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Applications of Generalized Hypergeometric Distribution on Comprehensive Families of Analytic Functions

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Abstract: A sequence of n trials from a finite population with no replacement is described by the hypergeometric distribution as the number of successes. Calculating the likelihood that factory-produced items would be defective is one of the most popular uses of the hypergeometric distribution in industrial quality control. Very recently, several researchers have applied this distribution on certain families of analytic functions. In this study, we provide certain adequate criteria for the generalized hypergeometric distribution series to be in two families of analytic functions defined in the open unit disk. Furthermore, we consider an integral operator for the hypergeometric distribution. Some corollaries will be implied from our main results.

Keywords: analytic function; geometric functions; hypergeometric distribution; poisson distribution

MSC: 30C45

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Citation: Al-Hawary, T.; Frasin, B.; Aldawish, I. Applications of Generalized Hypergeometric Distribution on Comprehensive Families of Analytic Functions. *Mathematics* **2024**, *12*, 2851. https://doi.org/10.3390/math12182851

Academic Editor: Gheorghe Oros

Received: 10 August 2024
Revised: 9 September 2024
Accepted: 12 September 2024
Published: 13 September 2024

1. Introduction and Preliminaries

A discrete probability distribution called the hypergeometric distribution is used to simulate the likelihood of receiving a given number of successes in a sample taken without replacement from a limited population. Unlike the binomial distribution, which uses replacement sampling and constant probabilities, it is frequently employed in situations when sampling is performed from a finite population and the probability of success varies with each item taken.

The hypergeometric distribution is frequently used in quality control settings, such as estimating the likelihood that a sample taken from a production batch will contain defective items, modeling the likelihood that a sample taken from a population with distinct characteristics will contain specific genetic traits or features, and in a classic example, estimating the likelihood of drawing a specific number of specific cards from a set without replacement.

Let Λ be the family of all functions of the form:

$$G(z) = z + \sum_{\wp=2}^{\infty} c_{\wp} z^{\wp}, \tag{1}$$

which are analytic in the open disk $\mathfrak{D} = \{z \in \mathbb{C} : |z| < 1\}$. Also, using Q, denote the subfamily of Λ made up of form functions:

$$G(z) = z - \sum_{\wp=2}^{\infty} c_{\wp} z^{\wp}, \ \ c_{\wp} \geq 0. \tag{2}$$

For complex numbers $s_1, s_2, \ldots, s_r$ and $t_1, t_2, \ldots, t_v$ with $t_q \neq 0, -1, -2, \ldots, q = 1, 2, \ldots, v$, the generalized hypergeometric functions ${}_rH_v(s_1, s_2, \ldots, s_r; t_1, t_2, \ldots, t_v; z)$ are characterized by

$${}_rH_v(s_1, s_2, \ldots, s_r; t_1, t_2, \ldots, t_v; z) = \sum_{\wp=0}^{\infty} \frac{(s_1)_{\wp} \ldots (s_r)_{\wp}}{(t_1)_{\wp} \ldots (t_v)_{\wp}} \frac{z^{\wp}}{\wp!}, \ \ |z| < 1, \tag{3}$$

where $r \leq v + 1$, and $(s)_{\wp}$ is the Pochhammer symbol.

The series defined by (3) has the following convergence conditions (see [1]):

1) If $r \leq v + 1$, then the series converges absolutely in $\mathbb{C}$.

2) If $r \leq v$, then for each finite z, the series converges absolutely.

3) If $r = v + 1$, then for $|z| < 1$, the series converges absolutely.

4) If $r = v + 1$ and $|z| = 1$, then the series converges at

$$Re\left(\sum_{q=1}^{v} t_q - \sum_{n=1}^{r} s_n\right) > 0.$$

Now, for $s_n, n = 1, 2, \ldots, r, t_q, q = 1, 2, \ldots, v$, and $\zeta > 0$, we define

$${}_rH_v(s_1, s_2, \ldots, s_r; t_1, t_2, \ldots, t_v; \zeta) = \sum_{\wp=0}^{\infty} \frac{(s_1)_{\wp} \ldots (s_r)_{\wp}}{(t_1)_{\wp} \ldots (t_v)_{\wp}} \frac{\zeta^{\wp}}{\wp!}, \tag{4}$$

assuming convergence of the series.

In this paper, we will utilize the notations

$${}_rH_v(z) =_r H_v(s, t, z)$$

and

$${}_rH_v(s_1 + \eta; t_1 + \eta; z) =_r H_v(s_1 + \eta, s_2 + \eta, \ldots, s_r + \eta; t_1 + \eta, t_2 + \eta, \ldots, t_v + \eta; z), \ \eta \in \mathbb{N}.$$

Now, we present the probability mass function-based generalized hypergeometric distribution.

$$\frac{(s_1)_{\wp} \ldots (s_r)_{\wp}}{(t_1)_{\wp} \ldots (t_v)_{\wp}} \frac{\zeta^{\wp}}{\wp!} \frac{1}{{}_rH_v(\zeta)}, \quad \wp = 0, 1, 2, \cdots.$$

Remark 1. *The generalized hypergeometric distribution can be reduced to the following probability distributions by specializing its parameters:*

1) If $r = 2$ and $v = 1$, the distribution can be simplified to the hypergeometric-type probability distribution obtained by Porwal and Gupta [2].

2) If $r = v = 1$, it can be simplified to the confluent hypergeometric distribution obtained by Porwal [3].

3) If $r = v = 1$ and $s_1 = t_1$, it can be simplified to the Poisson distribution obtained by Porwal [4].

Themangani et al. [5] introduced the generalized hypergeometric distribution series ${}_rH_v(\zeta, z)$ as:

$${}_rH_v(\zeta, z) = z + \sum_{\wp=2}^{\infty} \frac{(s_1)_{\wp-1} \ldots (s_r)_{\wp-1}}{(t_1)_{\wp-1} \ldots (t_v)_{\wp-1}} \frac{\zeta^{\wp-1}}{(\wp-1)!} \frac{z^{\wp}}{{}_rH_v(\zeta)} \tag{5}$$

where $s_n, n = 1, 2, \ldots, r$ and $t_q, q = 1, 2, \ldots, v$.

The convolution of $G(z) = z + \sum_{\wp=2}^{\infty} c_{\wp} z^{\wp}$ and $L(z) = z + \sum_{\wp=2}^{\infty} d_{\wp} z^{\wp}$ is defined as

$$(G * L)(z) = \sum_{\wp=0}^{\infty} c_{\wp} d_{\wp} z^{\wp}, \quad |z| < 1.$$

Now, we examine the linear operator $\Pi(r, v, \zeta) : \Lambda \to \Lambda$ defined by [5]

$$\Pi(r, v, \zeta)G(z) =_r H_v(\zeta, z) * G(z) = z + \sum_{\wp=2}^{\infty} \frac{(s_1)_{\wp-1} \dots (s_r)_{\wp-1}}{(t_1)_{\wp-1} \dots (t_v)_{\wp-1}} \frac{\zeta^{\wp-1}}{(\wp-1)!} \frac{c_{\wp} z^{\wp}}{{}_r H_v(\zeta)}. \tag{6}$$

In this paper, we are mainly interested in the families $P^*_{\varpi}(\sigma, \rho)$ and $\wp^*_{\varpi}(\sigma, \rho)$, defined as follows:

Definition 1. *For some $\sigma(0 \le \sigma < 1)$, $\rho(\rho \ge 0)$, and $\varpi(0 \le \varpi \le 1)$ and $G(z)$ of the form (1), let the family $P^*_{\varpi}(\sigma, \rho)$ consist of functions in Λ, satisfying the inequality*

$$Re\left(\frac{zG'(z)}{(1-\varpi)z + \varpi G(z)} - \sigma\right) > \rho\left|\frac{zG'(z)}{(1-\varpi)z + \varpi G(z)} - 1\right|, \ z \in \mathfrak{D}, \tag{7}$$

and the family $\wp^*_{\varpi}(\sigma, \rho)$ consists of functions that meet the inequality

$$Re\left(\frac{zG'(z) + z^2 G''(z)}{(1-\varpi)z + \varpi z G'(z)} - \sigma\right) > \rho\left|\frac{zG'(z) + z^2 G''(z)}{(1-\varpi)z + \varpi z G'(z)} - 1\right|, \ z \in \mathfrak{D}. \tag{8}$$

Example 1 ([6,7])**.** *For some $\sigma(0 \le \sigma < 1)$, $\rho(\rho \ge 0)$ and choosing $\varpi = 1$, let the family $P^*_1(\sigma, \rho)$ consist of functions that meet the inequality*

$$Re\left(\frac{zG'(z)}{G(z)} - \sigma\right) > \rho\left|\frac{zG'(z)}{G(z)} - 1\right|, \ z \in \mathfrak{D},$$

and the family $\wp^*_1(\sigma, \rho)$ consists of functions that meet the inequality

$$Re\left(\frac{zG''(z)}{G'(z)} + 1 - \sigma\right) > \rho\left|\frac{zG''(z)}{G'(z)}\right|, \ z \in \mathfrak{D}.$$

Example 2 ([8])**.** *For some $\sigma(0 \le \sigma < 1)$, $\rho(\rho \ge 0)$ and choosing $\varpi = 0$, let the family $P^*_0(\sigma, \rho)$ consist of functions that meet the inequality*

$$Re\big(G'(z) - \sigma\big) > \rho\big|G'(z) - 1\big|, \ z \in \mathfrak{D},$$

and the family $\wp^*_0(\sigma, \rho)$ consists of functions that meet the inequality

$$Re\Big(\big(zG'(z)\big)' - \sigma\Big) > \rho\Big|\big(zG'(z)\big)' - 1\Big|, \ z \in \mathfrak{D}.$$

Example 3 ([8])**.** *For some $\sigma(0 \le \sigma < 1)$, and choosing $\rho = 0$, $\varpi = 1$, let the family $P^*_1(\sigma, 0) \equiv ST^*(\sigma)$ consist of functions that meet the inequality*

$$Re\left(\frac{zG'(z)}{G(z)}\right) > \sigma, \ z \in \mathfrak{D},$$

and the family $\wp^*_1(\sigma, 0) \equiv CV(\sigma)$ consists of functions that meet the inequality

$$Re\left(\frac{zG''(z)}{G'(z)} + 1\right) > \sigma, \ z \in \mathfrak{D}.$$

Remark 2. *The families* $P_1^*(\sigma,0) \equiv ST^*(\sigma)$ *and* $\wp_1^*(\sigma,0) \equiv CV(\sigma)$ *are well known families of starlike and convex functions of order* σ*, respectively.*

Dixit and Pal [9] introduced the family $\mathcal{R}^\alpha(M,K)$, which includes the function $G(z)$ from (1), which satisfies the inequality

$$\left|\frac{G'(z)-1}{(M-K)\alpha - K(G'(z)-1)}\right| < 1, \alpha \in \mathbb{C}\backslash\{0\}, \quad -1 \le M < K \le 1, \quad z \in \mathfrak{D}.$$

Porwal [4] established a connection between probability distribution and complex analysis and opened up a new direction of research in Geometric Function Theory by introducing a power series whose coefficients are probabilities of Poisson distribution. Recently, several authors have followed his work to obtain certain necessary and sufficient conditions by using other important distribution series, like the hypergeometric distribution series [10], Poisson distribution [11], Pascal distribution [12], Mittag-Leffler-type Poisson distribution [13,14], binomial distribution [15,16], generalized distribution [17], confluent hypergeometric distribution [3], and hypergeometric-type probability distribution [2], see also [18]. Motivating with the above mentioned works, we obtain sufficient conditions for the generalized hypergeometric distribution series to be in the families $P_\varpi^*(\sigma,\rho)$ and $\wp_\varpi^*(\sigma,\rho)$. Also we derive a few inclusion connections between the families $\mathcal{R}^\alpha(M,K)$ and $\wp_\varpi^*(\sigma,\rho)$, and examined an integral operator for the generalized distribution series.

2. Preliminaries Lemmas

For functions G of the form (2), we need the following sufficient conditions in order to prove our results.

Lemma 1 ([8])**.** *A function* $G \in \Lambda$ *of the form (1) belongs to the family* $P_\varpi^*(\sigma,\rho)$ *if*

$$\sum_{\wp=2}^{\infty} [(\wp(\rho+1) - \varpi(\sigma+\rho)]|c_\wp| \le 1-\sigma. \tag{9}$$

The result (9) is sharp for

$$G(z) = z + \frac{1-\sigma}{\wp(\rho+1-\varpi(\sigma+\rho)} z^\wp, \qquad \wp=2,3,\cdots.$$

Moreover, $G \in \wp_\varpi^*(\sigma,\rho)$ if

$$\sum_{\wp=2}^{\infty} \wp[(\wp(\rho+1) - \varpi(\sigma+\rho)]|c_\wp| \le 1-\sigma. \tag{10}$$

The result (10) is sharp for

$$G(z) = z + \frac{1-\sigma}{\wp[(\wp(\rho+1) - \varpi(\sigma+\rho)]} z^\wp, \qquad \wp=2,3,\cdots.$$

Lemma 2 ([6,7])**.** *A function* $G \in \Lambda$ *of the form (1) belongs to the family* $P_1^*(\sigma,\rho)$ *if*

$$\sum_{\wp=2}^{\infty} [(\wp(\rho+1) - (\sigma+\rho)]|c_\wp| \le 1-\sigma, \tag{11}$$

and $G \in \wp_1^*(\sigma,\rho)$ *if*

$$\sum_{\wp=2}^{\infty} \wp[(\wp(\rho+1) - (\sigma+\rho)]|c_\wp| \le 1-\sigma. \tag{12}$$

Lemma 3 ([8])**.** *A function $G \in \Lambda$ of the form (1) belongs to the family $P_0^*(\sigma,\rho)$ if*

$$\sum_{\wp=2}^{\infty} \wp(\rho+1)|c_\wp| \leq 1-\sigma, \tag{13}$$

and $G \in \wp_0^*(\sigma,\rho)$ if

$$\sum_{\wp=2}^{\infty} \wp^2(\rho+1)|c_\wp| \leq 1-\sigma. \tag{14}$$

Remark 3. *The conditions (9)–(14) are also necessary for functions $G(z)$ of the form (2).*

Lemma 4 ([9])**.** *Let the function $G \in \Lambda$ of the form (1) belong to the family $\mathcal{R}^\alpha(M,K)$. Then,*

$$|c_\wp| \leq \frac{(M-K)|\alpha|}{\wp}, \quad \alpha \in \mathbb{C}\backslash\{0\};\ \wp \in \mathbb{N}\backslash\{1\}. \tag{15}$$

The bounds (15) is sharp.

3. Main Results

3.1. Sufficient Conditions

In this section, we derive a suitable condition that allows for the generalized hypergeometric distribution ${}_rH_v(\zeta,z)$ to be in the families $P_\varpi^*(\sigma,\rho)$ and $\wp_\varpi^*(\sigma,\rho)$.

Theorem 1. *Let $s_n, t_q > 0 (n = 1,2,\cdots,r; q = 1,2,\cdots,v)$. Suppose that the inequality*

$$\begin{aligned} &\frac{1}{{}_rH_v(\zeta)}\left[(\rho+1)\frac{s_1\cdots s_r}{t_1\cdots t_v}\zeta\,{}_rH_v(s_1+1;t_1+1;\zeta)+(\rho-\varpi(\sigma+\rho)+1)({}_rH_v(\zeta)-1)\right] \\ \leq\ & 1-\sigma, \end{aligned} \tag{16}$$

occurs under one of the following conditions:

$$\begin{cases} r \leq v \text{ and } \zeta > 0, \\ r = v+1 \text{ and } \zeta < 1, \\ r = v+1, \zeta = 1 \text{ and } \sum_{q=0}^{v} t_q > \sum_{n=0}^{r} s_n + 1, \end{cases} \tag{17}$$

then ${}_rH_v(\zeta,z)$ in the family $P_\varpi^*(\sigma,\rho)$.

Proof. To prove that ${}_rH_v(\zeta,z) \in P_\varpi^*(\sigma,\rho)$, by inequality (9), it suffices to prove that

$$\sum_{\wp=2}^{\infty}[(\wp(\rho+1)-\varpi(\sigma+\rho)]\frac{(s_1)_{\wp-1}\cdots(s_r)_{\wp-1}}{(t_1)_{\wp-1}\cdots(t_v)_{\wp-1}}\frac{\zeta^{\wp-1}}{(\wp-1)!}\frac{1}{{}_rH_v(\zeta)} \leq 1-\sigma$$

We have

$$\begin{aligned} &\sum_{\wp=2}^{\infty}[(\wp(\rho+1)-\varpi(\sigma+\rho)]\frac{(s_1)_{\wp-1}\cdots(s_r)_{\wp-1}}{(t_1)_{\wp-1}\cdots(t_v)_{\wp-1}}\frac{\zeta^{\wp-1}}{(\wp-1)!}\frac{1}{{}_rH_v(\zeta)} \\ =\ & \frac{1}{{}_rH_v(\zeta)}\left[\sum_{\wp=2}^{\infty}[(\rho+1)(\wp-1)+\rho-\varpi(\sigma+\rho)+1]\frac{(s_1)_{\wp-1}\cdots(s_r)_{\wp-1}}{(t_1)_{\wp-1}\cdots(t_v)_{\wp-1}}\frac{\zeta^{\wp-1}}{(\wp-1)!}\right] \\ =\ & \frac{1}{{}_rH_v(\zeta)}\left[(\rho+1)\sum_{\wp=2}^{\infty}(\wp-1)\frac{(s_1)_{\wp-1}\cdots(s_r)_{\wp-1}}{(t_1)_{\wp-1}\cdots(t_v)_{\wp-1}}\frac{\zeta^{\wp-1}}{(\wp-1)!}\right. \\ &\left.+(\rho-\varpi(\sigma+\rho)+1)\sum_{\wp=2}^{\infty}\frac{(s_1)_{\wp-1}\cdots(s_r)_{\wp-1}}{(t_1)_{\wp-1}\cdots(t_v)_{\wp-1}}\frac{\zeta^{\wp-1}}{(\wp-1)!}\right] \end{aligned}$$

$$
\begin{aligned}
= & \frac{1}{{}_rH_v(\zeta)}\Bigg[(\rho+1)\frac{s_1\cdots s_r}{t_1\cdots t_v}\zeta\sum_{\wp=2}^{\infty}\frac{(s_1+1)_{\wp-2}\cdots(s_r+1)_{\wp-2}}{(t_1+1)_{\wp-2}\cdots(t_v+1)_{\wp-2}}\frac{\zeta^{\wp-2}}{(\wp-2)!} \\
& +(\rho-\varpi(\sigma+\rho)+1)\sum_{\wp=2}^{\infty}\frac{(s_1)_{\wp-1}\cdots(s_r)_{\wp-1}}{(t_1)_{\wp-1}\cdots(t_v)_{\wp-1}}\frac{\zeta^{\wp-1}}{(\wp-1)!}\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
= & \frac{1}{{}_rH_v(\zeta)}\left[(\rho+1)\frac{s_1\cdots s_r}{t_1\cdots t_v}\zeta{}_rH_v(s_1+1;t_1+1;\zeta)+(\rho-\varpi(\sigma+\rho)+1)({}_rH_v(\zeta)-1)\right] \\
\le & 1-\sigma.
\end{aligned}
$$

□

Theorem 2. *Let $s_n, t_q > 0 (n = 1, 2, \cdots, r; q = 1, 2, \cdots, v)$. Suppose that the inequality*

$$
\begin{aligned}
& \frac{1}{{}_rH_v(\zeta)}\Bigg[(\rho+1)\frac{s_1(s_1+1)\cdots s_r(s_r+1)}{t_1(t_1+1)\cdots s_v(s_v+1)}\zeta^2_r H_v(s_1+2;t_1+2;\zeta) \\
& +(3(\rho+1)-\varpi(\sigma+\rho))\frac{s_1\cdots s_r}{t_1\cdots t_v}\zeta_r H_v(s_1+1;t_1+1;\zeta) \\
& +(\rho-\varpi(\sigma+\rho)+1)({}_rH_v(\zeta)-1)\Bigg] \\
\le & 1-\sigma
\end{aligned}
$$

occurs under one of the following conditions:

$$
\begin{cases}
r \le v \text{ and } \zeta > 0, \\
r = v+1 \text{ and } \zeta < 1, \\
r = v+1, \zeta = 1 \text{ and } \sum_{q=0}^{v} t_q > \sum_{n=0}^{r} s_n + 2,
\end{cases} \tag{18}
$$

then ${}_rH_v(\zeta, z)$ in the family $\wp^*_\varpi(\sigma, \rho)$.

Proof. To prove that ${}_rH_v(\zeta, z) \in \wp^*_\varpi(\sigma, \rho)$, by inequality (10), it suffices to prove that

$$
\sum_{\wp=2}^{\infty}\wp[(\wp(\rho+1)-\varpi(\sigma+\rho)]\frac{(s_1)_{\wp-1}\cdots(s_r)_{\wp-1}}{(t_1)_{\wp-1}\cdots(t_v)_{\wp-1}}\frac{\zeta^{\wp-1}}{(\wp-1)!}\frac{1}{{}_rH_v(\zeta)} \le 1-\sigma
$$

We have

$$
\begin{aligned}
& \sum_{\wp=2}^{\infty}\wp[(\wp(\rho+1)-\varpi(\sigma+\rho)]\frac{(s_1)_{\wp-1}\cdots(s_r)_{\wp-1}}{(t_1)_{\wp-1}\cdots(t_v)_{\wp-1}}\frac{\zeta^{\wp-1}}{(\wp-1)!}\frac{1}{{}_rH_v(\zeta)} \\
= & \frac{1}{{}_rH_v(\zeta)}\Bigg[\sum_{\wp=2}^{\infty}[(\rho+1)(\wp-1)(\wp-2)]\frac{(s_1)_{\wp-1}\cdots(s_r)_{\wp-1}}{(t_1)_{\wp-1}\cdots(t_v)_{\wp-1}}\frac{\zeta^{\wp-1}}{(\wp-1)!} \\
& +\sum_{\wp=2}^{\infty}(3(\rho+1)-\varpi(\sigma+\rho))(\wp-1)\frac{(s_1)_{\wp-1}\cdots(s_r)_{\wp-1}}{(t_1)_{\wp-1}\cdots(t_v)_{\wp-1}}\frac{\zeta^{\wp-1}}{(\wp-1)!} \\
& +(\rho-\varpi(\sigma+\rho)+1)\sum_{\wp=2}^{\infty}\frac{(s_1)_{\wp-1}\cdots(s_r)_{\wp-1}}{(t_1)_{\wp-1}\cdots(t_v)_{\wp-1}}\frac{\zeta^{\wp-1}}{(\wp-1)!}\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
= & \frac{1}{{}_rH_v(\zeta)}\Bigg[(\rho+1)\frac{s_1(s_1+1)\cdots s_r(s_r+1)}{t_1(t_1+1)\cdots s_v(s_v+1)}\zeta^2\sum_{\wp=3}^{\infty}\frac{(s_1+2)_{\wp-3}\cdots(s_r+2)_{\wp-3}}{(t_1+2)_{\wp-3}\cdots(t_v+2)_{\wp-3}}\frac{\zeta^{\wp-3}}{(\wp-3)!} \\
& +(3(\rho+1)-\varpi(\sigma+\rho))\frac{s_1\cdots s_r}{t_1\cdots t_v}\zeta\sum_{\wp=2}^{\infty}\frac{(s_1+1)_{\wp-2}\cdots(s_r+1)_{\wp-2}}{(t_1+1)_{\wp-2}\cdots(t_v+1)_{\wp-2}}\frac{\zeta^{\wp-2}}{(\wp-2)!}
\end{aligned}
$$

$$+(\rho-\varpi(\sigma+\rho)+1)\sum_{\wp=2}^{\infty}\frac{(s_1)_{\wp-1}\cdots(s_r)_{\wp-1}}{(t_1)_{\wp-1}\cdots(t_v)_{\wp-1}}\frac{\zeta^{\wp-1}}{(\wp-1)!}\Bigg]$$

$$\begin{aligned}
&= \frac{1}{{}_rH_v(\zeta)}\bigg[(\rho+1)\frac{s_1(s_1+1)\cdots s_r(s_r+1)}{t_1(t_1+1)\cdots s_v(s_v+1)}\zeta_r^2H_v(s_1+2;t_1+2;\zeta)\\
&\quad+(3(\rho+1)-\varpi(\sigma+\rho))\frac{s_1\cdots s_r}{t_1\cdots t_v}\zeta_rH_v(s_1+1;t_1+1;\zeta)\\
&\quad+(\rho-\varpi(\sigma+\rho)+1)({}_rH_v(\zeta)-1)]\\
&\leq 1-\sigma.
\end{aligned}$$

□

3.2. Inclusion Properties

In view of Lemma 4, we prove Theorem 3.

Theorem 3. *Let $s_n, t_q > 0(n=1,2,\cdots,r;q=1,2,\cdots,v)$ and $G\in\mathcal{R}^{\alpha}(M,K)$. Suppose that the inequality*

$$\begin{aligned}
&\frac{(M-K)|\alpha|}{{}_rH_v(\zeta)}\bigg[(\rho+1)\frac{s_1\cdots s_r}{t_1\cdots t_v}\zeta_rH_v(s_1+1;t_1+1;\zeta)\\
&\quad+(\rho-\varpi(\sigma+\rho)+1)({}_rH_v(\zeta)-1)]\\
&\leq 1-\sigma,
\end{aligned}$$

occurs with one of the conditions given by (17), then $\Pi(r,v,\zeta)H(z)\in\wp^*_{\varpi}(\sigma,\rho)$.

Proof. Since

$$\Pi(r,v,\zeta)H(z)=z+\sum_{\wp=2}^{\infty}\frac{(s_1)_{\wp-1}\cdots(s_r)_{\wp-1}}{(t_1)_{\wp-1}\cdots(t_v)_{\wp-1}}\frac{\zeta^{\wp-1}}{(\wp-1)!}\frac{c_{\wp}z^{\wp}}{{}_rH_v(\zeta)},$$

to prove that $\Pi(r,v,\zeta)H(z)\in\wp^*_{\varpi}(\sigma,\rho)$, by inequality (10), it suffices to prove that

$$\sum_{\wp=2}^{\infty}\wp[(\wp(\rho+1)-\varpi(\sigma+\rho)]\frac{(s_1)_{\wp-1}\cdots(s_r)_{\wp-1}}{(t_1)_{\wp-1}\cdots(t_v)_{\wp-1}}\frac{\zeta^{\wp-1}}{(\wp-1)!}\frac{|c_{\wp}|}{{}_rH_v(\zeta)}\leq 1-\sigma.$$

Using inequality (15), we have

$$\begin{aligned}
&\sum_{\wp=2}^{\infty}\wp[(\wp(\rho+1)-\varpi(\sigma+\rho)]\frac{(s_1)_{\wp-1}\cdots(s_r)_{\wp-1}}{(t_1)_{\wp-1}\cdots(t_v)_{\wp-1}}\frac{\zeta^{\wp-1}}{(\wp-1)!}\frac{|c_{\wp}|}{{}_rH_v(\zeta)}\\
&= \frac{(M-K)|\alpha|}{{}_rH_v(\zeta)}\sum_{\wp=2}^{\infty}[(\wp(\rho+1)-\varpi(\sigma+\rho)]\frac{(s_1)_{\wp-1}\cdots(s_r)_{\wp-1}}{(t_1)_{\wp-1}\cdots(t_v)_{\wp-1}}\frac{\zeta^{\wp-1}}{(\wp-1)!}
\end{aligned}$$

$$\begin{aligned}
&= \frac{(M-K)|\alpha|}{{}_rH_v(\zeta)}\Bigg[(\rho+1)\sum_{\wp=2}^{\infty}\frac{(s_1)_{\wp-1}\cdots(s_r)_{\wp-1}}{(t_1)_{\wp-1}\cdots(t_v)_{\wp-1}}\frac{\zeta^{\wp-1}}{(\wp-1)!}\\
&\quad+(\rho-\varpi(\sigma+\rho)+1)\sum_{\wp=2}^{\infty}\frac{(s_1)_{\wp-1}\cdots(s_r)_{\wp-1}}{(t_1)_{\wp-1}\cdots(t_v)_{\wp-1}}\frac{\zeta^{\wp-1}}{(\wp-1)!}\Bigg]
\end{aligned}$$

$$= \frac{(M-K)|\alpha|}{{}_rH_v(\zeta)}\Bigg[(\rho+1)\frac{s_1\cdots s_r}{t_1\cdots t_v}\zeta\sum_{\wp=2}^{\infty}\frac{(s_1)_{\wp-2}\cdots(s_r)_{\wp-2}}{(t_1)_{\wp-2}\cdots(t_v)_{\wp-2}}\frac{\zeta^{\wp-2}}{(\wp-2)!}$$

$$+(\rho-\varpi(\sigma+\rho)+1)\sum_{\wp=2}^{\infty}\frac{(s_1)_{\wp-1}\cdots(s_r)_{\wp-1}}{(t_1)_{\wp-1}\cdots(t_v)_{\wp-1}}\frac{\zeta^{\wp-1}}{(\wp-1)!}\Bigg]$$

$$\begin{aligned}&= \frac{(M-K)|\alpha|}{{}_rH_v(\zeta)}\left[(\rho+1)\frac{s_1\cdots s_r}{t_1\cdots t_v}\zeta_r H_v(s_1+1;t_1+1;\zeta)\right.\\ &\quad \left.+(\rho-\varpi(\sigma+\rho)+1)({}_rH_v(\zeta)-1)\right]\\ &\leq 1-\sigma.\end{aligned}$$

□

3.3. An Integral Operator

This section is dedicated to obtaining similar results regarding the specific integral

$$\Upsilon(r,v,\zeta,z)=\int_0^z\frac{{}_rH_v(\zeta,t)}{t}dt. \tag{19}$$

Theorem 4. *Let $s_n,t_q>0(n=1,2,\cdots,r;q=1,2,\cdots,v)$. Suppose that the inequality (16) holds under one of the conditions given by (17), then the integral operator $\Upsilon(r,v,\zeta,z)$ belongs to the family $\wp^*_\varpi(\sigma,\rho)$.*

Proof. From (19), we have

$$\Upsilon(r,v,\zeta,z)=z+\sum_{\wp=2}^{\infty}\frac{(s_1)_{\wp-1}\cdots(s_r)_{\wp-1}}{(t_1)_{\wp-1}\cdots(t_v)_{\wp-1}}\frac{\zeta^{\wp-1}}{\wp!}\frac{z^{\wp}}{{}_rH_v(\zeta)}.$$

To prove that $\Upsilon(r,v,\zeta,z)\in\wp^*_\varpi(\sigma,\rho)$, by (10), it suffices to prove that

$$\sum_{\wp=2}^{\infty}\wp[(\wp(\rho+1)-\varpi(\sigma+\rho)]\frac{(s_1)_{\wp-1}\cdots(s_r)_{\wp-1}}{(t_1)_{\wp-1}\cdots(t_v)_{\wp-1}}\frac{\zeta^{\wp-1}}{\wp!}\frac{1}{{}_rH_v(\zeta)}\leq 1-\sigma.$$

We have

$$\begin{aligned}&\sum_{\wp=2}^{\infty}\wp[(\wp(\rho+1)-\varpi(\sigma+\rho)]\frac{(s_1)_{\wp-1}\cdots(s_r)_{\wp-1}}{(t_1)_{\wp-1}\cdots(t_v)_{\wp-1}}\frac{\zeta^{\wp-1}}{\wp!}\frac{1}{{}_rH_v(\zeta)}\\ &= \sum_{\wp=2}^{\infty}[(\wp(\rho+1)-\varpi(\sigma+\rho)]\frac{(s_1)_{\wp-1}\cdots(s_r)_{\wp-1}}{(t_1)_{\wp-1}\cdots(t_v)_{\wp-1}}\frac{\zeta^{\wp-1}}{(\wp-1)!}\frac{1}{{}_rH_v(\zeta)}\\ &= \frac{1}{{}_rH_v(\zeta)}\left[\sum_{\wp=2}^{\infty}[(\rho+1)(\wp-1)+\rho-\varpi(\sigma+\rho)+1]\frac{(s_1)_{\wp-1}\cdots(s_r)_{\wp-1}}{(t_1)_{\wp-1}\cdots(t_v)_{\wp-1}}\frac{\zeta^{\wp-1}}{(\wp-1)!}\right]\end{aligned}$$

$$\begin{aligned}&= \frac{1}{{}_rH_v(\zeta)}\left[(\rho+1)\sum_{\wp=2}^{\infty}\frac{(s_1)_{\wp-1}\cdots(s_r)_{\wp-1}}{(t_1)_{\wp-1}\cdots(t_v)_{\wp-1}}\frac{\zeta^{\wp-1}}{(\wp-2)!}\right.\\ &\quad\left.+(\rho-\varpi(\sigma+\rho)+1)\sum_{\wp=2}^{\infty}\frac{(s_1)_{\wp-1}\cdots(s_r)_{\wp-1}}{(t_1)_{\wp-1}\cdots(t_v)_{\wp-1}}\frac{\zeta^{\wp-1}}{(\wp-1)!}\right]\\ &= \frac{1}{{}_rH_v(\zeta)}\left[(\rho+1)\frac{s_1\cdots s_r}{t_1\cdots t_v}\zeta_r H_v(s_1+1;t_1+1;\zeta)+(\rho-\varpi(\sigma+\rho)+1)({}_rH_v(\zeta)-1)\right]\\ &\leq 1-\sigma.\end{aligned}$$

□

3.4. Some Corollaries

If we take $\varpi = 1$ in Theorems 1–4, we obtain the following corollaries for families $P_1^*(\sigma,\rho)$ and $\wp_1^*(\sigma,\rho)$, obtained by Themangani et al. [5].

Corollary 1. *Let $s_n, t_q > 0 (n = 1,2,\cdots,r; q = 1,2,\cdots,v)$. Suppose that the inequality*

$$(\rho+1)\frac{s_1\cdots s_r}{t_1\cdots t_v}\zeta_r H_v(s_1+1;t_1+1;\zeta) \le 1-\sigma, \tag{20}$$

occurs with one of the conditions given by (17), then ${}_rH_v(\zeta,z) \in P_1^*(\sigma,\rho)$.

Corollary 2. *Let $s_n, t_q > 0 (n = 1,2,\cdots,r; q = 1,2,\cdots,v)$. Suppose that the inequality*

$$(\rho+1)\frac{s_1(s_1+1)\cdots s_r(s_{r+1})}{t_1(t_1+1)\cdots s_v(s_{v+1})}\zeta_r^2 H_v(s_1+2;t_1+2;\zeta)$$
$$+(3\rho-\sigma+3)\frac{s_1\cdots s_r}{t_1\cdots t_v}\zeta_r H_v(s_1+1;t_1+1;\zeta) \le 1-\sigma,$$

occurs with one of the conditions given by (18), then ${}_rH_v(\zeta,z) \in \wp_1^*(\sigma,\rho)$.

Corollary 3. *Let $s_n, t_q > 0 (n = 1,2,\cdots,r; q = 1,2,\cdots,v)$ and $G \in \mathcal{R}^{\alpha}(M,K)$. Suppose that the inequality*

$$\begin{aligned} &\frac{(M-K)|\alpha|}{{}_rH_v(\zeta)}\Big[(\rho+1)\frac{s_1\cdots s_r}{t_1\cdots t_v}\zeta_r H_v(s_1+1;t_1+1;\zeta) \\ &+(1-\sigma)({}_rH_v(\zeta)-1)\Big] \\ \le\ & 1-\sigma, \end{aligned}$$

occurs with one of the conditions given by (17), then $\Pi(r,v,\zeta)H(z) \in \wp_1^*(\sigma,\rho)$.

Corollary 4. *Let $s_n, t_q > 0 (n = 1,2,\cdots,r; q = 1,2,\cdots,v)$. Suppose that the inequality (20) holds under one of the conditions given by (17), then $\Upsilon(r,v,\zeta,z) \in \wp_1^*(\sigma,\rho)$.*

Remark 4. *If we put $r = 2$ and $v = 1$ in Corollaries 1–4, then we acquire outcomes matching those of Porwal and Gupta [2].*

Also, if we take $\varpi = 1$ and ρ=0 in Theorems 1–4, we obtain the following corollaries for families $ST^*(\sigma)$ and $CV(\sigma)$.

Corollary 5. *Let $s_n, t_q > 0 (n = 1,2,\cdots,r; q = 1,2,\cdots,v)$. Suppose that the inequality*

$$\frac{s_1\cdots s_r}{t_1\cdots t_v}\zeta_r H_v(s_1+1;t_1+1;\zeta) \le 1-\sigma, \tag{21}$$

occurs with one of the conditions given by (17), then ${}_rH_v(\zeta,z) \in ST^*(\sigma)$.

Corollary 6. *Let $s_n, t_q > 0 (n = 1,2,\cdots,r; q = 1,2,\cdots,v)$. Suppose that the inequality*

$$\frac{s_1(s_1+1)\cdots s_r(s_{r+1})}{t_1(t_1+1)\cdots s_v(s_{v+1})}\zeta_r^2 H_v(s_1+2;t_1+2;\zeta)$$
$$+(3-\sigma)\frac{s_1\cdots s_r}{t_1\cdots t_v}\zeta_r H_v(s_1+1;t_1+1;\zeta) \le 1-\sigma,$$

occurs with one of the conditions given by (18), then ${}_rH_v(\zeta,z) \in CV(\sigma)$.

Corollary 7. *Let $s_n, t_q > 0(n = 1, 2, \cdots, r; q = 1, 2, \cdots, v)$ and $G \in \mathcal{R}^\alpha(M, K)$. Suppose that the inequality*

$$\begin{aligned} &\frac{(M-K)|\alpha|}{{}_rH_v(\zeta)}\Big[\frac{s_1\cdots s_r}{t_1\cdots t_v}\zeta_r H_v(s_1+1;t_1+1;\zeta) \\ &+(1-\sigma)({}_rH_v(\zeta)-1)\Big] \\ \leq\ & 1-\sigma, \end{aligned}$$

occurs with one of the conditions given by (17), then $\Pi(r, v, \zeta)H(z) \in CV(\sigma)$.

Corollary 8. *Let $s_n, t_q > 0(n = 1, 2, \cdots, r; q = 1, 2, \cdots, v)$. Suppose that the inequality (21) holds under one of the conditions given by (17), then the integral operator $\Upsilon(r, v, \zeta, z) \in CV(\sigma)$.*

4. Conclusions

Several researchers have employed specific distribution series, including the Mittag-Leffler-type Poisson distribution, the Poisson distribution series, the Pascal distribution series, and the hypergeometric distribution series, to derive the conditions necessary for these distributions to be in particular families of analytic functions defined in the open disk $\mathfrak{D}$. In our study, using the generalized hypergeometric distribution series ${}_rH_v(\zeta, z)$ and the linear operator $\Pi(r, v, \zeta)H(z)$ defined in (6), we found sufficient conditions for these functions to be in the families $P^*_\varpi(\sigma, \rho)$ and $\wp^*_\varpi(\sigma, \rho)$. Furthermore, we derived some inclusion connections between the integral operator $\Upsilon(r, v, \zeta, z)$ given in (19) and the family$\wp^*_\varpi(\sigma, \rho)$, as well as between the families $\mathcal{R}^\alpha(M, K)$ and $\wp^*_\varpi(\sigma, \rho)$. This study could inspire researchers to obtain new conditions for generalized hypergeometric distribution series ${}_rH_v(\zeta, z)$ to be in various families of analytic functions defined in $\mathfrak{D}$.

Author Contributions: Conceptualization, T.A.-H. and I.A.; methodology and validation, B.F.; formal analysis, T.A.-H.; investigation and resources, I.A.; data curation, writing—review and editing, T.A.-H.; visualization and supervision, B.F. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: Data are contained within the article.

Conflicts of Interest: The authors declare that they have no conflicts of interest.

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