symmetry

Article

Some Estimates for Certain q-analogs of Gamma Integral Transform Operators

Shrideh Al-Omari [1,*], Wael Salameh [2] and Sharifah Alhazmi [3]

[1] Department of Scientific Basic Sciences, Faculty of Engineering Technology, Al-Balqa Applied University, Amman 11134, Jordan
[2] Faculty of Information Technology, Abu Dhabi University, Abu Dhabi 59911, United Arab Emirates; wael.salameh@adu.ac.ae
[3] Mathematics Department, Al-Qunfudah University College, Umm Al-Qura University, Mecca 24382, Saudi Arabia
* Correspondence: shridehalomari@bau.edu.jo

Abstract: The aim of this work is to examine some q-analogs and differential properties of the gamma integral operator and its convolution products. The q-gamma integral operator is introduced in two versions in order to derive pertinent conclusions regarding the q-exponential functions. Also, new findings on the q-trigonometric, q-sine, and q-cosine functions are extracted. In addition, novel results for first and second-order q-differential operators are established and extended to Heaviside unit step functions. Lastly, three crucial convolution products and extensive convolution theorems for the q-analogs are also provided.

Keywords: q-gamma operator; integral operator; q-analogs; differential operator; symmetrical polynomials

check for updates

Citation: Al-Omari, S.; Salameh, W.; Alhazmi, S. Some Estimates for Certain q-analogs of Gamma Integral Transform Operators. *Symmetry* **2024**, *16*, 1368. https://doi.org/10.3390/sym16101368

Academic Editor: Ioan Rașa

Received: 19 September 2024
Revised: 30 September 2024
Accepted: 12 October 2024
Published: 15 October 2024

1. Introduction

Within the subject of the classical mathematical analysis, one essential topic of study is the quantum (or q-) calculus. It focuses on a useful theoretical generalization of differentiation and integration processes. Notably, Bernoulli and Euler's functions are the roots of a lengthy heritage in quantum calculus. But due to its many uses, it has drawn the attention of modern mathematicians in recent years [1,2]. An application of the said theory spans various symmetrical mathematical domains including number theory, symmetry of orthogonal polynomials, combinatorics, relativity theory, fractional calculus and mechanics [3–11]. The intriguing relationship between quantum calculus and these areas continues to captivate scholars from around the world. In the field of literature, significant advancements and practical implementations of the q-calculus theory have emerged, particularly in the realm of mathematical physics [12–17]. These developments revolve around hypergeometric functions, polynomials and many others, which have extensive applications in several areas such as partitions, symmetry, integral transforms and number theory [3,18–28]. This article discusses two q-analogs of a gamma integral operator and their application to various classes of polynomials and special functions. It also establishes some convolution theorems for the given analogs. Below, are some definitions and concepts from the q-calculus theory. For $0 < q < 1$, the concept of the q-analog $d_q\vartheta(\xi) = \vartheta(\xi) - \vartheta(q\xi)$ of the differential of a function ϑ was the first stone in the quantum calculus theory, which led to the idea of the q-derivative [29]

$$\frac{d_q\vartheta(\xi)}{d_q\xi} = \frac{\vartheta(\xi) - \vartheta(q\xi)}{(1-q)\xi}.$$

While it is highly beneficial to think about an object's unique q-analog, scientists occasionally think about several q-analogs of the same object. The factorial of an integer j, the integer j, and the binomial coefficient $\binom{j}{n}$ each have q-analogs defined as follows [30]

$$[j]_q = \frac{1-q^j}{1-q},\ \left([j]_q\right)! = \begin{cases} \prod_{k=1}^{j} [k]_q, & j = 1,2,3,\dots \\ 1, & j = 0 \end{cases} \text{ and } \begin{bmatrix} j \\ n \end{bmatrix}_q = \prod_{k=1}^{n} \frac{1-q^{j-k+1}}{1-q^k}.$$

The examination of the q-analogs of the classical integral transforms is a subject that garners attention from both physicists and mathematicians [15,31]. Starting from the q-Jackson's definition [1], several authors including Purohit and Kalla [32], Atici [33], Salem et al. [31], Hahn [14], Albayrak et al. [29], Ucar [34], Al-Omari [35–37], Won Sang et al. [38], Al-salam [39] have explored different aspects of the q-integral theory. The gamma integral transform is defined for a function ϑ by [40]

$$\hat{g}_j(\vartheta;\epsilon) = \frac{1}{\epsilon^j \Gamma(j)} \int_0^\infty \vartheta\left(\frac{\xi}{j}\right) \xi^{j-1} \exp\left(\frac{-\xi}{\epsilon}\right) d\xi,\ \epsilon \in [0,\infty), \tag{1}$$

where ϑ is a mapping of certain exponential growth conditions. The gamma function and its approximation properties to absolutely continuous and locally bounded functions are given in [40]. The first type q-analog of the gamma operator $\hat{g}_j$ given in (1) is defined by [41]

$$\hat{g}_{j,q}(\vartheta;\epsilon) = \frac{1}{\epsilon^j \Gamma_q(j)} \int_0^\infty \vartheta\left(\frac{\xi}{[j]_q}\right) \xi^{j-1} E_q\left(\frac{-q\xi}{\epsilon}\right) d_q\xi, \tag{2}$$

where $\vartheta \in C[0,\infty), 0 < q < 1$ and j is a natural number. The operators $\hat{g}_{j,q}$ are positive, linear and $\hat{g}_{j,q} \to \hat{g}_j$ as $q \to 1^-$. By making a change on variables, (2) can be written as

$$\hat{g}_{j,q}(\vartheta;\epsilon) = \frac{j^j}{\epsilon^j \Gamma_q(j)} \int_0^\infty \vartheta(\xi) \xi^{j-1} E_q\left(\frac{-qj\xi}{\epsilon}\right) d_q\xi, \tag{3}$$

while the q-analog of the gamma operator of type two can be defined as

$$g_{j,q}(\vartheta;\epsilon) = \frac{j^j}{\epsilon^j \Gamma_q(j)} \int_0^\infty \vartheta(\xi) \xi^{j-1} e_q\left(\frac{-qj\xi}{\epsilon}\right) d_q\xi. \tag{4}$$

Indeed, $E_q(\xi)$ and $e_q(\xi)$, ξ is a real number, are the q-analogs of the exponential function defined by [29]

$$E_q(\xi) = \sum_{k=0}^{\infty} q^{\frac{k(k-1)}{2}} \frac{\xi^k}{[k]_q!} = (\xi;q)_\infty \text{ and } e_q(\xi) = \sum_{k=0}^{\infty} \frac{\xi^k}{[k]_q!} = \frac{1}{(\xi;q)_\infty}, |\xi| < |1-q|^{-1}. \tag{5}$$

Therefore, the q-analogs of the remarkable gamma function are, respectively, expressed in terms of the q-exponential functions E_q and e_q as [30]

$$\Gamma_q(\xi) = \int_0^\infty \gamma^{\xi-1} E_q(-q\gamma) d_q\gamma \text{ and } \hat{\Gamma}_q(\xi) = \int_0^\infty \gamma^{\xi-1} e_q(-\gamma) d_q\gamma. \tag{6}$$

The preliminary result that is necessary for the sequel is as follows [30]

Lemma 1. $\Gamma_q(\xi+1) = [\xi]_q \Gamma_q(\xi), \Gamma_q(j+1) = [j]_q!$ *and* $\hat{\Gamma}_q(j) = q^{-j\frac{(j-1)}{2}} \Gamma_q(j), j \in \mathbb{N}$.

By a benefit of the facts

$$\sin_q(r\xi) = \frac{1}{2i}\left(e_q(ir\xi) - e_q(-ir\xi)\right) \text{ and } Sin_q(r\xi) = \frac{1}{2i}\left(E_q(ir\xi) - E_q(-ir\xi)\right), \tag{7}$$

and making use of the standard properties of the q-exponential functions it is noted from (7) that

$$\sin_q(r\xi) = \sum_{k=0}^{\infty} (-1)^k \frac{q^{\frac{k(k-1)}{2}}}{[2k+1]_q!}(r\xi)^{2k+1} \text{ and } Sin_q(r\xi) = \sum_{k=0}^{\infty} \frac{(-1)^k}{[2k+1]_q!}(r\xi)^{2k+1}. \quad (8)$$

Similarly, by taking into account the facts

$$\cos_q(r\xi) = \frac{1}{2}\left(e_q(ir\xi) + e_q(-ir\xi)\right) \text{ and } Cos_q(r\xi) = \frac{1}{2}\left(E_q(ir\xi) + E_q(-ir\xi)\right) \quad (9)$$

and utilizing the standard properties of q-exponential functions, (9) gives rise to

$$\cos_q(r\xi) = \sum_{k=0}^{\infty} (-1)^k \frac{1}{[2k]_q!}(r\xi)^{2k} \text{ and } Cos_q(r\xi) = \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{2k}{k}}}{[2k]_q!}(r\xi)^{2k}. \quad (10)$$

This article is divided into three sections, each with its own set of results. In Section 1, we provided definitions related to q-analogs of specific functions and presented some preliminary findings. Section 2 utilizes the q-gamma integral operators, specifically those of the first and second types, on specific sets of polynomials, q-sine and q-cosine functions and q-exponential functions as well. Section 3 presents the introduction of a specific q-differential operator and delves into the discussion surrounding q-gamma integrals.

2. q-Gamma Operators of Certain Functions

Within this section, we will acquire outcomes related to the first type of gamma operators applied to various classes of power functions, polynomials and different forms of trigonometric functions.

Theorem 1. *Let r be an arbitrary real number and $m \in \mathbb{N}$. Then, we have*

$$(i)\hat{g}_{j,q}(\xi^r;\epsilon) = \frac{\epsilon^r}{j^r\Gamma_q(j)}\Gamma_q(r+j), j \in \mathbb{N}. \quad (ii)\hat{g}_{j,q}(\xi^m;\epsilon) = \frac{\epsilon^m}{j^m\Gamma_q(j)}[m+j-1]_q!, j \in \mathbb{N}.$$

Proof. It is satisfactory to establish the validity of the first part as the proof for the second part of this result follows a similar line of reasoning taking into account Lemma 1. By making use of (3), we write

$$\hat{g}_{j,q}(\xi^r;\epsilon) = \frac{j^j}{\epsilon^j\Gamma_q(j)} \int_0^{\infty} \xi^{r+j-1} E_q\left(\frac{-qj\xi}{\epsilon}\right) d_q\xi. \quad (11)$$

Through the application of variable transformations, $\frac{j\xi}{\epsilon} = \gamma, \xi = \frac{\epsilon\gamma}{j}, d_q\xi = \frac{\epsilon}{j}d_q\gamma$, the integral Equation (11) can be modified to yield

$$\begin{aligned}
\hat{g}_{j,q}(\xi^r;\epsilon) &= \frac{j^j}{\epsilon^j\Gamma_q(j)} \int_0^{\infty} \left(\frac{\epsilon\gamma}{j}\right)^{r+j-1} E_q(-q\gamma)\frac{\epsilon}{j}d_q\gamma \\
&= \frac{j^j}{\epsilon^j\Gamma_q(j)} \int_0^{\infty} \frac{\epsilon^{r+j-1}\gamma^{r+j-1}}{j^{r+j-1}} E_q(-q\gamma)\frac{\epsilon}{j}d_q\gamma \\
&= \frac{\epsilon^r}{j^r\Gamma_q(j)} \int_0^{\infty} \gamma^{r+j-1} E_q(-q\gamma)\frac{\epsilon}{j}d_q\gamma \\
&= \frac{\epsilon^r}{j^r}\frac{\Gamma_q(r+j)}{\Gamma_q(j)}.
\end{aligned}$$

□

Theorem 1 is, therefore, proved.

Theorem 2. *Let r be an arbitrary real number and $m \in \mathbb{N}$. Then, the following assertions hold*
$(i) g_{j,q}(\xi^r;\epsilon) = \frac{\epsilon^r}{j^r \Gamma_q(j)} q^{\frac{-(r+j)(r+j-1)}{2}} \Gamma_q(r+j)$. $(ii) g_{j,q}(\xi^m;\epsilon) = \frac{\epsilon^m}{j^m \Gamma_q(j)} q^{\frac{-(m+j)(m+j-1)}{2}} [m+j-1]!$.

Proof. It is enough to establish the validity of the first part as the proof for the second part of the theorem follows a similar logical technique and Lemma 1. By (3) and (5) we write

$$g_{j,q}(\xi^r;\epsilon) = \frac{j^j}{\epsilon^j \Gamma_q(j)} \int_0^\infty \xi^{r+j-1} e_q\left(\frac{-qj\xi}{\epsilon}\right) d_q\xi.$$

Hence, by assuming $\xi = \frac{\epsilon\gamma}{j}$, and using Lemma 1, $\hat{\Gamma}_q(j) = q^{\frac{-j(j-1)}{2}} \Gamma_q(j), j \in \mathbb{N}$, we obtain

$$\begin{aligned} g_{j,q}(\xi^r;\epsilon) &= \frac{j^j}{\epsilon^j \Gamma_q(j)} \int_0^\infty \gamma^{r+j-1} e_q(-q\gamma) \frac{\epsilon}{j} d_q\gamma \\ &= \frac{\epsilon^r}{j^r \Gamma_q(j)} \int_0^\infty \gamma^{r+j-1} e_q(-q\gamma) \frac{\epsilon}{j} d_q\gamma \\ &= \frac{\epsilon^r}{j^r \Gamma_q(j)} q^{\frac{-(r+j)(r+j-1)}{2}} \Gamma_q(r+j). \end{aligned}$$

□

This ends the proof of our result.

Theorem 3. *Let r be a positive real number. Then, the following statements are true.*
$(i)\ \hat{g}_{j,q}(E_q(r\xi);\epsilon) = \frac{1}{\Gamma_q(j)} \sum_{k=0}^{\infty} q^{\frac{k(k-1)}{2}} \left(\frac{r\epsilon}{j}\right)^k \frac{[k+j-1]!}{[k]_q!}, j \in \mathbb{N}.$
$(ii)\ \hat{g}_{j,q}(e_q(r\xi);\epsilon) = \frac{1}{\Gamma_q(j)} \sum_{k=0}^{\infty} \left(\frac{r\epsilon}{j}\right)^k \frac{[k+j-1]!}{[k]_q!}, j \in \mathbb{N}.$

Proof. In order to demonstrate part (i), we utilize (3) to write

$$\hat{g}_{j,q}(e_q(r\xi);\epsilon) = \frac{j^j}{\epsilon^j \Gamma_q(j)} \int_0^\infty e_q(r\xi) \xi^{j-1} E_q\left(\frac{-qj\xi}{\epsilon}\right) d_q\xi. \tag{12}$$

Therefore, following ([29], (5)) and employing the change in variables $\gamma = \frac{j\xi}{\epsilon}, d_q\xi = \frac{\epsilon}{j} d_q\gamma$ on (12) yield

$$\begin{aligned} \hat{g}_{j,q}(e_q(r\xi);\epsilon) &= \frac{j^j}{\epsilon^j \Gamma_q(j)} \int_0^\infty \sum_{k=0}^{\infty} \frac{r^k \xi^k}{[k]_q!} \xi^{j-1} E_q\left(\frac{-qj\xi}{\epsilon}\right) d_q\xi \\ &= \frac{j^j}{\epsilon^j \Gamma_q(j)} \int_0^\infty \sum_{k=0}^{\infty} \frac{r^k \epsilon^k \gamma^k \epsilon}{n^k [k]_q! j} \frac{\epsilon^{j-1}}{j^{j-1}} E_q(-q\gamma) d_q\gamma \\ &= \frac{\epsilon^j}{\Gamma_q(j)} \sum_{k=0}^{\infty} \frac{1}{j^k [k]_q!} \int_0^\infty \gamma^{2j-1} E_q(-q\gamma) d_q\gamma. \end{aligned}$$

Hence, motivating the above integral equation indicates to have

$$\hat{g}_{j,q}(e_q(r\xi);\epsilon) = \frac{1}{\Gamma_q(j)} \sum_{k=0}^{\infty} \frac{r^k \epsilon^k}{[k]_q! j^k} \int_0^\infty \gamma^{k+j-1} E_q(-q\gamma) d_q\gamma. \tag{13}$$

According to the definition of $\Gamma_q(j)$, we can conclude that (13) has the form

$$\hat{g}_{j,q}(e_q(r\xi);\epsilon) = \frac{1}{\Gamma_q(j)} \sum_{k=0}^{\infty} \frac{r^k \epsilon^k}{[k]_q! j^k} \Gamma_q(k+j). \tag{14}$$

By applying ([29], Theorem 1), we rewrite (14) in the combined infinite series form

$$\hat{g}_{j,q}(e_q(r\xi);\epsilon) = \frac{1}{\Gamma_q(j)} \sum_{k=0}^{\infty} \frac{r^k \epsilon^k}{[k]_q! j^k} [k+j-1]_q! = \frac{1}{\Gamma_q(j)} \sum_{k=0}^{\infty} \left(\frac{r\epsilon}{j}\right)^k \frac{[k+j-1]!}{[k]_q!}.$$

The proof for the validity of (ii) is analogous. Hence, the proof is finished. □

Theorem 4. *Let r represent any positive real number, and $g_{j,q}$ denote the q-gamma integral of the second type. Then, the following statements hold true.*

$$(i) g_{j,q}(e_q(r\xi);\epsilon) = \frac{j^j}{\Gamma_q(j)} \sum_{k=0}^{\infty} \frac{q^{k(k-1)}}{[k]_q!} \left(\frac{r\epsilon}{j}\right) [k+j-1]_q!.$$

$$(ii)\ g_{j,q}(E_q(r\xi);\epsilon) = \frac{1}{\Gamma_q(j)} \sum_{k=0}^{\infty} \left(\frac{r\epsilon}{j}\right)^k \frac{[k+j-1]_q!}{[k]_q!}.$$

Proof. We establish Part (i) as proving the second equation is analogous. By employing (4) and (5) we write

$$g_{j,q}(e_q(r\xi);\epsilon) = \frac{j^j}{\epsilon^j \Gamma_q(j)} \int_0^{\infty} e_q(r\xi) \xi^{j-1} e_q\left(\frac{-qj\xi}{\epsilon}\right) d_q\xi.$$

That is,

$$g_{j,q}(e_q(r\xi);\epsilon) = \frac{j^j}{\epsilon^j \Gamma_q(j)} \int_0^{\infty} \sum_{k=0}^{\infty} q^{\frac{k(k-1)}{2}} \frac{(r\xi)^k}{[k]_q!} \xi^{j-1} e_q\left(\frac{-qj\xi}{\epsilon}\right) d_q\xi. \tag{15}$$

Changing the variables as $\frac{j\xi}{\epsilon} = \gamma, d_q\xi = \frac{\epsilon}{j} d_q\gamma$ in (15) and the following simple motivation reveals

$$\begin{aligned} g_{j,q}(e_q(r\xi);\epsilon) &= \frac{j^j}{\epsilon^j \Gamma_q(j)} \sum_{k=0}^{\infty} \frac{q^{\frac{k(k-1)}{2}}}{[k]_q!} \int_0^{\infty} \frac{r^k \epsilon^k \gamma^k}{j^k} \left(\frac{\epsilon\gamma}{j}\right)^{j-1} e_q(-q\gamma) \frac{\epsilon}{j} d_q\gamma \\ &= \frac{1}{\Gamma_q(j)} \sum_{k=0}^{\infty} \frac{q^{\frac{k(k-1)}{2}}}{[k]_q!} \frac{r^k \epsilon^k}{j^k} \int_0^{\infty} \gamma^{k+j-1} e_q(-q\gamma) \frac{\epsilon}{j} d_q\gamma \\ &= \frac{1}{\Gamma_q(j)} \sum \frac{q^{k(k-1)}}{[k]_q!} \left(\frac{r\epsilon}{j}\right)^k \Gamma_q(k+j). \end{aligned}$$

Hence, the theorem is proved. □

Theorem 5. *Let r be an arbitrary positive real number and $j \in \mathbb{N}$. Then, the assertions that follow hold true.*

$$(i)\ \hat{g}_{j,q}\big(Cos_q(r\xi);\epsilon\big) = \frac{1}{\Gamma_q(j)}\sum_{k=0}^{\infty}\frac{(-1)^k}{[2k]_q!}\left(\frac{r\epsilon}{j}\right)^{2k}\Gamma_q(2k+j).$$

$$(ii)\ \hat{g}_{j,q}\big(Sin_q(r\xi);\epsilon\big) = \frac{1}{\Gamma_q(j)}\sum_{k=0}^{\infty}\frac{(-1)^k}{[2k+1]_q!}\left(\frac{r\epsilon}{j}\right)^{2k+1}\Gamma_q(2k+j+1).$$

$$(iii)\ \hat{g}_{j,q}\big(\cos_q(r\xi);\epsilon\big) = \frac{1}{\Gamma_q(j)}\sum_{k=0}^{\infty}\frac{(-1)^k q^{\frac{k(k-1)}{2}}}{[2k]_q!}\left(\frac{r\epsilon}{j}\right)^{2k}\Gamma_q(2k+j).$$

$$(iv)\ \hat{g}_{j,q}\big(\sin_q(r\xi);\epsilon\big) = \frac{1}{\Gamma_q(j)}\sum_{k=0}^{\infty}(-1)^k\frac{q^{\frac{k(k-1)}{2}}}{[2k+1]_q!}\left(\frac{r\epsilon}{j}\right)^{2k+1}\Gamma_q(2k+j-1).$$

Proof. To demonstrate the first part of the theorem, we make use of (3) and utilize (8) to yield

$$\hat{g}_{j,q}\big(Cos_q(r\xi);\epsilon\big) = \frac{j^j}{\epsilon^j\Gamma_q(j)}\int_0^{\infty} Cos_q(r\xi)\xi^{j-1}E_q\left(\frac{-qj\xi}{\epsilon}\right)d_q\xi.$$

This can be simply explained as

$$\hat{g}_{j,q}\big(Cos_q(r\xi);\epsilon\big) = \frac{j^j}{\epsilon^j\Gamma_q(j)}\int_0^{\infty}\sum_{k=0}^{\infty}(-1)^k\frac{(r\xi)^{2k}}{[2k]_q!}\xi^{j-1}E_q\left(-q\frac{j\xi}{\epsilon}\right)d_q\xi. \tag{16}$$

Hence, by using an appropriate change in variables, $\xi = \frac{\epsilon\gamma}{j}$, (16) can be expressed as

$$\begin{aligned}
\hat{g}_{j,q}\big(Cos_q(r\xi);\epsilon\big) &= \frac{j^j}{\epsilon^j\Gamma_q(j)}\int_0^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k}{[2k]_q!}\left(\frac{r\epsilon\gamma}{j}\right)^{2k}\left(\frac{\epsilon\gamma}{j}\right)^{j-1}E_q(-q\gamma)\frac{\epsilon}{j}d_q\gamma \\
&= \frac{1}{\Gamma_q(j)}\sum_{k=0}^{\infty}\frac{(-1)^k}{[2k]_q!}\left(\frac{r\epsilon}{j}\right)^{2k}\int_0^{\infty}\gamma^{2k+j-1}E_q(-q\gamma)d_q\gamma \\
&= \frac{1}{\Gamma_q(j)}\sum_{k=0}^{\infty}\frac{(-1)^k}{[2k]_q!}\left(\frac{r\epsilon}{j}\right)^{2k}\Gamma_q(2k+j).
\end{aligned}$$

Following analogous proof, which is similar to that employed above, we derive a proof for (ii). Hence, the proof of the result is finished. □

In the following, we declare the following result without proof. The provided result has an easy demonstration that is comparable to the proof of Theorem 5.

Theorem 6. *Let r be a positive real number and $j \in \mathbb{N}$. Then, the assertions that follow hold true.*

$$(i)\ g_{j,q}\big(\cos_q(r\xi);\epsilon\big) = \frac{1}{\Gamma_q(j)}\sum_{k=0}^{\infty}\frac{(-1)^k q^{\frac{k(k-1)}{2}}}{[2k]_q!}\left(\frac{r\epsilon}{j}\right)^{2k}\Gamma_q(2k+j).$$

$$(ii)\ g_{j,q}\big(\sin_q(r\xi);\epsilon\big) = \frac{1}{\Gamma_q(j)}\sum_{k=0}^{\infty}(-1)^k\frac{q^{\frac{k(k-1)}{2}}}{[2k+1]_q!}\left(\frac{r\epsilon}{j}\right)^{2k+1}\Gamma_q(2k+j-1).$$

Theorem 7. *Let r be a positive real number and $j \in \mathbb{N}$. Then, it follows that*

$$(i)\ g_{j,q}\big(Cos_q(r\xi);\epsilon\big) = \frac{1}{\Gamma_q(j)}\sum_{k=0}^{\infty}\frac{(-1)^k}{[2k]_q!}\left(\frac{r\epsilon}{j}\right)^{2k}q^{\frac{-(2k+j)(2k+j-1)}{2}}\Gamma_q(2k+j).$$

$$(ii)\ g_{j,q}\big(Sin_q(r\xi);\epsilon\big) = \frac{1}{\Gamma_q(j)}\sum_{k=0}^{\infty}\frac{(-1)^k}{[2k+1]_q!}\left(\frac{r\epsilon}{j}\right)^{2k+1}q^{\frac{-(2k+j-1)(2k+j)}{2}}\Gamma_q(2k+j-1).$$

Proof. Proof of (i). By using a similar reasoning to the proof of Theorem 5 and Lemma 1 $\left(\hat{\Gamma}_q(j) = q^{-j\frac{(j-1)}{2}}\Gamma_q(j)\right)$ we write

$$g_{j,q}\left(Cos_q(r\xi);\epsilon\right) = \frac{1}{\Gamma_q(j)}\sum_{k=0}^{\infty}\frac{(-1)^k}{[2k]_q!}\left(\frac{r\epsilon}{j}\right)^{2k}\Gamma_q(2k+j).$$

Therefore, we have

$$g_{j,q}\left(Cos_q(r\xi);\epsilon\right) = \frac{1}{\Gamma_q(j)}\sum_{k=0}^{\infty}\frac{(-1)^k}{[2k]_q!}\left(\frac{r\epsilon}{j}\right)^{2k} q^{\frac{-(2k+j)(2k+j-1)}{2}}\Gamma_q(2k+j). \tag{17}$$

Thus, (17) proves Part (i). To prove Part (ii), we use the same reasoning as before to demonstrate that

$$g_{j,q}\left(Sin_q(r\xi);\epsilon\right) = \frac{1}{\Gamma_q(j)}\sum_{k=0}^{\infty}\frac{(-1)^k}{[2k+1]_q!}\left(\frac{r\epsilon}{j}\right)^{2k+1}\hat{\Gamma}_q(2k+j-1).$$

Therefore, we obtain

$$g_{j,q}\left(Sin_q(r\xi);\epsilon\right) = \frac{1}{\Gamma_q(j)}\sum_{k=0}^{\infty}\frac{(-1)^k}{[2k+1]_q!}\left(\frac{r\epsilon}{j}\right)^{2k+1} q^{\frac{-(2k+j-1)(2k+j)}{2}}\Gamma_q(2k+j-1). \tag{18}$$

Hence, (18) completes the proof of the theorem. □

However, it is interesting to note that the findings in Theorem 7 for $g_{j,q}\left(\cos_q(r\xi);\epsilon\right)$ and $g_{j,q}\left(\sin_q(r\xi);\epsilon\right)$ may be achieved in a similar way. Thus, we would rather disregard the comparable proofs.

3. The *q*-Gamma Integral for a Class of *q*-Differential Operators

This section covers the q-analogs of a few provided differential operators, namely $\hat{g}_{j,q}$ and $g_{j,q}$. We start our study by proving the practical lemma that follows.

Lemma 2. *Let $\epsilon > 0$ and $j \in \mathbb{N}$. If $D_{q,\xi}$ denotes the q-derivative with respect to ξ, then we have*

$$(i) D_{q,\xi}E_q\left(-q\frac{j\xi}{\epsilon}\right) = \xi^{-1}\sum_{k=0}^{\infty}(-1)^k q^{\frac{-(k+1)}{2}} j^k q^k \epsilon^{-k}.$$

$$(ii) D_{q,\xi}e_q\left(-q\frac{j\xi}{\epsilon}\right) = \xi^{-1}\sum_{k=0}^{\infty}(-1)^k q^k j^k \xi^k \epsilon^{-k}\frac{[k]_q}{[k]_q!}.$$

Proof. To prove (i), we employ Equation (5) and differentiate inside the summation to obtain

$$\begin{aligned} D_{q,\xi}E_q\left(-q\frac{j\xi}{\epsilon}\right) &= D_q^{\xi}\sum_{k=0}^{\infty} q^{\frac{-(k-1)}{2}}\frac{\left(-q\frac{j\xi}{\epsilon}\right)^k}{[k]_q!} \\ &= \sum_{k=0}^{\infty} q^{\frac{-(k-1)}{2}}\frac{(-1)^k j^k q^k \epsilon^{-k}[k]_q \xi^{k-1}}{[k]_q!} \\ &= \xi^{-1}\sum_{k=0}^{\infty}(-1)^k q^{\frac{-(k+1)}{2}} j^k q^k \epsilon^{-k}. \end{aligned}$$

To prove Part (ii), it is very natural to write

$$D_{q,\xi}e_q\left(-q\frac{j\xi}{\epsilon}\right) = \sum_{k=0}^{\infty}(-1)^k q^k \frac{j^k}{\epsilon^k}\xi^{k-1} = \xi^{-1}\sum_{k=0}^{\infty}(-1)^k q^k j^k \xi^k \epsilon^{-k}\frac{[k]_q}{[k]_q!}.$$

This finishes the proof of our result. □

Theorem 8. *Let the differential operator be defined as* $\Delta_q^{\xi} = \xi^{1-j} D_{q,\xi}$. *Then, we have*

$$\hat{g}_{j,q}\left(\Delta_q^{\xi}\vartheta;\epsilon\right) = -\frac{j^j}{\epsilon^j\Gamma_q(j)}\vartheta(0) - \hat{g}_{j,q}\left(\xi^{-j}\vartheta;q\epsilon\right), j \in \mathbb{N}.$$

Proof. By applying the definition given in (3), we derive the following integral equation

$$\left(\hat{g}_{j,q}\Delta_q^{\xi}\vartheta;\epsilon\right) = \frac{j^j}{\epsilon^j\Gamma_q(j)}\int_0^{\infty} D_{q,\xi}\vartheta(\xi)E_q\left(-qj\xi\epsilon^{-1}\right)d_q\xi. \tag{19}$$

Hence, employing the concept of the q-integration by parts for (19) (see [30]) reveals that

$$\left(\hat{g}_{j,q}\Delta_q^{\xi}\vartheta;\epsilon\right) = \frac{j^j}{\epsilon^j\Gamma_q(j)}\left(\vartheta(\xi)E_q\left(-qj\xi\epsilon^{-1}\right)\right)_0^{\infty} - \int_0^{\infty}\vartheta(q\xi)D_{q,\xi}E_q\left(-qj\xi\epsilon^{-1}\right)d_q\xi.$$

Therefore, by invoking Lemma 2, the previous equation becomes

$$\left(\hat{g}_{j,q}\Delta_q^{\xi}\vartheta;\epsilon\right) = \frac{j^j}{\epsilon^j\Gamma_q(j)}\left(-\vartheta(0) - \int_0^{\infty}\vartheta(q\xi)\xi^{-1}\sum_{k=0}^{\infty}(-1)^k q^{\frac{k(k+1)}{2}} j^k\xi^k\epsilon^{-k}d_q\xi\right). \tag{20}$$

Thus, the change in variables $\gamma = q\xi, d_q\gamma = qd_q\xi$, $\left(\xi = q^{-1}\gamma\right)$, suggests to write (20) in the form

$$\begin{aligned}\left(\hat{g}_{j,q}\Delta_q^{\xi}\vartheta;\epsilon\right) &= \frac{j^j}{\epsilon^j\Gamma_q(j)}\left(-\vartheta(0) - \int_0^{\infty}\vartheta(\gamma)\left(q^{-1}\gamma\right)^{-1}\sum_{k=0}^{\infty}(-1)^k q^{\frac{k(k+1)}{2}} j^k q^{-k}\gamma^k\epsilon^{-k}q^{-1}d_q\gamma\right)\\ &= \frac{j^j}{\epsilon^j\Gamma_q(j)}\left(-\vartheta(0) - \int_0^{\infty}\vartheta(\gamma)\gamma^{-1}\sum_{k=0}^{\infty}(-1)^k q^{\frac{k(k-1)}{2}} j^k q^k\gamma^k\epsilon^{-k}q^{-1}d_q\gamma\right).\end{aligned}$$

The gained equation can be simplified as

$$\left(\hat{g}_{j,q}\Delta_q^{\xi}\vartheta;\epsilon\right) = \frac{j^j}{\epsilon^j\Gamma_q(j)}\left(-\vartheta(0) - \int_0^{\infty}\vartheta(\gamma)\gamma^{-1}E_q\left(-q\frac{j\gamma}{q\epsilon}\right)^k d_q\gamma\right).$$

Equivalently, it can be transferred into the simplest form

$$\left(\hat{g}_{j,q}\Delta_q^{\xi}\vartheta;\epsilon\right) = \frac{j^j}{\epsilon^j\Gamma_q(j)}\left(-\vartheta(0) - \int_0^{\infty}\left(\gamma^{-j}\vartheta(\gamma)\right)\gamma^{j-1}E_q\left(-q\frac{j\gamma}{q\epsilon}\right)d_q\gamma\right). \tag{21}$$

Hence, from (21) we have established that

$$\left(\hat{g}_{j,q}\Delta_q^{\xi}\vartheta;\epsilon\right) = \frac{j^j}{\epsilon^j\Gamma_q(j)}\vartheta(0) - G_{j,q}\left(\gamma^{-j}\vartheta(\gamma);q\epsilon\right).$$

This ends the proof. □

Theorem 9. *Let* $\Delta_q^{\xi} = \xi^{1-j}D_{q,\xi}$ *and* $j \in \mathbb{N}$. *Then, we have*

$$g_{j,q}\left(\Delta_q^{\xi}\vartheta;\epsilon\right) = \frac{-j^j}{\epsilon^j\Gamma_q(j)}\vartheta(0) - g_{j,q}\left(\gamma^{-j}\vartheta;q\epsilon\right).$$

Proof. By using (4) and pursuing an argument alike to that in Theorem 8 and Lemma 2, imply

$$g_{j,q}\left(\Delta_q^{\xi}\vartheta;\epsilon\right) = \frac{j^j}{\epsilon^j\Gamma_q(j)}\left(-\vartheta(0) - \int_0^{\infty}\vartheta(q\xi)\xi^{-1}\sum_{k=0}^{\infty}(-1)^q q^k j^k \xi^k \epsilon^{-k} d_q\xi\right). \tag{22}$$

Hence, by inserting the value $q\xi = \gamma$ in (22), we obtain

$$\begin{aligned} g_{j,q}\left(\Delta_q^{\xi}\vartheta;\epsilon\right) &= \frac{j^j}{\epsilon^j\Gamma_q(j)}\left(-\vartheta(0) - \int_0^{\infty}\vartheta(\gamma)\gamma^{-1}\sum_{k=0}^{\infty}(-1)^k q^k j^k q^{-1}\gamma^k\epsilon^{-k}d_q\gamma\right) \\ &= \frac{j^j}{\epsilon^j\Gamma_q(j)}\left(-\vartheta(0) - \int_0^{\infty}\vartheta(\gamma)\gamma^{-1}\sum_{k=0}^{\infty}(-1)^k j^k q^k \gamma^k q^{-k}\epsilon^{-k}d_q\gamma\right). \end{aligned}$$

Thus, we have obtained that

$$g_{j,q}\left(\Delta_q^{\xi}\vartheta;\epsilon\right) = \frac{j^j}{\epsilon^j\Gamma_q(j)}\vartheta(0) - g_{j,q}\left(\gamma^{-j}\vartheta;q\epsilon\right).$$

The proof is ended. □

Theorem 10. *Let $\epsilon > 0$, $\Delta_q^{\xi} = \xi^{1-j}D_{q,\xi}$, $\Delta_q^{\xi,2} = \left(\Delta_q^{\xi}\right)^2$ and $j \in \mathbb{N}$. Then, we have*

$$\hat{g}_{j,q}\left(\Delta_q^{\xi,2}\vartheta;\epsilon\right) = \frac{-j^j}{\epsilon^j\Gamma_q(j)}\Delta_q^{\xi}\vartheta(0) + \hat{g}_{j,q}\left(\gamma^{-2j}\vartheta;q^2\epsilon\right).$$

Proof. Let the hypothesis of the theorem hold. Then, with the aid of definitions, we write

$$\begin{aligned} \hat{g}_{j,q}\left(\Delta_q^{\xi}\vartheta;\epsilon\right) &= \hat{g}_{j,q}\left(\Delta_q^{\xi}\left(\Delta_q^{\xi}\vartheta\right);\epsilon\right) \\ &= \frac{j^j}{\epsilon^j\Gamma_q(j)}\Delta_q^{\xi}\vartheta(0) - \hat{g}_{j,q}\left(\gamma^{-j}\Delta_q^{\xi}\left(\gamma^{-j}\vartheta\right);q\epsilon\right). \end{aligned}$$

Making reductions, thus, provides

$$\begin{aligned} \hat{g}_{j,q}\left(\Delta_q^{\xi}\vartheta;\epsilon\right) &= \frac{-j^j}{\epsilon^j\Gamma_q(j)}\Delta_q^{\xi}\vartheta(0) - \left(\frac{-j^j}{\epsilon^j\Gamma_q(j)}\left(\gamma^{-j}\vartheta\right)(0) - \hat{g}_{j,q}\left(\gamma^{-j}\left(\gamma^{-j}\vartheta\right);q\epsilon\right)\right) \\ &= \frac{-j^j}{\epsilon^j\Gamma_q(j)}\Delta_q^{\xi}\vartheta(0) + \frac{-j^j}{\epsilon^j\Gamma_q(j)}\left(\gamma^{-j}\vartheta(\gamma)\right)(0) + \hat{g}_{j,q}\left(\gamma^{-2j}\vartheta;q^2\epsilon\right). \end{aligned}$$

Therefore, we have obtained that

$$\hat{g}_{j,q}\left(\Delta_q^{\xi,2}\vartheta;\epsilon\right) = \frac{-j^j}{\epsilon^j\Gamma_q(j)}\Delta_q^{\xi}\vartheta(0) + \hat{g}_{j,q}\left(\gamma^{-2j}\vartheta;q^2\epsilon\right).$$

This ends the proof. □

In what follows, we state without proof the following theorem. The proof can be derived by employing Theorem 10. Therefore, we delete the details.

Theorem 11. $g_{j,q}\left(\Delta_q^{\xi,2}\vartheta;\epsilon\right) = \dfrac{-j^j}{\epsilon^j\Gamma_q(j)}\Delta_q^{\xi}\vartheta(0) + g_{j,q}(\gamma^{-2j}\vartheta;q^2\epsilon), j \in \mathbb{N}.$

4. The q-Gamma Operators for Heaviside Functions

Let us now discuss the $g_{j,q}$ and $\hat{g}_{j,q}$ of the Heaviside function for the q-gamma operators.

Definition 1. *Let a be an arbitrary real number. Then, the Heaviside unit function is defined by*

$$u(\xi - a) \; = u_a(\xi) = \begin{cases} 1 & , \; \xi \geq a \\ 0 & , \; 0 \leq \xi < a \end{cases} . \tag{23}$$

Theorem 12. *Let a be an arbitrary real number and u_a denote the Heaviside function. Then, for $\epsilon > 0$, we have*

$$g_{j,q}(u_a;\epsilon) = q^{-j\frac{(j+1)}{2}} + \frac{j^j a^{j-1}}{\epsilon^j \Gamma_q(j)} \sum_{k=0}^{\infty} (-1)^k \frac{(aqj)^k}{\epsilon^k [k]_q! [j+k]_q!}. \tag{24}$$

Proof. By making use of the second type q-analog (4) of the gamma operator and employing (23) and (24) yields

$$g_{j,q}(u_a;\epsilon) = \frac{j^j}{\epsilon^j \Gamma_q(j)} \int_a^{\infty} \xi^{j-1} e_q\left(\frac{-qj\xi}{\epsilon}\right) d_q\xi.$$

By utilizing the properties of integration, we rewrite the previous formula in the expanded form

$$g_{j,q}(u_a;\epsilon) = \frac{j^j}{\epsilon^j \Gamma_q(j)} \int_0^{\infty} \xi^{j-1} e_q\left(\frac{-qj\xi}{\epsilon}\right) d_q\xi - \frac{j^j}{\epsilon^j \Gamma_q(j)} \int_0^{a} \xi^{j-1} e_q\left(\frac{-qj\xi}{\epsilon}\right) d_q\xi. \tag{25}$$

By employing a change in variables as $\dfrac{qj\xi}{\epsilon} = t$ to the above improper integral, (25) can be reformulated as

$$g_{j,q}(u_a;\epsilon) = \frac{j^j}{\epsilon^j \Gamma_q(j)} \int_0^{\infty} \left(\frac{\epsilon}{qj}\right)^{j-1} t^{j-1} e_q(-t) \frac{\epsilon}{qj} d_q t - \frac{j^j}{\epsilon^j \Gamma_q(j)} \int_0^{a} \xi^{j-1} e_q\left(\frac{-qj\xi}{\epsilon}\right) d_q\xi.$$

Therefore, by using the series form of the exponential function e_q and the definition of the q-analogs of the gamma function given in (6), we write

$$g_{j,q}(u_a;\epsilon) = \frac{j^j}{\epsilon^j \Gamma_q(j)} \left(\frac{\epsilon}{qj}\right)^j \hat{\Gamma}_q(j) - \frac{j^j}{\epsilon^j \Gamma_q(j)} \int_0^{a} \xi^{j-1} \sum_{k=0}^{\infty} (-1)^k \left(\frac{qj}{\epsilon}\right)^k \frac{1}{[k]_q!} d_q\xi. \tag{26}$$

Integrating the right hand side of (26) inside the summation and applying simple computations reveal that

$$\begin{aligned} g_{j,q}(u_a;\epsilon) &= \frac{1}{q^j \Gamma_q(j)} \hat{\Gamma}_q(j) - \frac{j^j}{\epsilon^j \Gamma_q(j)} \sum_{k=0}^{\infty} (-1)^k \left(\frac{qj}{\epsilon}\right)^k \frac{1}{[k]_q!} \int_0^a \xi^{j+k-1} d_q\xi \\ &= \frac{1}{q^j \Gamma_q(j)} \hat{\Gamma}_q(j) - \frac{j^j}{\epsilon^j \Gamma_q(j)} \sum_{k=0}^{\infty} (-1)^{k+1} \frac{(qj)^k}{\epsilon^k [k]_q! [j+k]_q!} a^{j+k-1}. \end{aligned}$$

Therefore, by employing Lemma 1 ($\hat{\Gamma}_q(j) = q^{-j\frac{(j-1)}{2}} \Gamma_q(j), j \in \mathbb{N}$) and rearranging the terms, we write

$$g_{j,q}(u_a;\epsilon) = q^{-j\frac{(j+1)}{2}} + \frac{j^j a^{j-1}}{\epsilon^j \Gamma_q(j)} \sum_{k=0}^{\infty} (-1)^k \frac{(aqj)^k}{\epsilon^k [k]_q! [j+k]_q!}.$$

□

The proof is, therefore, finished. The proof for the subsequent theorem is comparable to that used for Theorem 12. Hence, we omit the details.

Theorem 13. *If u denotes the Heaviside function and $\epsilon > 0$. Then, we have*

$$\hat{g}_{j,q}(u_a;\epsilon) = q^j + \frac{j^j a^{j-1}}{\epsilon^j \Gamma_q(j)} \sum_{k=0}^{\infty} (-1)^k q^{\frac{k(k-1)}{2}} \frac{(aqj)^k}{\epsilon^k [k]_q! [j+k]_q!}.$$

5. q-Convolution Results

This section is dedicated to setting up the convolution theorem for the q-gamma operator. Consequently, it recommends two convolution products that justify the aimed theorem of the q-integral operator.

Definition 2. *Let ϵ be a positive real number and ϑ_1 and ϑ_2 be two real-valued functions. Then, we define two convolution products between ϑ_1 and ϑ_2 as follows*

$$(\vartheta_1 * \vartheta_2)(\epsilon) = \int_0^\infty \vartheta_1\left(\epsilon t^{-1}\right)\vartheta_2(t) t^{-1} d_q t \tag{27}$$

and

$$(\vartheta_1 \times \vartheta_2)(\epsilon) = \int_0^\infty t^{j-1} \vartheta_2(t) \vartheta_1\left(\frac{\epsilon}{t}\right) d_q t, \tag{28}$$

provided the integral parts of (27) and (28) exist.

Now, we derive a convolution theorem for the q-gamma function as follows.

Theorem 14 (Convolution theorem)**.** *Let $*$ be defined by (27). Then, the convolution theorem for $\hat{g}_{j,q}$ is given by*

$$\hat{g}_{j,q}(\vartheta_1 * \vartheta_2)(\epsilon) = \left(\hat{g}_{j,q}\vartheta_1 \times \vartheta_2\right)(\epsilon).$$

Proof. By making use of (3) and (27) we derive

$$\begin{aligned}
\hat{g}_{j,q}(\vartheta_1 * \vartheta_2)(\epsilon) &= \frac{j^j}{\epsilon^j \Gamma_q(j)} \int_0^\infty (\vartheta_1 * \vartheta_2)(\xi) \xi^{j-1} E_q\left(\frac{-jq\xi}{\epsilon}\right) d_q \xi \\
&= \frac{j^j}{\epsilon^j \Gamma_q(j)} \int_0^\infty \left(\int_0^\infty t^{-1} \vartheta_1\left(\frac{\xi}{t}\right) \vartheta_2(t) d_q t\right) \xi^{j-1} E_q\left(\frac{-jq\xi}{\epsilon}\right) d_q \xi.
\end{aligned}$$

Hence, by utilizing the change in variables $\frac{\xi}{t} = w$ and the product (28) we obtain

$$\begin{aligned}
\hat{g}_{j,q}(\vartheta_1 * \vartheta_2)(\epsilon) &= \frac{j^j}{\epsilon^j \Gamma_q(j)} \int_0^\infty \int_0^\infty t^{-1} \vartheta_1(w) \vartheta_2(t) d_q t w^{j-1} E_q\left(\frac{-jqwt}{\epsilon}\right) d_q w \\
\text{i.e.,} \quad &= \frac{j^j}{\epsilon^j \Gamma_q(j)} \int_0^\infty t^{j-1} \vartheta_2(t) \left[\int_0^\infty \vartheta_1(w) w^{j-1} E_q\left(\frac{\frac{-jqwt}{\epsilon}}{t}\right) d_q w\right] d_q t \\
\text{i.e.,} \quad &= \frac{j^j}{\epsilon^j \Gamma_q(j)} \int_0^\infty t^{j-1} \vartheta_2(t) \left[\int_0^\infty \vartheta_1(w) w^{j-1} E_q\left(\frac{\frac{-jqwt}{\epsilon}}{t}\right) d_q w\right] d_q t \\
\text{i.e.,} \quad &= \int_0^\infty t^{j-1} \vartheta_2(t) \hat{g}_{j,q} \vartheta_1\left(\frac{\epsilon}{t}\right) d_q t \\
\text{i.e.,} \quad &= \left(\hat{g}_{j,q}\vartheta_1 \times \vartheta_2\right)(\epsilon).
\end{aligned}$$

□

This ends the proof.

The proof of the following convolution theorem is similar to that of Theorem 14. We, therefore, remove the details.

Theorem 15. *Let $*$ be defined by* (27)*. Then, the convolution theorem for $g_{j,q}$ is given by*

$$g_{j,q}(\vartheta_1 * \vartheta_2)(\epsilon) = \left(g_{j,q}\vartheta_1 \dagger \vartheta_2\right)(\epsilon),$$

where

$$(\vartheta_2 \dagger \vartheta_1)(\epsilon) = \int_0^\infty t^{j-1}\vartheta_2(t)\vartheta_1\left(\frac{\epsilon}{t}\right)d_qt \ \ and \ \ (\vartheta_1 * \vartheta_2)(\epsilon) = \int_0^\infty \vartheta_1\left(\epsilon t^{-1}\right), \vartheta_2(t)t^{-1}d_qt$$

provided the integrals converge.

Proof. Using a method similar to that of Theorem 14, the proof of this theorem can be achieved by using the definition of (4) for the products $\dagger$ and $*$. Thus, we remove the comparable information. □

6. Conclusions

This work presented two q-analogs of the gamma integral operator and discussed some of the new generalized operators' properties. Certain fundamental polynomials and q-analogs of the sine and cosine functions are applied to the q-analogs. Additionally, the q-analogs are employed for certain suitable classes of first- and second-order q-differential operators. Convolution theorems and two convolution products are also examined. However, we intend to use our differential results in a subsequent study to solve specific q-difference operators and provide some applications.

Author Contributions: Conceptualization, S.A.-O.; methodology, S.A.; software, S.A.; validation, S.A.-O.; investigation, W.S.; resources, S.A.-O.; writing—original draft preparation, S.A.-O.; writing—review and editing, S.A.; funding acquisition, W.S. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: Data are contained within the article.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Jackson, F.H. On q-definite integrals. *Q. J. Pure Appl. Math.* **1910**, *41*, 193–203.
2. Jackson, F.H. The application of basic numbers to Bessel's and Legendre's functions. *Proc. Lond. Math. Soc.* **1905**, *2*, 192–220. [CrossRef]
3. Vyas, V.; AL-Jarrah, A.; Purohit, S.; Araci, S.; Nisar, K. q -Laplace transform for product of general class of q-polynomials and q-analog of L-function. *J. Inequ. Appl.* **2020**, *11*, 21–28.
4. Youm, D. q-deformed conformal quantum mechanics. *Phys. Rev. D* **2000**, *62*, 095009. [CrossRef]
5. Lavagno, A.; Swamy, P.N. q-Deformed structures and nonextensive statistics: A comparative study. *Physica A* **2002**, *305*, 310–315. [CrossRef]
6. Miller, K.; Ross, B. *An Introduction to the Fractional Calculus and Fractional Differential Equations*; John Willy and Sons, Inc.: New York, NY, USA, 1993.
7. Srivastava, H. Operators of Basic (or q-)Calculus and fractional q-calculus and their applications in geometric function theory of complex analysis. *Iran. J. Sci. Tech. Trans. A Sci.* **2020**, *44*, 327–344. [CrossRef]
8. Zhao, K.; Liu, J.; Lv, X. A unified approach to solvability and stability of multipoint BVPs for Langevin and Sturm–Liouville equations with CH–fractional derivatives and impulses via coincidence theory. *Fractal Fract.* **2024**, *8*, 111. [CrossRef]
9. Zhao, K. Study on the stability and its simulation algorithm of a nonlinear impulsive ABC-fractional coupled system with a Laplacian operator via F-contractive mapping. *Adv. Contin. Discret. Model.* **2024**, *2024*, 5. [CrossRef]
10. Zhao, K. Solvability, Approximation and stability of periodic boundary value problem for a nonlinear Hadamard fractional differential equation with p-Laplacian. *Axioms* **2023**, *12*, 733. [CrossRef]
11. Zhao, K. Existence and uh-stability of integral boundary problem for a class of nonlinear higher-order Hadamard fractional Langevin equation via Mittag-Leffler functions. *Filomat* **2023**, *37*, 1053–1063. [CrossRef]
12. Al-salam, W.A.; Verma, A. A fractional Leibniz q-formula. *Pac. J. Math.* **1975**, *60*, 1–9. [CrossRef]
13. Rajkovic, P.M.; Marinkovic, S.; Stankovic, M.S. Fractional integrals and derivatives in q-calculus. *Appl. Anal. Discret. Math.* **2007**, *1*, 311–323.

14. Hahn, W. Beitrage Zur Theorie Der Heineschen Reihen, die 24 Integrale der hypergeometrischen q-diferenzengleichung, das q-Analog on der Laplace transformation. *Math. Nachrichten* **1949**, *2*, 340–379. [CrossRef]
15. Al-Omari, S.; Baleanu, D.; Purohit, D. Some results for Laplace-type integral operator in quantum calculus. *Adv. Diff. Equ.* **2018**, *124*, 1–10.
16. Mangontarum, M. On a q-analog of the Elzaki transform Called Mangontarum q-Transform. *Dis. Dyn. Nat. Soc.* **2014**, *2014*, 825618.
17. Albayrak, D. On q-Laplace and q-Sumudu transforms of a product of generalized q-Bessel functions. *Math. Eng. Sci. Aerosp.* **2020**, *11*, 355.
18. Araci, S.; Ugur, D.; Mehmet, A. On weighted q-Daehee polynomials with their applications. *Indag. Math.* **2019**, *30*, 365–374. [CrossRef]
19. Uçar, F.; Albayrak, D. On q-Laplace type integral operators and their applications. *J. Diff. Equ. Appl.* **2011**, *18*, 1001–1014. [CrossRef]
20. Al-Omari, S. On q-analogs of the Mangontarum transform for certain q-Bessel functions and some application. *J. King Saud Univ.-Sci.* **2016**, *28*, 375–379. [CrossRef]
21. Naik, S.; Haubold, H. On the q-Laplace Transform and Related Special Functions. *Axioms* **2016**, *5*, 24. [CrossRef]
22. Dammak, M.; El Ghord, M.; Kharrat, S. On the extremal solutions of superlinear Helmholtz problems. *Bol. Soc. Parana. Mat.* **2022**, *40*, 1–8. [CrossRef]
23. Mennouni, A.; Ramdani, N.; Zennir, K. A new class of Fredholm integral equations of the second kind with non symmetric kernel: Solving by wavelets method. *Bol. Soc. Paran. Mat.* **2021**, *39*, 67–80. [CrossRef]
24. Nasri, N.; Aissaoui, F.; Bouhali, K.; Frioui, A.; Meftah, B.; Zennir, K. Radwan, T. Fractional weighted midpoint-type inequalities for s-convex functions. *Symmetry* **2023**, *15*, 612. [CrossRef]
25. Kim, T.; Kim, D.; Chung, W.S.; Kwon, H. Some families of q-sums and q-products. *Filomat* **2017**, *31*, 1611–1618. [CrossRef]
26. Chung, W.S.; Kim, T.; Mansour, T. The q-deformed gamma function and q-deformed polygamma function. *Bull. Korean Math. Soc.* **2014**, *51*, 1–8. [CrossRef]
27. Kim, T.; Rim, S.; Park, S. Daehee formula associated with the q-extensions of trigonometric functions. *arXiv* **2007**, arXiv:math/0701369.
28. Araci, S.; Bagdasaryan, A.; Ozel, C.; Srivastava, H. New symmetric identities involving q-zeta type functions. *Appl. Math. Inf. Sci.* **2014**, *8*, 2803–2808. [CrossRef]
29. Albayrak, D.; Purohit, S.D.; Ucar, F. On q-Sumudu transforms of certain q-polynomials. *Filomat* **2013**, *27*, 413–429. [CrossRef]
30. Kac, V.; Cheung, P. *Quantum Calculus*; Springer: Berlin, Germany, 2001.
31. Salem, A.; Ucar, F. The q-analog of the $E_{f_{2;1}}$ g-transform and its applications. *Turk. J. Math.* **2016**, *40*, 98–107. [CrossRef]
32. Purohit, S.D.; Kalla, S.L. On q-Laplace transforms of the q-Bessel functions. *Fract. Calc. Appl. Anal.* **2007**, *10*, 189–196.
33. Atici, F.M.; Eloe, P.W. Fractional q-calculus on a time scale. *J. Nonlinear Math. Phys.* **2007**, *14*, 333–344. [CrossRef]
34. Ucar, F. q-Sumudu transforms of q-analogs of Bessel functions. *Sci. World J.* **2014**, *2014*, 327019. [CrossRef] [PubMed]
35. Al-Omari, S. The q-Sumudu transform and its certain properties in a generalized q-calculus theory. *Adv. Diff. Equ.* **2021**, *2021*, *10*. [CrossRef]
36. Al-Omari, S. On q-analogs of Mangontarum transform of some polynomials and certain class of H-functions. *Nonlinear Stud.* **2016**, *23*, 51–61.
37. Al-Omari, S. On a q-Laplace-type integral operator and certain class of series expansion. *Math. Meth. Appl. Sci.* **2021**, *44*, 8322–8332. [CrossRef]
38. Chung, W.S.; Taekyun, K.; Hyuck, I.K. On the q-Analog of the Laplace transform. *Russ. J. Math. Phys.* **2014**, *21*, 156–168. [CrossRef]
39. Al-salam, W.A. Some fractional q-integrals and q -derivatives. *Proc. Edinb. Math. Soc.* **1969**, *15*, 135–140. [CrossRef]
40. Zeng, X.M. approximation properties of Gamma operators. *J. Math. Anal. Appl.* **2005**, *311*, 389–401. [CrossRef]
41. Qing-Bo, C. properties of convergence for a class of generalized q-Gamma operators. *J. Math. Study* **2014**, *47*, 388–395. [CrossRef]