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Improved Hille-Type and Ohriska-Type Criteria for Half-Linear Third-Order Dynamic Equations

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Abstract: In this paper, we examine the oscillatory behavior of solutions to a class of half-linear third-order dynamic equations with deviating arguments $\left\{\alpha_2(\eta)\phi_{\delta_2}\left(\left[\alpha_1(\eta)\phi_{\delta_1}\left(u^{\Delta}(\eta)\right)\right]^{\Delta}\right)\right\}^{\Delta}+p(\eta)\phi_{\delta}(u(g(\eta)))=0$, on an arbitrary unbounded-above time scale $\mathbb{T}$, where $\eta\in[\eta_0,\infty)_{\mathbb{T}}:=[\eta_0,\infty)\cap\mathbb{T}$, $\eta_0\geq 0$, $\eta_0\in\mathbb{T}$ and $\phi_{\zeta}(w):=|w|^{\zeta}\operatorname{sgn}w$, $\zeta>0$. Using the integral mean approach and the known Riccati transform methodology, several improved Hille-type and Ohriska-type oscillation criteria have been derived that do not require some restrictive assumptions in the relevant results. Illustrative examples and conclusions show that these criteria are sharp for all third-order dynamic equations compared to the previous results in the literature.

Keywords: oscillation criteria; Hille-type; Ohriska-type; differential equations; dynamic equations; time scales

MSC: 39A10; 39A21; 39A99; 34C10; 34C15; 34K11; 34K42; 34N05

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1. Introduction

Stefan Hilger [1] introduced a theory of dynamic equations on time scales, aiming to unify continuous and discrete analysis. Different time scales can be used in a variety of applications. The theory of dynamic equations consist of the classical theories of differential and difference equations and other cases that lie between these classical cases. The difference equations q, which have critical applications in quantum theory (see [2]), can be considered when $\mathbb{T}=q^{\mathbb{N}_0}:=\{q^n:n\in\mathbb{N}_0 \text{ for } q>1\}$, as well as other time scales, such as $\mathbb{T}=h\mathbb{N}$, $\mathbb{T}=\mathbb{N}^2$, and $\mathbb{T}=\mathbb{T}_n$, where $\mathbb{T}_n$ is the set of harmonic numbers. See [3–5] for more details on time-scale calculus.

The oscillation phenomenon, with its significant applications in various fields of engineering and science and its roots in mechanical vibrations, has attracted considerable interest from researchers across multiple applied disciplines. Oscillation models can incorporate advanced terms or delays to account for the impact of temporal contexts on their solutions. Numerous studies have been conducted on oscillation in delay equations, as demonstrated by the works of [6–12]. However, research has focused on advanced oscillation topics in the literature, such as that found in [13–16].

Various models are used to explore oscillation phenomena, with widespread practical applications. Mathematical models have been enhanced in biology by including cross-diffusion factors to reflect delay and oscillation effects better, as discussed in Refs. [17,18]. Current research focuses on dynamic equations essential for analyzing various real-world phenomena. This study examines the turbulent flow of a polytrophic gas through porous materials and non-Newtonian fluid theory, both of which require a solid understanding of the underlying mathematics. For more information, refer to articles [19–23]. Consequently, we are interested in the oscillatory behavior of solutions of the third-order functional half-linear dynamic equation

$$\left\{\alpha_2(\eta)\phi_{\delta_2}\left(\left[\alpha_1(\eta)\phi_{\delta_1}\left(u^\Delta(\eta)\right)\right]^\Delta\right)\right\}^\Delta + p(\eta)\phi_\delta(u(g(\eta))) = 0, \tag{1}$$

on an arbitrary time scale $\mathbb{T}$ with $\sup \mathbb{T} = \infty$, where $\eta \in [\eta_0, \infty)_\mathbb{T} := [\eta_0, \infty) \cap \mathbb{T}$, $\eta_0 \geq 0$, $\eta_0 \in \mathbb{T}$; $\phi_\zeta(w) := |w|^\zeta \operatorname{sgn} w$, $\zeta > 0$; $\delta_1, \delta_2, \delta := \delta_1\delta_2 > 0$; $g : \mathbb{T} \to \mathbb{T}$ is an $rd-$continuous nondecreasing function such that $\lim_{\eta\to\infty} g(\eta) = \infty$; and p, α_j, $j = 1, 2$, are positive rd-continuous functions on $\mathbb{T}$ such that

$$\int_{\eta_0}^{\infty} \frac{\Delta s}{\alpha_j^{1/\delta_j}(s)} = \infty, \; j = 1, 2. \tag{2}$$

and the function $u^\Delta : \mathbb{T} \to \mathbb{R}$ is said to be the derivative of u on $\mathbb{T}$ and is defined by

$$u^\Delta(\eta) = \lim_{s\to\eta} \frac{u(\sigma(\eta)) - u(s)}{\sigma(\eta) - s};$$

A solution of (1) is a nontrivial real-valued function $u \in C^1_{\mathrm{rd}}[T_u, \infty)_\mathbb{T}$ for some $T_u \geq \eta_0$ for a positive constant $\eta_0 \in \mathbb{T}$ such that $\alpha_1(\eta)\phi_{\delta_1}(u^\Delta(\eta)), \alpha_2(\eta)\phi_{\delta_2}([\alpha_1(\eta)\phi_{\delta_1}(u^\Delta(\eta))]^\Delta) \in C^1_{\mathrm{rd}}[T_u, \infty)_\mathbb{T}$, and $u(\eta)$ satisfying (1) on $[T_u, \infty)_\mathbb{T}$, where C_{rd} is the space of right-dense continuous functions. A solution u of (1) is called oscillatory if it is neither eventually positive nor eventually negative; otherwise, it is nonoscillatory. Solutions vanishing in the neighborhood of infinity will not be taken into account. In the following, we present oscillation criteria for differential/dynamic equations related to our main findings and results for Equation (1), and we explain the significant contributions of this work. Fite [24] showed that every solution of the second-order linear differential equation

$$u''(\eta) + p(\eta)u(\eta) = 0, \tag{3}$$

oscillates if

$$\int_{\eta_0}^{\infty} p(s)\mathrm{d}s = \infty. \tag{4}$$

Hille [25] improved (4), proving that if

$$\liminf_{\eta\to\infty} \eta \int_{\eta}^{\infty} p(s)\mathrm{d}s > \frac{1}{4}, \tag{5}$$

then every solution of Equation (3) oscillates. Erbe [26] extended (5) and demonstrated that every solution of the delay second-order linear differential equation

$$u''(\eta) + p(\eta)u(g(\eta)) = 0, \tag{6}$$

oscillates if

$$\liminf_{\eta\to\infty} \eta \int_{\eta}^{\infty} \left(\frac{g(s)}{s}\right) p(s)\mathrm{d}s > \frac{1}{4},$$

where $g(\eta) \leq \eta$. Ohriska [27] states that every solution of Eq. (6) oscillates if

$$\limsup_{\eta\to\infty} \eta \int_{\eta}^{\infty} \left(\frac{g(s)}{s}\right) p(s) \mathrm{d}s > 1.$$

The Hille-type criteria for various forms of second-order dynamic equations was generalized by the results in [28–30]. Regarding the third-order dynamic equations, Erbe et al. [31] formulated the Hille oscillation criteria for the third-order dynamic equation

$$u^{\Delta\Delta\Delta}(\eta) + p(\eta)u(\eta) = 0. \tag{7}$$

The main finding of [31] is that every solution of Equation (7) oscillates or converges to zero if

$$\int_{\eta_0}^{\infty}\int_{\omega}^{\infty}\int_{\tau}^{\infty} p(s)\Delta s\,\Delta\tau\,\Delta\omega = \infty, \tag{8}$$

and

$$\liminf_{\eta\to\infty} \eta \int_{\eta}^{\infty} \frac{h_2(s)}{\sigma(s)} p(s)\Delta s > \frac{1}{4}; \tag{9}$$

where $h_2(s)$ is the Taylor monomial of degree 2; see ([4] Section 1.6). Saker [32] examined the third-order delay dynamic equation

$$\left\{\alpha_2(\eta)\left[u^{\Delta\Delta}(\eta)\right]^{\delta_2}\right\}^{\Delta} + p(\eta)u^{\delta_2}(g(\eta)) = 0, \tag{10}$$

where $g(\eta) \leq \eta$, δ_2 is a quotient of odd positive integers and α_2 is a nondecreasing function on $\mathbb{T}$; one such result is that every solution of Equation (10) oscillates or converges to zero, provided that

$$\int_{\eta_0}^{\infty} \frac{\Delta s}{\alpha_2^{1/\delta_2}(s)} = \infty; \tag{11}$$

$$\int_{\eta_0}^{\infty}\int_{\omega}^{\infty}\left[\frac{1}{\alpha_2(\tau)}\int_{\tau}^{\infty} p(s)\Delta s\right]^{1/\delta_2}\Delta\tau\Delta\omega = \infty; \tag{12}$$

and

$$\liminf_{\eta\to\infty} \frac{\eta^{\delta_2}}{\alpha_2(\eta)} \int_{\sigma(\eta)}^{\infty} \left(\frac{h_2(g(s))}{\sigma(s)}\right)^{\delta_2} p(s)\Delta s > \frac{\delta_2^{\delta_2}}{l^{\delta_2^2}(1+\delta_2)^{1+\delta_2}}; \tag{13}$$

where $l := \liminf_{\eta\to\infty} \dfrac{\eta}{\sigma(\eta)}$. When $\alpha_2(\eta) = 1$, $\delta_2 = 1$, and $g(\eta) = \eta$, criterion (13) reduces to

$$\liminf_{\eta\to\infty} \eta \int_{\sigma(\eta)}^{\infty} \frac{h_2(s)}{\sigma(s)} p(s)\Delta s > \frac{1}{4l}. \tag{14}$$

By comparing (9) and (14), it is obvious that [31] improves [32] for Equation (7) since

$$\frac{1}{4l} \geq \frac{1}{4} \quad \text{and} \quad \eta \int_{\sigma(\eta)}^{\infty} \frac{h_2(s)}{\sigma(s)} p(s)\Delta s \leq \eta \int_{\eta}^{\infty} \frac{h_2(s)}{\sigma(s)} p(s)\Delta s.$$

Wang and Xu, in [33], studied the third-order dynamic equation

$$\left(\alpha_2(\eta)\left[(\alpha_1(\eta)u^{\Delta}(\eta))^{\Delta}\right]^{\delta_2}\right)^{\Delta} + p(\eta)u(\eta) = 0,$$

under specific restrictive conditions related to the time scales. Agarwal et al. [34] proposed Hille-type oscillation criteria for the third-order delay dynamic equation

$$\left(\alpha_2(\eta)(\alpha_1(\eta)u^{\Delta}(\eta))^{\Delta}\right)^{\Delta} + p(\eta)u(g(\eta)) = 0, \tag{15}$$

where $g(\eta) \leq \eta$ on $[\eta_0, \infty)_{\mathbb{T}}$, and under the assumptions

$$\int_{\eta_0}^{\infty} \frac{\Delta s}{\alpha_i(s)} = \infty,\ i = 1,2, \tag{16}$$

and

$$\int_{\eta_0}^{\infty} \frac{1}{\alpha_1(\omega)} \int_{\omega}^{\infty} \frac{1}{\alpha_2(\tau)} \int_{\tau}^{\infty} p(s)\Delta s\, \Delta\tau\, \Delta\omega = \infty. \tag{17}$$

One of the results presented in [34] states that every solution of Equation (15) oscillates or converges to zero if (16) and (17) hold, and

$$\liminf_{\eta\to\infty} D_1(\eta) \int_{\eta}^{\infty} \frac{D_2(g(s))}{D_1(\sigma(s))} p(s)\Delta s > \frac{1}{4}, \tag{18}$$

where

$$D_j(\eta) := \int_{\eta_0}^{\eta} \frac{D_{j-1}(s)}{\alpha_{3-j}(s)}\Delta s,\ j = 1,2, \text{ with } D_0(\eta) := 1, \tag{19}$$

The results in [34] included the results that were established in [31]. We note that the results obtained in [32,34] are proved only when $g(\eta) \leq \eta$ and cannot be applied when $g(\eta) \geq \eta$. In the following, we let

$$\psi(\eta) := \begin{cases} \eta, & g(\eta) \geq \eta, \\ g(\eta), & g(\eta) \leq \eta. \end{cases}$$

Agarwal et al. [35] examined a third-order delay dynamic Equation (1) and gave some new oscillation criteria under the conditions (2) and

$$\int_{\eta_0}^{\infty} \left(\frac{1}{\alpha_1(\omega)} \int_{\omega}^{\infty} \left(\frac{1}{\alpha_2(\tau)} \int_{\tau}^{\infty} p(s)\Delta s \right)^{1/\delta_2} \Delta\tau \right)^{1/\delta_1} \Delta\omega = \infty. \tag{20}$$

and showed that if (2) and (20) hold, and

$$\liminf_{\eta\to\infty} D_1^{\delta_2}(\eta,\eta_0) \int_{\sigma(\eta)}^{\infty} \left(\frac{D_2^{\delta_1}(\psi(s))}{D_1(\sigma(s))} \right)^{\delta_2} p(s)\Delta s > \frac{\delta_2^{\delta_2}}{l^{\delta_2^2}(1+\delta_2)^{1+\delta_2}}, \tag{21}$$

where $l := \liminf_{\eta\to\infty} \dfrac{D_1(\eta)}{D_1(\sigma(\eta))} > 0$ and

$$D_j(\eta) := \int_{\eta_0}^{\eta} \left(\frac{D_{j-1}(s)}{\alpha_{3-j}(s)} \right)^{1/\delta_{3-j}} \Delta s,\ j = 1,2, \text{ with } D_0(\eta) := 1, \tag{22}$$

then every solution of Equation (1) oscillates or converges to zero. We note that the critical constant in (18) is $\frac{1}{4}$ and in (21) is $\dfrac{\delta_2^{\delta_2}}{l^{\delta_2^2}(1+\delta_2)^{1+\delta_2}}$, which is $\dfrac{1}{4l} \geq \dfrac{1}{4}$ if $\delta_2 = 1$ and depends on a concrete time scale; so the critical constant in [34] is better than the one in [35].

Hassan et al. [36] improved the results of [31–35] for Equation (15) and proved that if (16) and (17) hold, and

$$\liminf_{\eta\to\infty} D_1(\eta,\eta_0) \int_{\eta}^{\infty} \frac{D_2(\psi(s))}{D_1(s)} p(s)\Delta s > \frac{1}{4}, \tag{23}$$

where D_j, $j = 0,1,2$ is defined as in (19), then every solution of Equation (15) oscillates or converges to zero. We note that when $g(\eta) = \eta$ and $\alpha_1(\eta) = \alpha_2(\eta) = 1$, condition (23) improves condition (9); when $g(\eta) \leq \eta$ and $\alpha_1(\eta) = 1$, condition (23) improves condition (13); and when $g(\eta) \leq \eta$, condition (23) improves condition (18). In addition, the critical constant in (23) does not depend on a concrete time scale. Hassan et al. [37] extended the results in [34,36] for the half-linear dynamic Equation (1) and obtained that every solution of Equation (15) oscillates or converges to zero if (2) and (20) hold, and for $0 < \delta_2 \le 1$,

$$\liminf_{\eta\to\infty} D_1^{\gamma_2}(\eta)\int_\eta^\infty \left(\frac{D_2^{\delta_1}(\psi(s))}{D_1(\sigma(s))}\right)^{\delta_2} p(s)\Delta s > \frac{\delta_2^{\delta_2}}{l^{\delta_2(1-\delta_2)}(1+\delta_2)^{1+\delta_2}}, \tag{24}$$

and for $\delta_2 \ge 1$,

$$\liminf_{\eta\to\infty} D_1^{\gamma_2}(\eta)\int_\eta^\infty \left(\frac{D_2^{\delta_1}(\psi(s))}{D_1(s))}\right)^{\delta_2} p(s)\Delta s > \frac{\delta_2^{\delta_2}}{l^{\delta_2(\delta_2-1)}(1+\delta_2)^{1+\delta_2}}, \tag{25}$$

where $l := \liminf_{\eta\to\infty} \dfrac{D_1(\eta)}{D_1(\sigma(\eta))} > 0$ and D_j, $j = 0,1,2$ are defined as in (22).

The summary of what was previously mentioned and explained is that several Hille-type oscillation criteria were established for different forms of third-order dynamic equations under some restrictive times, which ensure that the solutions are either oscillatory or nonoscillatory and converge to zero under various restrictive conditions, for an excellent comparison of these results; see ([37], discussions and conclusions section). Reducing third-order dynamic equations to second-order dynamic equations is the technique used in Refs. [31–37].

Recently, Hassan et al. [38] proved an interesting Hille-type and Ohriska-type oscillation criteria for (1) as follows.

Theorem 1 (see [38])**.** *Every solution of Equation* (1) *oscillates or converges to zero if* (2) *and* (20) *hold, and either*

$$\liminf_{\eta\to\infty} D_2^{\delta}(\eta)\int_\eta^\infty \left(\frac{D_2(\psi(s))}{D_2(\sigma(s))}\right)^{\delta} p(s)\Delta s > \frac{\delta^{\delta}}{L^{\delta^2}(1+\delta)^{1+\delta}}, \tag{26}$$

or

$$\limsup_{\eta\to\infty} D_2^{\delta}(\eta)\int_\eta^\infty \left(\frac{D_2(\psi(s))}{D_2(\sigma(s))}\right)^{\delta} p(s)\Delta s > 1, \tag{27}$$

where $L := \liminf_{\eta\to\infty} \dfrac{D_2(\eta)}{D_2(\sigma(\eta))} > 0$ *and* D_j, $j = 0,1,2$ *are defined as in* (22).

It should be noted that the work in [38] had substantial effects on this work. Obtaining some sharp Hille-type and Ohriska-type oscillation criteria for (1) in both cases $g(\eta) \le \eta$ and $g(\eta) \ge \eta$ are our purpose in this study.

The reader is recommended to read references [39–43]; additionally, the list of the papers mentioned within.

This paper is structured as follows: After this introduction, we present preliminaries of the main results in Section 2 and the main results in Equation (1) in Section 3. Section 4 provides examples of the main results, and the conclusions are presented in Section 5.

2. Preliminaries

Throughout this paper, we assume the following:

$$D_j(\eta) := \int_{\eta_0}^{\eta} \left(\frac{D_{j-1}(s)}{\alpha_{3-j}(s)}\right)^{1/\delta_{3-j}} \Delta s,\ j = 1,2,\ \text{with } D_0(\eta) := 1,$$

$$u^{[j]}(\eta) := \alpha_j(\eta)\phi_{\delta_j}([u^{[j-1]}(\eta)]^{\Delta}),\ j = 1,2,3,$$

with
$$u^{[0]}(\eta) = u,\ \alpha_3(\eta) = 1, \text{ and } \delta_3 = 1,$$
and
$$\psi(\eta) := \begin{cases} \eta, & g(\eta) \geq \eta, \\ g(\eta), & g(\eta) \leq \eta, \end{cases}$$
and for nonoscillatory solutions of (1), we let
$$\mathcal{M}_1 := \left\{ u(\eta) :\ u^{[j-1]}(\eta) u^{[j]}(\eta) > 0,\ j = 1, 2, \text{ eventually} \right\} \tag{28}$$
and
$$\mathcal{M}_2 := \left\{ u(\eta) :\ u^{[j-1]}(\eta) u^{[j]}(\eta) < 0,\ j = 1, 2, \text{ eventually} \right\}. \tag{29}$$

This work needs the next preliminary lemmas.

Lemma 1. *If $u(\eta) \in \mathcal{M}_1$, then*
$$\left(\frac{\left| u^{[j]}(\eta) \right|}{D_{2-j}(\eta)} \right)^{\Delta} < 0,\ j = 0, 1, 2, \tag{30}$$
eventually.

Proof. Suppose, without losing generality, that $u(\eta) > 0$ and $u(g(\eta)) > 0$ on $[\eta_0, \infty)_{\mathbb{T}}$. From Equation (1), we conclude that for $\eta \in [\eta_0, \infty)_{\mathbb{T}}$,
$$u^{[3]}(\eta) = -p(\eta) \phi_\delta(u(g(\eta))) < 0.$$

This proves that (30) holds for $j = 2$. Since $u^{[3]}(\eta) < 0$, we obtain
$$\begin{aligned} u^{[1]}(\eta) &\geq \phi_{\delta_2}^{-1}\left(u^{[2]}(\eta)\right) \int_{\eta_0}^{\eta} \frac{\Delta s}{\alpha_2^{1/\delta_2}(s)} \\ &= \phi_{\delta_2}^{-1}\left(u^{[2]}(\eta)\right) D_1(\eta), \end{aligned} \tag{31}$$
that implies
$$\left(\frac{u^{[1]}(\eta)}{D_1(\eta)} \right)^{\Delta} = \frac{D_1(\eta) \phi_{\delta_2}^{-1}\left(u^{[2]}(\eta)\right) - u^{[1]}(\eta)}{\alpha_2^{1/\delta_2}(\eta) D_1(\eta) D_1^{\sigma}(\eta)} < 0 \quad \text{on } (\eta_0, \infty)_{\mathbb{T}}. \tag{32}$$

This proves that (30) holds for $j = 1$. In view of $\left(\frac{u^{[1]}(\eta)}{D_1(\eta)} \right)^{\Delta} < 0$, we obtain, for $\eta \in (\eta_0, \infty)_{\mathbb{T}}$,
$$\begin{aligned} u(\eta) &\geq \phi_{\delta_1}^{-1}\left(\frac{u^{[1]}(\eta)}{D_1(\eta)} \right) \int_{\eta_0}^{\eta} \left(\frac{D_1(s)}{\alpha_1(s)} \right)^{1/\delta_1} \Delta s \\ &= \phi_{\delta_1}^{-1}\left(\frac{u^{[1]}(\eta)}{D_1(\eta)} \right) D_2(\eta). \end{aligned} \tag{33}$$

From (33), we have

$$\left(\frac{u(\eta)}{D_2(\eta)}\right)^{\Delta} = \frac{1}{D_2(\eta)D_2^{\sigma}(\eta)}\left\{D_2(\eta)u^{\Delta}(\eta) - \left(\frac{D_1(\eta)}{\alpha_1(\eta)}\right)^{1/\delta_1} u(\eta)\right\}$$
$$= \frac{1}{D_2(\eta)D_2^{\sigma}(\eta)}\left(\frac{D_1(\eta)}{\alpha_1(\eta)}\right)^{1/\delta_1}\left\{\phi_{\delta_1}^{-1}\left(\frac{u^{[1]}(\eta)}{D_1(\eta)}\right)D_2(\eta) - u(\eta)\right\} < 0.$$

This proves that (30) holds for $j = 0$. This completes the proof. □

The proof of the next lemma is straightforward, and so is omitted.

Lemma 2. *If $u(\eta) \in \mathcal{M}_2$, then $u^{[j]}(\eta)$, $j = 0, 1, 2$ converge.*

The proof of the following result is similar to that of ([44], Theorem 2.1), and we will state for completeness.

Lemma 3. *Assume that*

either

$$\int_{\eta_0}^{\infty} p(s)\Delta s = \infty;$$

$$\int_{\eta_0}^{\infty}\left(\frac{1}{\alpha_2(\tau)}\int_{\tau}^{\infty} p(s)\Delta s\right)^{1/\delta_2}\Delta\tau = \infty;$$

or

$$\int_{\eta_0}^{\infty}\left[\frac{1}{\alpha_1(\omega)}\int_{\omega}^{\infty}\left(\frac{1}{\alpha_2(\tau)}\int_{\tau}^{\infty} p(s)\Delta s\right)^{1/\delta_2}\Delta\tau\right]^{1/\delta_1}\Delta\omega = \infty. \tag{34}$$

If $u(\eta) \in \mathcal{M}_2$, then $u(\eta)$ converges to zero.

Proof. Suppose, without losing generality, that $u(\eta) > 0$ and $u(g(\eta)) > 0$ on $[\eta_0, \infty)_{\mathbb{T}}$. Hence, there is $\eta_1 \in [\eta_0, \infty)_{\mathbb{T}}$ such that

$$u^{[1]}(\eta) < 0 \quad \text{and} \quad u^{[2]}(\eta) > 0 \quad \text{for } \eta \in [\eta_1, \infty)_{\mathbb{T}}.$$

In this case, $u^{\Delta}(\eta) < 0$ eventually. Hence,

$$\lim_{\eta\to\infty} u(\eta) = k \geq 0.$$

Assume $k > 0$. Then, for sufficiently large $\eta_2 \in [\eta_1, \infty)_{\mathbb{T}}$, we have $u(g(\eta)) \geq k$ for $\eta \geq \eta_2$. Integrating (1) from η to $\tau \in [\eta, \infty)_{\mathbb{T}}$, we obtain

$$\begin{aligned} -u^{[2]}(\tau) + u^{[2]}(\eta) &= \int_{\eta}^{\tau} p(s)u^{\delta}(g(s))\Delta s \\ &\geq k^{\delta}\int_{\eta}^{\tau} p(s)\Delta s. \end{aligned}$$

Due to $u^{[2]} > 0$ and taking limits as $\tau \to \infty$, we have

$$u^{[2]}(\eta) \geq k^{\delta}\int_{\eta}^{\infty} p(s)\Delta s$$

If $\int_{\eta}^{\infty} p(s)\Delta s = \infty$, we obtain a contradiction. Otherwise,

$$\left[u^{[1]}(\eta)\right]^{\Delta} \geq k^{\delta_1}\left(\frac{1}{\alpha_2(\eta)}\int_{\eta}^{\infty} p(s)\Delta s\right)^{1/\delta_2}.$$

Again, integrating this inequality from η to ∞ and noting that $u^{[1]} < 0$ eventually, we obtain

$$-u^{[1]}(\eta) \geq k^{\delta_1} \int_{\eta}^{\infty} \left(\frac{1}{\alpha_2(\tau)} \int_{\tau}^{\infty} p(s)\Delta s \right)^{1/\delta_2} \Delta\tau,$$

If $\int_{\eta}^{\infty} \left(\frac{1}{\alpha_2(\tau)} \int_{\tau}^{\infty} p(s)\Delta s \right)^{1/\delta_2} \Delta\tau = \infty$, we obtain a contradiction. Otherwise,

$$-u^{\Delta}(\eta) \geq k \left[\frac{1}{\alpha_1(\eta)} \int_{\eta}^{\infty} \left(\frac{1}{\alpha_2(\tau)} \int_{\tau}^{\infty} p(s)\Delta s \right)^{1/\delta_2} \Delta\tau \right]^{1/\delta_1},$$

Finally, integrating the last inequality from η_2 to η, we obtain

$$-u(\eta) + u(\eta_2) \geq k \int_{\eta_2}^{\eta} \left[\frac{1}{\alpha_1(\omega)} \int_{\omega}^{\infty} \left(\frac{1}{\alpha_2(\tau)} \int_{\tau}^{\infty} p(s)\Delta s \right)^{1/\delta_2} \Delta\tau \right]^{1/\delta_1} \Delta\omega.$$

Consequently, by (34), we have $\lim_{\eta\to\infty} u(\eta) = -\infty$, which contradicts the fact that $u(\eta)$ is a positive solution of Equation (1). This indicates that $\lim_{\eta\to\infty} u(\eta) = 0$, thereby completing the proof. □

3. Main Results

In this section, the main results of this paper are presented. The next theorems deals with the non-existence criteria for nonoscillatory solutions in class $\mathcal{M}_1$.

Theorem 2. *If*

$$\int_{\eta_0}^{\infty} p(s)\Delta s = \infty, \tag{35}$$

then $\mathcal{M}_1 = \varnothing$.

Proof. Assume that Equation (1) has a nonoscillatory solution $u(\eta) \in \mathcal{M}_1$. Without losing generality, we can assume that $u(\eta) > 0$ and $u(g(\eta)) > 0$ eventually. We find from (1) that $u^{[3]}(\eta) < 0$, and by (28) we obtain $u^{[j]}(\eta) > 0$, $j = 1,2$ eventually. Therefore, there is $\eta_1 \in [\eta_0, \infty)_{\mathbb{T}}$ such that for $[\eta_1, \infty)_{\mathbb{T}}$,

$$u^{[j]}(\eta) > 0,\ j = 0,1,2, \text{ and } u^{[3]}(\eta) < 0.$$

Integrating (1) from $\eta \geq \eta_1$ to $t \in [\eta, \infty)_{\mathbb{T}}$, we obtain

$$\begin{aligned} u^{[2]}(\eta) &> -u^{[2]}(v) + u^{[2]}(\eta) = \int_{\eta}^{v} p(s)\phi_\delta(u(g(s)))\Delta s \\ &\geq \phi_\delta(u(g(\eta))) \int_{\eta}^{v} p(s)\Delta s. \end{aligned}$$

Dividing by $\phi_\delta(u(g(\eta))) > 0$ and letting $v \to \infty$ yields

$$\int_{\eta}^{\infty} p(s)\Delta s \leq \frac{u^{[2]}(\eta)}{\phi_\delta(u(g(\eta)))} < \infty.$$

This contradicts (35). □

Now, we will consider that

$$\int_{\eta_0}^{\infty} \left(\frac{D_2(\psi(s))}{D_2(s)} \right)^{\delta} p(s)\Delta s < \infty.$$

Otherwise, meaning that $\mathcal{M}_1 = \varnothing$ according to Theorem 3 since $\frac{D_2(\psi(s))}{D_2(s)} \leq 1$.

Theorem 3. *Let* $0 < \delta \leq 1$. *If* $L > 0$ *and*

$$\liminf_{\eta\to\infty} D_2^\delta(\eta) \int_\eta^\infty \left(\frac{D_2(\psi(s))}{D_2(s)}\right)^\delta p(s)\Delta s > \frac{\delta^\delta}{L^{\delta(1-\delta)}(1+\delta)^{1+\delta}}, \tag{36}$$

then $\mathcal{M}_1 = \varnothing$.

Proof. Suppose that Equation (1) has a nonoscillatory solution $u(\eta) \in \mathcal{M}_1$. Without losing generality, we assume that $u(\eta) > 0$ and $u(g(\eta)) > 0$ eventually. As demonstrated in the proof of Theorem 2, we have $u^{[3]}(\eta) < 0$ and $u^{[j]}(\eta) > 0$, $j = 1,2$ eventually. Thus, from Lemma 1, we obtain

$$\left(\frac{u^{[j]}(\eta)}{D_{2-j}(\eta)}\right)^\Delta < 0,\ j = 0,1,2,$$

eventually. Therefore, there is $\eta_1 \in [\eta_0,\infty)_{\mathbb{T}}$ such that for $\eta \in [\eta_1,\infty)_{\mathbb{T}}$,

$$\left(\frac{u^{[j]}(\eta)}{D_{2-j}(\eta)}\right)^\Delta < 0,\ u^{[j]}(\eta) > 0,\ j = 0,1,2,\ \text{and } u^{[3]}(\eta) < 0. \tag{37}$$

Define

$$x(\eta) := \frac{u^{[2]}(\eta)}{u^\delta(\eta)}. \tag{38}$$

Hence,

$$\begin{aligned} x^\Delta(\eta) &= \left(\frac{1}{u^\delta(\eta)} u^{[2]}(\eta)\right)^\Delta \\ &= \frac{1}{u^\delta(\eta)} u^{[3]}(\eta) - \frac{\left(u^\delta(\eta)\right)^\Delta}{u^\delta(\eta) u^\delta(\sigma(\eta))} u^{[2]}(\sigma(\eta)) \\ &\overset{(1)}{=} -\left(\frac{u(g(\eta))}{u(\eta)}\right)^\delta p(\eta) - \frac{\left(u^\delta(\eta)\right)^\Delta}{u^\delta(\eta)} x(\sigma(\eta)). \end{aligned}$$

Consider the case where $g(\eta) \leq \eta$ on $[\eta_1,\infty)_{\mathbb{T}}$. Since $\left(\frac{u(\eta)}{D_2(\eta)}\right)^\Delta < 0$, we obtain

$$\frac{u(g(\eta))}{u(\eta)} \geq \frac{D_2(g(\eta))}{D_2(\eta)} \quad \text{for } \eta \in [\eta_1,\infty)_{\mathbb{T}}. \tag{39}$$

While the case where $g(\eta) \geq \eta$ on $[\eta_1,\infty)_{\mathbb{T}}$. In view of the fact that $u^\Delta(\eta) > 0$, we see that

$$\frac{u(g(\eta))}{u(\eta)} \geq 1 \quad \text{for } \eta \in [\eta_1,\infty)_{\mathbb{T}}. \tag{40}$$

It follows from (39) and (40) that

$$\frac{u(g(\eta))}{u(\eta)} \geq \frac{D_2(\psi(\eta))}{D_2(\eta)} \quad \text{for } \eta \in [\eta_1,\infty)_{\mathbb{T}}.$$

Hence, we conclude that for $\eta \in [\eta_1,\infty)_{\mathbb{T}}$,

$$x^\Delta(\eta) \leq -\left(\frac{D_2(\psi(\eta))}{D_2(\eta)}\right)^\delta p(\eta) - \frac{\left(u^\delta(\eta)\right)^\Delta}{u^\delta(\eta)} x(\sigma(\eta)). \tag{41}$$

The Pötzsche chain rule ([4] Theorem 1.90) and $u^{[1]}(\eta) > 0$ yields

$$\begin{aligned}\frac{\left(u^{\delta}(\eta)\right)^{\Delta}}{u^{\delta}(\eta)} &= \delta\left(\int_0^1 [(1-h)u(\eta) + hu(\sigma(\eta))]^{\delta-1}\mathrm{dh}\right)\frac{u^{\Delta}(\eta)}{u^{\delta}(\eta)} \\ &\geq \delta\frac{u^{\Delta}(\eta)u^{\delta-1}(\sigma(\eta))}{u^{\delta}(\eta)} \\ &= \delta\frac{u^{\Delta}(\eta)}{u(\eta)}\left(\frac{u(\eta)}{u(\sigma(\eta))}\right)^{1-\delta},\end{aligned}$$

and by $\left(\frac{u(\eta)}{D_2(\eta)}\right)^{\Delta} < 0$, we have for $\eta \in [\eta_1, \infty)_{\mathbb{T}}$,

$$\frac{\left(u^{\delta}(\eta)\right)^{\Delta}}{u^{\delta}(\eta)} \geq \delta\frac{u^{\Delta}(\eta)}{u(\eta)}\left(\frac{D_2(\eta)}{D_2(\sigma(\eta))}\right)^{1-\delta}. \tag{42}$$

From (31), we see that

$$u^{\Delta}(\eta) \geq \phi_{\delta}^{-1}\left(u^{[2]}(\eta)\right)\left(\frac{D_1(\eta)}{\alpha_1(\eta)}\right)^{1/\delta_1}. \tag{43}$$

Substituting (43) into (42), we obtain

$$\begin{aligned}\frac{\left(u^{\delta}(\eta)\right)^{\Delta}}{u^{\delta}(\eta)} &\geq \delta\phi_{\delta}^{-1}\left(\frac{u^{[2]}(\eta)}{u^{\delta}(\eta)}\right)\left(\frac{D_1(\eta)}{\alpha_1(\eta)}\right)^{1/\delta_1}\left(\frac{D_2(\eta)}{D_2(\sigma(\eta))}\right)^{1-\delta} \\ &= \delta\left(\frac{D_1(\eta)}{\alpha_1(\eta)}\right)^{1/\delta_1}\left(\frac{D_2(\eta)}{D_2(\sigma(\eta))}\right)^{1-\delta} x^{1/\delta}(\eta).\end{aligned} \tag{44}$$

Using (44) in (41), we obtain

$$\begin{aligned}x^{\Delta}(\eta) \leq\; &-\left(\frac{D_2(\psi(\eta))}{D_2(\eta)}\right)^{\delta} p(\eta) \\ &-\delta\left(\frac{D_1(\eta)}{\alpha_1(\eta)}\right)^{1/\delta_1}\left(\frac{D_2(\eta)}{D_2(\sigma(\eta))}\right)^{1-\delta} x^{1/\delta}(\eta)x(\sigma(\eta)).\end{aligned} \tag{45}$$

Integrating (45) from η to v, we obtain

$$\begin{aligned}x(v) - x(\eta) \leq\; &-\int_{\eta}^{v}\left(\frac{D_2(\psi(s))}{D_2(s)}\right)^{\delta} p(s)\Delta s \\ &-\delta\int_{\eta}^{v}\left(\frac{D_1(s)}{\alpha_1(s)}\right)^{1/\delta_1}\left(\frac{D_2(s)}{D_2(\sigma(s))}\right)^{1-\delta} x^{1/\delta}(s)x(\sigma(s))\Delta s.\end{aligned}$$

Since $x > 0$ and as $v \to \infty$, we obtain

$$\begin{aligned}\int_{\eta}^{\infty}\left(\frac{D_2(\psi(s))}{D_2(s)}\right)^{\delta} p(s)\Delta s \leq\; &x(\eta) \\ &-\delta\int_{\eta}^{\infty}\left(\frac{D_1(s)}{\alpha_1(s)}\right)^{1/\delta_1}\left(\frac{D_2(s)}{D_2(\sigma(s))}\right)^{1-\delta} x^{1/\delta}(s)x(\sigma(s))\Delta s.\end{aligned}$$

Let

$$R := \liminf_{s\to\infty} D_2^{\delta}(s)x(s).$$

In view of (38) and (43), we have $0 \leq R \leq 1$. Then, for any $\varepsilon_1 > 0$, there exists a $\eta_2 \in [\eta_1, \infty)_{\mathbb{T}}$ such that for $\eta \in [\eta_2, \infty)_{\mathbb{T}}$,

$$D_2^{\delta}(s)x(s) \geq R - \varepsilon_1 \quad \text{and} \quad \frac{D_2(s)}{D_2(\sigma(s))} \geq L - \varepsilon_1, \tag{46}$$

Therefore,

$$\begin{aligned} & \int_{\eta}^{\infty} \left(\frac{D_2(\psi(s))}{D_2(s)} \right)^{\delta} p(s)\Delta s \\ = \quad & x(\eta) - \delta \int_{\eta}^{\infty} \left(\frac{D_1(s)}{\alpha_1(s)} \right)^{1/\delta_1} \left(\frac{D_2(s)}{D_2(\sigma(s))} \right)^{1-\delta} \frac{\left(D_2^{\delta}(s)x(s)\right)^{1/\delta} \left(D_2^{\delta}(s)x(s)\right)^{\sigma}}{D_2(s)D_2^{\delta}(\sigma(s))} \Delta s \\ \leq \quad & x(\eta) - (L - \varepsilon_1)^{1-\delta}(R - \varepsilon_1)^{1+1/\delta} \int_{\eta}^{\infty} \left(\frac{D_1(s)}{\alpha_1(s)} \right)^{1/\delta_1} \frac{\delta}{D_2(s)D_2^{\delta}(\sigma(s))} \Delta s. \end{aligned} \tag{47}$$

Since

$$\begin{aligned} \left(\frac{-1}{D_2^{\delta}(s)} \right)^{\Delta} &= \frac{\left(D_2^{\delta}(s)\right)^{\Delta}}{D_2^{\delta}(s)D_2^{\delta}(\sigma(s))} \\ &= \frac{\delta \int_0^1 [(1-h)D_2(s) + hD_2(\sigma(s))]^{\delta-1} \mathrm{dh}}{D_2^{\delta}(s)D_2^{\delta}(\sigma(s))} \left(\frac{D_1(s)}{\alpha_1(s)} \right)^{1/\delta_1} \\ &\leq \frac{\delta}{D_2(s)D_2^{\delta}(\sigma(s))} \left(\frac{D_1(s)}{\alpha_1(s)} \right)^{1/\delta_1}. \end{aligned}$$

Hence, (45) yields

$$\begin{aligned} \int_{\eta}^{\infty} \left(\frac{D_2(\psi(s))}{D_2(s)} \right)^{\delta} p(s)\Delta s &\leq x(\eta) - (L - \varepsilon_1)^{1-\delta}(R - \varepsilon_1)^{1+1/\delta} \int_{\eta}^{\infty} \left(\frac{-1}{D_2^{\delta}(s)} \right)^{\Delta} \Delta s \\ &= x(\eta) - (L - \varepsilon_1)^{1-\delta}(R - \varepsilon_1)^{1+1/\delta} \frac{1}{D_2^{\delta}(\eta)}, \end{aligned}$$

which implies

$$D_2^{\delta}(\eta) \int_{\eta}^{\infty} \left(\frac{D_2(\psi(s))}{D_2(s)} \right)^{\delta} p(s)\Delta s \leq D_2^{\delta}(\eta)x(\eta) - (L - \varepsilon_1)^{1-\delta}(R - \varepsilon_1)^{1+1/\delta}. \tag{48}$$

Taking the lim inf of the inequality (48) as $\eta \to \infty$, we obtain

$$\liminf_{\eta \to \infty} D_2^{\delta}(\eta) \int_{\eta}^{\infty} \left(\frac{D_2(\psi(s))}{D_2(s)} \right)^{\delta} p(s)\Delta s \leq R - (L - \varepsilon_1)^{1-\delta}(R - \varepsilon_1)^{1+1/\delta}.$$

By dint of $\varepsilon_1 > 0$ being arbitrary, we have

$$\liminf_{\eta \to \infty} D_2^{\delta}(\eta) \int_{\eta}^{\infty} \left(\frac{D_2(\psi(s))}{D_2(s)} \right)^{\delta} p(s)\Delta s \leq R - L^{1-\delta}R^{1+1/\delta}. \tag{49}$$

Setting

$$\lambda := 1 + 1/\delta, \; A := L^{\delta(1-\delta)/(1+\delta)}R, \text{ and } B := \left(\frac{\delta}{1+\delta} \right)^{\delta} \frac{1}{L^{\delta^2(1-\delta)/(1+\delta)}}.$$

From the inequality (see [45])

$$\lambda AB^{\lambda-1} - A^{\lambda} \leq (\lambda-1)B^{\lambda}, \tag{50}$$

we conclude that

$$R - L^{1-\delta}R^{1+1/\delta} \leq \frac{\delta^{\delta}}{L^{\delta(1-\delta)}(1+\delta)^{1+\delta}}.$$

Thus, (49) becomes

$$\liminf_{\eta\to\infty} D_2^{\delta}(\eta)\int_{\eta}^{\infty}\left(\frac{D_2(\psi(s))}{D_2(s)}\right)^{\delta}p(s)\Delta s \leq \frac{\delta^{\delta}}{L^{\delta(1-\delta)}(1+\delta)^{1+\delta}}.$$

That contradicts (36). □

Theorem 4. *Let $\delta \geq 1$. If $L > 0$ and*

$$\liminf_{\eta\to\infty} D_2^{\delta}(\eta)\int_{\eta}^{\infty}\left(\frac{D_2(\psi(s))}{D_2(s)}\right)^{\delta}p(s)\Delta s > \frac{\delta^{\delta}}{L^{\delta(\delta-1)}(1+\delta)^{1+\delta}}, \tag{51}$$

then $\mathcal{M}_1 = \varnothing$.

Proof. Assume that Equation (1) has a nonoscillatory solution $u(\eta) \in \mathcal{M}_1$. Without losing generality, we can assume that $u(\eta) > 0$ and $u(g(\eta)) > 0$ eventually. As shown in the proof of Theorem 3, there is $\eta_1 \in [\eta_0,\infty)_{\mathbb{T}}$ such that for $\eta \in [\eta_1,\infty)_{\mathbb{T}}$,

$$\left(\frac{u^{[j]}(\eta)}{D_{2-j}(\eta)}\right)^{\Delta} < 0,\ u^{[j]}(\eta) > 0,\ j = 0,1,2, \text{ and } u^{[3]}(\eta) < 0,$$

$$u^{\Delta}(\eta) \geq \phi_{\delta}^{-1}\left(u^{[2]}(\eta)\right)\left(\frac{D_1(\eta)}{\alpha_1(\eta)}\right)^{1/\delta_1}, \tag{52}$$

and

$$x^{\Delta}(\eta) \leq -\left(\frac{D_2(\psi(\eta))}{D_2(\eta)}\right)^{\delta}p(\eta) - \frac{\left(u^{\delta}(\eta)\right)^{\Delta}}{u^{\delta}(\eta)}x(\sigma(\eta)),$$

and for any $\varepsilon_1 > 0$, there exists a $\eta_2 \in [\eta_1,\infty)_{\mathbb{T}}$ such that for $\eta \in [\eta_2,\infty)_{\mathbb{T}}$,

$$D_2^{\delta}(s)x(s) \geq R - \varepsilon_1 \quad \text{and} \quad \frac{D_2(s)}{D_2(\sigma(s))} \geq L - \varepsilon_1, \tag{53}$$

where $x(\xi)$ is defined by (38). By the Pötzsche chain rule and $u^{[1]}(\eta) > 0$, we obtain

$$\begin{aligned}
\frac{\left(u^{\delta}(\eta)\right)^{\Delta}}{u^{\delta}(\eta)} &= \delta\left(\int_0^1[(1-h)u(\eta)+hu(\sigma(\eta))]^{\delta-1}\mathrm{dh}\right)\frac{u^{\Delta}(\eta)}{u^{\delta}(\eta)} \\
&\geq \delta\frac{u^{\Delta}(\eta)}{u(\eta)} \\
&\overset{(52)}{\geq} \delta\left(\frac{D_1(\eta)}{\alpha_1(\eta)}\right)^{1/\delta_1}\phi_{\delta}^{-1}\left(\frac{u^{[2]}(\eta)}{u^{\delta}(\eta)}\right) \\
&= \delta\left(\frac{D_1(\eta)}{\alpha_1(\eta)}\right)^{1/\delta_1}x^{1/\delta}(\eta).
\end{aligned}$$

Therefore,

$$x^{\Delta}(\eta) \leq -\left(\frac{D_2(\psi(\eta))}{D_2(\eta)}\right)^{\delta}p(\eta) - \delta\left(\frac{D_1(\eta)}{\alpha_1(\eta)}\right)^{1/\delta_1}x^{1/\delta}(\eta)x(\sigma(\eta)). \tag{54}$$

Integrating (54) from η to v, we obtain

$$x(v) - x(\eta) \leq -\int_{\eta}^{v} \left(\frac{D_2(\psi(s))}{D_2(s)} \right)^{\delta} p(s)\Delta s - \delta \int_{\eta}^{v} \left(\frac{D_1(s)}{\alpha_1(s)} \right)^{1/\delta_1} x^{1/\delta}(s) x(\sigma(s))\Delta s.$$

Due to $x > 0$ and letting $v \to \infty$, we obtain

$$-x(\eta) \leq -\int_{\eta}^{\infty} \left(\frac{D_2(\psi(s))}{D_2(s)} \right)^{\delta} p(s)\Delta s - \delta \int_{\eta}^{\infty} \left(\frac{D_1(s)}{\alpha_1(s)} \right)^{1/\delta_1} x^{1/\delta}(s) x(\sigma(s))\Delta s,$$

which implies that

$$\int_{\eta}^{\infty} \left(\frac{D_2(\psi(s))}{D_2(s)} \right)^{\delta} p(s)\Delta s \leq x(\eta) - \delta \int_{\eta}^{\infty} \left(\frac{D_1(s)}{\alpha_1(s)} \right)^{1/\delta_1} x^{1/\delta}(s) x(\sigma(s))\Delta s$$

$$= x(\eta) - \delta \int_{\eta}^{\infty} \left(\frac{D_1(s)}{\alpha_1(s)} \right)^{1/\delta_1} \left(\frac{D_2(s)}{D_2(\sigma(s))} \right)^{\delta-1} \frac{\left(D_2^{\delta}(s)x(s)\right)^{1/\delta} \left(D_2^{\delta}(s)x(s)\right)^{\sigma}}{D_2^{\delta}(s) D_2(\sigma(s))} \Delta s$$

$$\overset{(53)}{\leq} x(\eta) - (L-\varepsilon_1)^{\delta-1} (R-\varepsilon_1)^{1+1/\delta} \int_{\eta}^{\infty} \frac{\delta}{D_2^{\delta}(s) D_2(\sigma(s))} \left(\frac{D_1(s)}{\alpha_1(s)} \right)^{1/\delta_1} \Delta s.$$

Since

$$\begin{aligned} \left(\frac{-1}{D_2^{\delta}(s)} \right)^{\Delta} &= \frac{\left(D_2^{\delta}(s)\right)^{\Delta}}{D_2^{\delta}(s) D_2^{\delta}(\sigma(s))} \\ &= \frac{\int_0^1 [(1-h)D_2(s) + hD_2(\sigma(s))]^{\delta-1} \mathrm{dh}}{D_2^{\delta}(s) D_2^{\delta}(\sigma(s))} \left(\frac{D_1(s)}{\alpha_1(s)} \right)^{1/\delta_1} \\ &\leq \frac{\delta}{D_2^{\delta}(s) D_2(\sigma(s))} \left(\frac{D_1(s)}{\alpha_1(s)} \right)^{1/\delta_1}. \end{aligned}$$

Then,

$$\begin{aligned} \int_{\eta}^{\infty} \left(\frac{D_2(\psi(s))}{D_2(s)} \right)^{\delta} p(s)\Delta s &\leq x(\eta) - \delta (L-\varepsilon_1)^{\delta-1} (R-\varepsilon_1)^{1+1/\delta} \int_{\eta}^{\infty} \left(\frac{-1}{D_2^{\delta}(s)} \right)^{\Delta} \Delta s \\ &= x(\eta) - \delta (L-\varepsilon_1)^{\delta-1} (R-\varepsilon_1)^{1+1/\delta} \frac{1}{D_2^{\delta}(\eta)}. \end{aligned}$$

Hence,

$$D_2^{\delta}(\eta) \int_{\eta}^{\infty} \left(\frac{D_2(\psi(s))}{D_2(s)} \right)^{\delta} p(s)\Delta s \leq D_2^{\delta}(\eta) x(\eta) - \delta (L-\varepsilon_1)^{\delta-1} (R-\varepsilon_1)^{1+1/\delta}.$$

The rest of the proof is similar to that of Theorem 3. □

Theorem 5. *If*

$$\limsup_{\eta \to \infty} D_2^{\delta}(\eta) \int_{\eta}^{\infty} \left(\frac{D_2(\psi(s))}{D_2(s)} \right)^{\delta} p(s)\Delta s > 1, \tag{55}$$

then $\mathcal{M}_1 = \varnothing$.

Proof. Suppose that Equation (1) has a nonoscillatory solution $u(\eta) \in \mathcal{M}_1$. Without losing generality, we assume that $u(\eta) > 0$ and $u(g(\eta)) > 0$ eventually. As in the proof of Theorem 3, there is $\eta_1 \in [\eta_0, \infty)_{\mathbb{T}}$ such that for $\eta \in [\eta_1, \infty)_{\mathbb{T}}$,

$$u^{[j]}(\eta) > 0,\ j = 0, 1, 2, \text{ and } u^{[3]}(\eta) < 0, \tag{56}$$

$$u(g(\eta)) \geq \frac{D_2(\psi(\eta))}{D_2(\eta)} u(\eta), \tag{57}$$

and

$$u^{\Delta}(\eta) \geq \phi_{\delta}^{-1}\Big(u^{[2]}(\eta)\Big)\left(\frac{D_1(\eta)}{\alpha_1(\eta)}\right)^{1/\delta_1}. \tag{58}$$

Integrating (58) from η_1 to η, we obtain

$$\begin{aligned} u(\eta) - u(\eta_1) &\geq \int_{\eta_1}^{\eta} \phi_{\delta}^{-1}\Big(u^{[2]}(s)\Big)\left(\frac{D_1(s)}{\alpha_1(s)}\right)^{1/\delta_1} \Delta s \\ &\geq \phi_{\delta}^{-1}\Big(u^{[2]}(\eta)\Big) \int_{\eta_1}^{\eta} \left(\frac{D_1(s)}{\alpha_1(s)}\right)^{1/\delta_1} \Delta s \\ &= \phi_{\delta}^{-1}\Big(u^{[2]}(\eta)\Big) D_2(\eta). \end{aligned}$$

Then,

$$u(\eta) \geq \phi_{\delta}^{-1}\Big(u^{[2]}(\eta)\Big) D_2(\eta). \tag{59}$$

From (56), (57), and (59) we see that for $t \in [\eta, \infty)_{\mathbb{T}}$,

$$\begin{aligned} \phi_{\delta}(u(g(s))) &\geq \left(\frac{D_2(\psi(s))}{D_2(s)}\right)^{\delta} \phi_{\delta}(u(s)) \\ &\geq \left(\frac{D_2(\psi(s))}{D_2(s)}\right)^{\delta} \phi_{\delta}(u(\eta)) \\ &\geq D_2^{\delta}(\eta) \left(\frac{D_2(\psi(s))}{D_2(s)}\right)^{\delta} u^{[2]}(\eta). \end{aligned} \tag{60}$$

Integrating (1) from η to u, we obtain

$$\int_{\eta}^{u} p(s)\phi_{\delta}(u(g(s)))\Delta s = u^{[2]}(\eta) - u^{[2]}(u) \leq u^{[2]}(\eta). \tag{61}$$

Substituting (60) into (61), we obtain

$$D_2^{\delta}(\eta) \int_{\eta}^{u} \left(\frac{D_2(\psi(s))}{D_2(s)}\right)^{\delta} p(s)\Delta s \leq 1.$$

Let $u \to \infty$, we have

$$D_2^{\delta}(\eta) \int_{\eta}^{\infty} \left(\frac{D_2(\psi(s))}{D_2(s)}\right)^{\delta} p(s)\Delta s \leq 1.$$

Then,

$$\limsup_{\eta \to \infty} D_2^{\delta}(\eta) \int_{\eta}^{\infty} \left(\frac{D_2(\psi(s))}{D_2(s)}\right)^{\delta} p(s)\Delta s \leq 1,$$

which contradicts (55). □

It easy to see that if $u(\eta)$ is a nonoscillatory solution of the canonical Equation (1), then $u(\eta) \in \mathcal{M}_1 \cup \mathcal{M}_2$, eventually; see [44], Part $(\mathcal{I})$ of the proof of Theorem 2.1, for additional details.

The next results are obtained by combining the conclusions of Theorems 2–5 with Lemmas 2 and 3.

Theorem 6. *If* (35) *holds, then every solution of Equation* (1) *oscillates or converges to zero.*

Theorem 7. *Assume that either* (55) *or*

$$\liminf_{\eta\to\infty} D_2^{\delta}(\eta) \int_{\eta}^{\infty} \left(\frac{D_2(\psi(s))}{D_2(s)}\right)^{\delta} p(s)\Delta s > \frac{\delta^{\delta}}{L^{\delta|1-\delta|}(1+\delta)^{1+\delta}}, \tag{62}$$

holds. Then, every solution of Equation (1) *oscillates or converges.*

Theorem 8. *Assume that* (H) *and either* (55) *or* (62) *hold. Then, every solution of Equation* (1) *oscillates or converges to zero.*

4. Examples

Now, we provide illustrative examples to highlight the importance of our findings.

Example 1. *Consider the third-order dynamic equation*

$$\left\{\eta^{\delta_2}\phi_{\delta_2}\left(\left[\eta^{\delta_1}\phi_{\delta_1}\left(u^{\Delta}(\eta)\right)\right]^{\Delta}\right)\right\}^{\Delta} + \frac{1}{\eta^{1-1/\delta}}\phi_{\delta}(u(g(\eta))) = 0, \tag{63}$$

It is clear to see that

$$\int_{\eta_0}^{\infty} \frac{\Delta s}{\alpha_j^{1/\delta_j}(s)} = \int_{\eta_0}^{\infty} \frac{\Delta s}{s} = \infty,\ j = 1,2,$$

and

$$\int_{\eta_0}^{\infty} p(s)\Delta s = \int_{\eta_0}^{\infty} \frac{\Delta s}{s^{1-1/\delta}} = \infty,$$

by [5], *Example 5.60. According to Theorem* 6, *then every solution of Equation* (63) *oscillates or converges to zero.*

Example 2. *Consider the third-order delay dynamic equation*

$$\left\{\frac{1}{\eta^{\delta_2-1}}\phi_{\delta_2}\left(\left[\frac{1}{\eta^{\delta_1}}\phi_{\delta_1}\left(u^{\Delta}(\eta)\right)\right]^{\Delta}\right)\right\}^{\Delta} + \frac{\beta\eta D_1^{1/\delta_1}(\eta)}{D_2(\eta)D_2^{\delta}(g(\eta))}\phi_{\delta}(u(g(\eta))) = 0, \tag{64}$$

where $\beta > 0$. *Clearly, condition* (2) *is satisfied. Hence,*

$$\begin{aligned}
&\limsup_{\eta\to\infty} D_2^{\delta}(\eta) \int_{\eta}^{\infty} \left(\frac{D_2(\psi(s))}{D_2(s)}\right)^{\delta} p(s)\Delta s \\
&= \beta \limsup_{\eta\to\infty} D_2^{\delta}(\eta) \int_{\eta}^{\infty} \frac{sD_1^{1/\delta_1}(s)}{D_2^{\delta+1}(s)}\Delta s \\
&\geq \frac{\beta}{\delta} \limsup_{\eta\to\infty} D_2^{\delta}(\eta) \int_{\eta}^{\infty} \left(\frac{-1}{D_2^{\delta}(s)}\right)^{\Delta}\Delta s \\
&= \frac{\beta}{\delta}.
\end{aligned}$$

Then, according to Theorem 7, every solution of Equation (64) oscillates or converges if $\beta > \delta$.

Example 3. *Consider the third-order advanced dynamic equation*

$$\left\{\eta^{\delta_2-1}\phi_{\delta_2}\left(\left[\eta^{\delta_1}\phi_{\delta_1}\left(u^{\Delta}(\eta)\right)\right]^{\Delta}\right)\right\}^{\Delta}+\frac{\beta D_1^{1/\delta_1}(\eta)}{\eta D_2^{\delta}(\eta)D_2(\sigma(\eta))}\phi_{\delta}(u(g(\eta)))=0, \tag{65}$$

where $\delta \geq 1$ and $\beta > 0$. It is clear that condition (2) is fulfilled. Hence,

$$\begin{aligned}&\liminf_{\eta\to\infty} D_2^{\delta}(\eta)\int_{\eta}^{\infty}\left(\frac{D_2(\psi(s))}{D_2(s)}\right)^{\delta}p(s)\Delta s\\&=\beta\liminf_{\eta\to\infty} D_2^{\delta}(\eta)\int_{\eta}^{\infty}\frac{D_1^{1/\delta_1}(s)}{sD_2(\sigma(s))}\Delta s\\&\geq\frac{\beta}{\delta}\liminf_{\eta\to\infty} D_2^{\delta}(\eta)\int_{\eta}^{\infty}\left(\frac{-1}{D_2^{\delta}(s)}\right)^{\Delta}\Delta s\end{aligned}$$

Consequently, Theorem 7, implies that every solution of (65) oscillates or converges if

$$\beta > \frac{1}{L^{\delta(\delta-1)}}\left(\frac{\delta}{1+\delta}\right)^{1+\delta}.$$

5. Discussion and Conclusions

(1) In this paper, the findings presented are applicable across all time scales without any restrictive conditions, including $\mathbb{T}=\mathbb{R}$, $\mathbb{T}=\mathbb{N}$, and $\mathbb{T}=q^{\mathbb{N}_0}:=\{q^n : n\in\mathbb{N}_0 \text{ for } q>1\}$.

(2) In this paper, we present some sharp oscillation criteria of the Hille-type and Ohriska-type for third-order half-linear functional dynamic equations when $g(\eta)\leq\eta$ and $g(\eta)\geq\eta$. Our results represent an improvement over previously established Hille-type and Ohriska-type criteria, as detailed below. By virtue of

$$D_2^{\delta}(\eta)\int_{\eta}^{\infty}\left(\frac{D_2(\psi(s))}{D_2(s)}\right)^{\delta}p(s)\Delta s\geq D_2^{\delta}(\eta)\int_{\eta}^{\infty}\left(\frac{D_2(\psi(s))}{D_2(\sigma(s))}\right)^{\delta}p(s)\Delta s$$

and

$$\frac{\delta^{\delta}}{L^{\delta|1-\delta|}(1+\delta)^{1+\delta}}<\frac{\delta^{\delta}}{L^{\delta^2}(1+\delta)^{1+\delta}}\quad\text{for } 0<L<1 \text{ and } \delta>\frac{1}{2}.$$

Theorem 7 improves Theorem 1 (criterion (62) improves (26) and criterion (55) improves (27)).

(3) Establishing Hille-type oscillation criteria for a third-order dynamic Equation (1) would be interesting, assuming that

$$\int_{\eta_0}^{\infty}\frac{\Delta s}{\alpha_j^{1/\delta_j}(s)}<\infty,\ j=1,2.$$

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