cl_{\mathbb{E}_{\sigma}}(S) \setminus {}^{\mathcal{I}} int_{\mathbb{E}_{\sigma}}(S) = \{\mathsf{p}_5\}$, and

(iv) $^{\mathcal{I}}\lambda_{\delta_{\sigma}}(S) = \frac{|\underline{^{\mathcal{I}}\delta_{\sigma}}(S)|}{|\overline{^{\mathcal{I}}\delta_{\sigma}}(S)|} = \frac{4}{5}$.

Based on the above outcomes, it is evident that these computations align with the findings presented in Theorem 5 and Corollary 7. In summary, one can note that the rough-set models proposed in Sect. 3.2 enhance the lower and upper approximations, thereby improving the accuracy measures of subsets compared to rough-set models described in Al-shami et al. (2024c). Moreover, the paradigms of rough sets Sect. 3.2 generate the same approximation spaces induced by topological approach displayed in Sect. 4. Let's compare the computations given by the above four rough set models. We remark that the best approximations (lower and upper) are obtained by the model introduced in Sect. 3.1. However, this model suffers some failures such as preserving the main characteristics of approximation operators and illogical measures of accuracy.

In the remainder of this section, we utilize the topological spaces, generated earlier using ideals and cardinality neighborhoods, to identify the key symptoms for determining whether a patient is infected with dengue disease. The original topology is: $^{\mathcal{I}}\Omega_{\mathbb{E}\sigma} = \{\emptyset, \mathbb{X}, \{\mathsf{p}_2\}, \{\mathsf{p}_6\}, \{\mathsf{p}_2, \mathsf{p}_6\}, \{\mathsf{p}_4, \mathsf{p}_8\}, \{\mathsf{p}_2, \mathsf{p}_4, \mathsf{p}_8\}, \{\mathsf{p}_4, \mathsf{p}_6, \mathsf{p}_8\}, \{\mathsf{p}_2, \mathsf{p}_4, \mathsf{p}_6, \mathsf{p}_8\}, \{\mathsf{p}_1, \mathsf{p}_3, \mathsf{p}_7\}, \{\mathsf{p}_1, \mathsf{p}_2, \mathsf{p}_3, \mathsf{p}_7\}, \{\mathsf{p}_1, \mathsf{p}_3, \mathsf{p}_5, \mathsf{p}_7\}, \{\mathsf{p}_1, \mathsf{p}_2, \mathsf{p}_3, \mathsf{p}_5, \mathsf{p}_7\}, \{\mathsf{p}_1, \mathsf{p}_3, \mathsf{p}_5, \mathsf{p}_6, \mathsf{p}_7\}, \{\mathsf{p}_1, \mathsf{p}_2, \mathsf{p}_3, \mathsf{p}_5, \mathsf{p}_6, \mathsf{p}_7\}\{\mathsf{p}_1, \mathsf{p}_3, \mathsf{p}_6, \mathsf{p}_7\}, \{\mathsf{p}_1, \mathsf{p}_2, \mathsf{p}_3, \mathsf{p}_6, \mathsf{p}_7\}, \{\mathsf{p}_1, \mathsf{p}_3, \mathsf{p}_4, \mathsf{p}_7, \mathsf{p}_8\}, \{\mathsf{p}_1, \mathsf{p}_2, \mathsf{p}_3, \mathsf{p}_4, \mathsf{p}_7, \mathsf{p}_8\}, \{\mathsf{p}_1, \mathsf{p}_3, \mathsf{p}_4, \mathsf{p}_6, \mathsf{p}_7, \mathsf{p}_8\}, \{\mathsf{p}_1, \mathsf{p}_3, \mathsf{p}_4, \mathsf{p}_5, \mathsf{p}_6, \mathsf{p}_7, \mathsf{p}_8\}, \{\mathsf{p}_1, \mathsf{p}_2, \mathsf{p}_3, \mathsf{p}_4, \mathsf{p}_5, \mathsf{p}_7, \mathsf{p}_8\}, \{\mathsf{p}_1, \mathsf{p}_3, \mathsf{p}_4, \mathsf{p}_5, \mathsf{p}_7, \mathsf{p}_8\}\}$. We will now compare the original topology generated from the patient information system presented in Table 6, with the topologies generated from the same table after removing one symptom at a time. This process will be repeated for each symptom individually.

(i) If the symptom "rashes" is excluded from the input attributes, then $\mathfrak{L}_{rashes} = \{(\mathsf{p}_2, \mathsf{p}_2), (\mathsf{p}_6, \mathsf{p}_6), (\mathsf{p}_2, \mathsf{p}_6), (\mathsf{p}_6, \mathsf{p}_2)\}$. It is clear that $^{\mathcal{I}}\Omega_{\mathbb{E}\sigma} - rashes \neq {}^{\mathcal{I}}\Omega_{\mathbb{E}\sigma}$.

(ii) If symptom "fever" is removed from the input attributes, then we obtain $^{\mathcal{I}}\Omega_{\mathbb{E}\sigma} - fever \neq {}^{\mathcal{I}}\Omega_{\mathbb{E}\sigma}$.

(iii) If the symptom "headache" is neglected from the input attributes, then a relation $\mathfrak{L}$ is generated. Therefore, $^{\mathcal{I}}\Omega_{\mathbb{E}\sigma} - headache = {}^{\mathcal{I}}\Omega_{\mathbb{E}\sigma}$.

(iv) If the symptom "vomiting" is omitted from the input attributes, then $^{\mathcal{I}}\Omega_{\mathbb{E}\sigma} - vomiting \neq {}^{\mathcal{I}}\Omega_{\mathbb{E}\sigma}$.

(v) If the symptom "fatigue" is canceled from the input attributes, then we also obtain $^{\mathcal{I}}\Omega_{\mathbb{E}\sigma} - fatigue \neq {}^{\mathcal{I}}\Omega_{\mathbb{E}\sigma}$.

Based on the above computations, it can be concluded that rashes, fever, vomiting, and fatigue are the core attributes. In other words, these symptoms are identified as the key indicators for determining whether a patient is infected with dengue disease. In contrast, removing the symptom of headache does not alter the structure of the topology formed; therefore, this symptom can be omitted during examinations.

In Algorithm 2 below, we articulate how the core set of symptoms is calculated depending on the topology inspired by $\mathbb{E}\sigma$-neighborhoods and ideal structure.

Input : A set of patients $\mathbb{X}$, symptoms of dengue disease (attributes set) S, and report of each patient describes his/her case for each symptom.
Output: Specify the key symptoms to judge that the patient has dengue.

Build information systems describing symptoms of each patient as offered by his/her report;
Determine the relation $\mathfrak{L}$ to classify the patients (this is the task of experts);
Extract the elements of relation $\mathfrak{L}$ using the data displayed in the information system;
for *each patient* p **do**
    Compute $\mathcal{D}_\sigma(\mathsf{p})$
end
for *each patient* p **do**
    Compute $\mathbb{E}\sigma(\mathsf{p})$
end
Give the ideal structure as constructed by experts;
Structure the topology ${}^{\mathcal{I}}\Omega_{\mathbb{E}\sigma}$ using method provided by Theorem 6;
for *each attribute* s *of* S **do**
    Remove the attribute s from the information system;
    Extract the elements of relation $\mathfrak{L} - s$ using the data displayed in the information system;
    for *each patient* **do**
        Compute $\mathcal{D}_\sigma(\mathsf{p})$ with respect to $\mathfrak{L} - s$
    end
    for *each patient* p **do**
        Compute $\mathbb{E}\sigma(\mathsf{p})$ with respect to $\mathfrak{L} - s$
    end
    Structure the topology ${}^{\mathcal{I}}\Omega^{s}_{\mathbb{E}\sigma}$ using method provided by Theorem 6;
    if ${}^{\mathcal{I}}\Omega^{s}_{\mathbb{E}\sigma} = {}^{\mathcal{I}}\Omega_{\mathbb{E}\sigma}$ **then**
        an attribute s is one of the core symptoms;
        Put $s \in A$
    else
        an attribute s is not important symptom
    end
end
Print the core set of symptoms A.

Algorithm 2: Determine the core attributes to judge whether the patient has dengue disease or not

6 Discussions: strengths and limitations

Herein, we tackle the advantages and disadvantages of the proposed rough-set paradigms as follows.

- Advantages
  (i) The binary relation applied to define the types of approximation operators introduced herein is free from the restrict condition of an equivalence relation imposed in Pawlak

model. Also, the current models does not stipulate any type of relation such as those models introduced in Abo-Tabl (2013, 2011); Dai et al. (2018); Salama et al. (2022).

(*ii*) Two kinds of rough sets models presented in this work preserve most properties of Pawlak model (mentioned in Proposition 1) as clarified in Sects. 3.2 and 4; this matter is illustrated in Theorem 4, Theorem 8 and Theorem 9. Also, these two models overcome shortcomings of the previous models that appear in the formula of accuracy criteria or the illogical characteristics of lower and upper approximations; see, El-Bably et al. (2021). We draw the reader's attention to the fact that the property of the distribution of intersections between the lower approximations of subsets, and likewise, the property of the distribution of unions between the upper approximations of subsets, are missing in most previous models. Whereas, in the current models, these properties are preserved.

(*iii*) The rough set models suggested in this work provide an efficient instrument to address some real situations they focus on the cardinality number of $\mathcal{D}_\sigma$-neighborhoods such as those are applied in the social media or used to category the applicants according to the number of their qualities.

(*iv*) This work provides an alternative instrument inspired by topological structures, which helps a wide range of users to choose the suitable methods with their expertise. That is, users with abstract backgrounds prefer to deal with the topological approach because of the ease of computing the approximation operators from their corresponding interior and closure topological operators.

– Limitations

(*i*) The efficiency of the present rough set models is less than the rough-set paradigms produced by $\sharp$-neighborhoods (Atef et al. 2020; Mareay 2016) in terms of enlarging the upper approximation and shrinking the lower approximation.

(*ii*) Our model displayed in Sect. 3.1 loses the main characteristic of rough set models that reports that a set $S \subseteq \mathbb{X}$ is a superset of its lower approximation and a subset of its upper approximation; that is, ${}^{\mathcal{I}}\widetilde{\mathcal{H}}_{\mathbb{E}_\sigma}(S) \subseteq S \subseteq {}^{\mathcal{I}}\widetilde{\mathcal{H}}^{\mathbb{E}_\sigma}(S)$.

(*iii*) Maximization or minimization of the given relation will certainly lead to a change in the number of elements in some neighborhoods, and thus the cardinal rough neighborhoods of these elements will change in a way that cannot be determined. Therefore, the existing models fail to satisfy the monotonicity property.

7 Conclusion and future work

Rough set theory, proposed by Polish mathematician Pawlak in 1982, is a powerful mathematical tool for effectively transacting with imprecise and uncertain information. A key advantage of rough set theory is its ability to represent data using the granular structure without requiring any a priori information beyond the dataset itself. As we know, the granular structure represented by equivalence classes has been updated using neighborhood systems inspired by arbitrary relations, which assists in canceling a strict condition of an equivalence relation. However, the insufficiency of recent models has appeared in relation to keeping the main features of the original paradigm and the invalidity of some formulas applied to measure confirmed and possible knowledge.

Stimulated by the notions of cardinality neighborhoods and ideals, we have introduced novel types of generalized approximation spaces in this manuscript. The first type has proved its efficiency in extracting as much knowledge as possible from subsets of data. However, it suffers from weakness concerning the violation of some properties of the original model. To overcome this obstacle, we have presented the second type of approximation space which enlarges the obtained knowledge to an acceptable extent and preserves the characteristics of the original model. We have derived the related useful properties of these models and illustrate their validity to improve the approximation operators. After that, we have built a topological frame to represent the proposed rough set model. We have exhibited the characterizations of the rough topological model and elucidated its relationships with its counterpart defined without ideal structure. According to the analysis displayed in the medical example of dengue disease, we can say that the rough set models adopted herein outperform the existing models.

Our future plan is as follows:

– expand the current paradigms using a finite set of arbitrary relations and ideals aiming to minimize the upper approximation and maximize the lower approximation.
– explore novel rough set models generated by $\mathbb{E}_\sigma$-neighborhoods and an ideal structure $\mathcal{I}$ generated by two ideals $\mathcal{I}_1, \mathcal{I}_2$ as follows $\mathcal{I} = \{W \cup S : W \in \mathcal{I}_1, S \in \mathcal{I}_2\}$.
– discuss the notions presented herein in the frames of fuzzy and soft settings.

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Conflict of interest The authors declare that there is no Conflict of interest regarding the publication of this article.

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