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Advanced numerical simulation of hybrid nanofluid radiative flow with Cattaneo-Christov heat flux model over a rotating disk: Innovative iterative techniques

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A R T I C L E I N F O

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A B S T R A C T

This paper investigates the effect of nonlinear thermal radiation on SWCNT-TiO_2 and MWCNT-$CoFe_2O_4$ nanoparticles suspended in a water-based hybrid nanofluid, flowing past rotating disks. The study employs the Cattaneo-Christov heat flux model to capture the influence of non-Fourier heat conduction. The rotational motion of the disks generates the fluid flow, and the governing partial differential equations are transformed into dimensionless forms using similarity variables. These equations are then solved using a New Iterative Technique (NIT) in Mathematica, which is known for its rapid convergence and accuracy. The analysis focuses on the behavior of various parameters, including velocity components ($\hat{u}$, $\hat{v}$, $\hat{w}$), temperature ($\widehat{T}$), and thermal conductivity ($\widehat{k}$), under different heat transfer conditions. Graphical representations illustrate the effects of these parameters, providing insights into the thermal and fluid dynamic performance of the hybrid nanofluid. The study demonstrates that the NIT is highly effective for solving complex fluid dynamics problems, offering precise and swift solutions. NIM provide an efficient and accurate solution for complex nonlinear problems, overcoming the limitations of traditional methods. This approach enhances computational efficiency and solution accuracy in modeling hybrid nanofluid behavior. This research contributes to the understanding of hybrid nanofluids in

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engineering applications, particularly in optimizing heat transfer in systems involving rotating machinery.

Terminology	
$\hat{u}, \hat{v}, \hat{w}$	Vector components of velocity (ms^{-1})
$\hat{w}_0$	Suction/infusion
Ω	Rotational velocity
$\hat{T}$	Hybrid nanofluid thermal temperature (K)
σ	Electrical conductivity $(\Omega^{-1}m^{-1})$
$\hat{k}$	Thermal Conductive Ability
σ^*	Thermal Radiation Constant $(Wm^{-2}K^{-4})$
$\hat{B}_0$	Magnetic field strength $\left(\Omega^{-\frac{1}{2}}m^{-1}s^{-\frac{1}{2}}kg^{\frac{1}{2}}\right)$
$\theta_{\hat{w}}$	Temperature ratio parameter
$\hat{P}\hat{r}$	Prandtl number
$\hat{f}_{\hat{w}}$	Vacuum Action and pumping
$\hat{C}_g$	Tangential Shear Force
$\hat{C}_{g\hat{r}}$	Non-dimensional Surface Shear Stress
$\hat{N}\hat{u}_{\hat{r}}$	Normalized Nusselt Ratio
$\hat{r}, \varnothing, \hat{z}$	Spatial Reference System
$\hat{T}_{\hat{w}}$	Wall Boundary Temperature (K)
$\hat{T}_\infty$	Atmospheric Temperature (K)
$\rho\hat{c}_p$	Heat capacitance $(Jm^{-3}K^{-1})$
μ	Dynamic viscosity $(kgs^{-1}K^{-1})$
$\hat{q}_{\hat{r}}$	Radiative head flux (kgs^{-3})
$\hat{K}^*$	Mean Absorption Factor (m^{-1})
φ	Solid Particle Resistance
$\hat{M}$	Magnetic Factor
$\hat{R}\hat{d}$	Nonlinear Radiative Thermal Effect
$\hat{G}_{\hat{f}}$	Radial Shear Stress

1. Introduction

The impact of form considerations is crucial, as nanomaterials vary widely in shape, properties, and size. Carbon nanotubes (CNTs) are a prevalent type of nanomaterial, categorized into single-wall carbon nanotubes (SWCNTs) and multi-wall carbon nanotubes (MWCNTs). Carbon nanotubes may be used for energy storage, wetting, and sharpening fracture stiffness, among other things. Carbon nanotubes are widely employed in a variety of applications, including thermal stability and energy conservation, electromagnetic fields, and nanotube transistors by Choi et al. [1] investigated nanofluids as well. In comparison to nanofluids, hybrid nanofluids offer a wide range of applications. Increased efficiency is predicted to improve heat transmission in hybrid nanofluids. A hybrid is a nanofluid generated when two nanofluids are displaced in the base fluid. Heat transmission to nanosolvent units has been found to be improved by hybrid nanofluids. The current study reveals that hybrid-based nanofluids exhibit high heat transfer rates, even when thermal radiation is present. This “hybrid nano-fluid" has become a prominent topic among engineering researchers in practically every sector, such as medicine, microstructure, mechanics, microelectronics, ships, and shipbuilding. Aerospace, security, and propulsion by Suresh et al. [2] initially proposed the basic concept of a hybrid nanofluid to enhance essential heat transfer.

A hybrid nanofluid consists of two or more nanoparticles with unique thermodynamic and chemical properties, designed to enhance heat transfer efficiency by Sarkar et al. [3] Tian et al. [4] explored how two-dimensional and three-dimensional fin geometries affect slip and non-slip magnetic hybrid nanoparticle flows and their impact on thermal zinc efficiency by Roy et al. [5] measured the brittle boundary layer behavior of micropolar water-based hybrid nanofluids flowing over extended surfaces by Alizadeh et al. [6] discussed the influence of permeable discs on optimizing the dynamics of phase change mechanisms in phase change materials (PCMs) by.

Fazili et al. [7] investigated the forced convection and thermal characteristics of copper heat transfer in nano-fluids when activated in circular tubes. Due to effect of existing main fluid heterogeneous characteristics and thermal radiation consisting of hybrid nanofluids, stability research was conducted with the goal of increasing the current heat transmission studied by Kumar et al. [8] Powell-Eyring hybrid nanofluids are water-based hybrid nanofluids that flow with convection heat transfer, generating volumetric entropy on a uniformly permeable, horizontally extending surface as presented by Aziz et al. [9]. Hybrid nanofluid flows from the transparent space through a revolving disc with changeable characteristics examined by Haider et al. [10]. Fast slip effects were used to assess the normalized second-grade nanofluid flow in by Lee et al. [11]. Kaska et al. [12] explored heat transfer from a flat tube to a hybrid nanofluid. Golinia et al. [13] investigated the effects of carbon nanotubes (CNTs) on water-based hybrid nanofluids. By Farooq et al. and Hosseinzadeh et al. [14,15] examined entropy generation and heat radiation effects in nanofluids.

The recent application of technology to rotating discs has made them a leading domain in the study of fluid dynamics. This has been a topic of increasing interest ever since the honourable study by von Kerman in 1921 was considered an indefinite extension. The recorded flow between contemporaneously rotating discs by Brady, J.F., Durlofsky et al. [16]. Researchers have extensively studied heat transfer to spinning surfaces through fluid flow for various applications, including thermal power generation, gas turbines, geothermal systems, chemical processes, medical equipment, and computers by John J. Nelka et al. [17] initially solved the Navier-Stokes problem for fluid flow over a rotating disc, by Cochrane et al. [18] providing the first numerical solution for this problem by C.L.Tien et al. [19] extended Von Kerman's problem using two series methods by Nihan Uygun et al. [20] investigated heat transfer properties on a rotating disc with an applied radial electric field. Khan et al. [21] examined heat transfer in magnetic hydrodynamics (MHD) fluid flow on a rotating disc. Rashidi et al. [22] explored the impact of rotating disc geometry on MHD flow, focusing on the effects of perforations, Transmission of heat According to Darcy, copper and silver nanofluid flow between the foci of two revolving discs that may be stretched, by T. Hyatt et al. [23]. The influence of a fluid spinning disk's thermal conductivity was discussed. Khan et al. [24]. Nanofluid frictional behavior is examined by Shuaib et al [25]. Gold and silver nanoscale particles are suspended in a dazzling film coating on a revolving disc with several fluids studied Elahi et al. [26].

Heat transmission is extremely significant in both industrial and biological applications. Electronic cooling, nuclear reactive cooling, tissue thermal conductivity, energy manufacturing, and other topics are discussed. The traditional Fourier thermal conductivity rule has correctly defined the heat transmission process by J. B. J. Fourier et al. [27]. The primary limitation of traditional thermal models is the "paradox of thermal conductivity," where immediate reflection of the energy equation suggests an unrealistic obstruction in the medium. To address this issue, researchers have proposed modifications to Fourier's law of thermal conductivity. Cattaneo [28] introduced a relaxation period in Fourier's law to account for the time required for thermal conductivity adjustments when temperature changes are applied to an insulating element. This approach was expanded by V. Tibullo et al. [29] to include a modified heat flux model. The Cattaneo-Christov model was further applied to incompressible fluids by E. H. Rikitu [30], and Straughan [31] utilized it for horizontal layers of Newtonian fluids under gravitational influence. Shihao Han et al. [32] examined the specificity and stability of Cattaneo-Christov equations. The model has also been used to address heat transport and viscoelastic fluid coupling by R.M. Khan et al. [33] and M. Javaid et al. [34] applied it to rotating viscoelastic fluid flows surrounded by extended sheets.

Heat transfer fluid transport is critical in various industrial and engineering processes, including advanced system heat management, device design, cooling and heating operations, gas turbine production, and nuclear applications. Despite its significance, there is limited research on the unstable magnetohydrodynamic (MHD) flow of non-Newtonian fluids in porous media under high velocity and elevated temperature conditions. The analytical handling of these complex nonlinear relationships holds significant practical implications. For instance, by B. Kunda [35] explored thermodynamic conditions for effective cancer treatment and minimal side effects from adjacent thermotherapy. McIntosh et al. [36] highlighted rapid warming as a strategy for mitigating global warming in high-temperature chemical processes. R. Nandkeolyar et al. [37] and G.S. Seth et al. [38] compared different plate movements in natural convection flows influenced by magnetic fields and varied acceleration conditions. G.S. Seth et al. [39] and P. Chandran et al. [40] investigated Hall currents, thermal radiation, and heat consumption in relation to plate temperatures and velocities. Chandran et al. [41]studied the effect of wall temperature on viscous fluid flow, later extended by Seth et al. [42] to porous media. T. Hayat et al. [43] examined fluid flow and wall temperature impacts on free convection with heat radiation and magnetic fields, with further research by Maqbool et al. [44] on finite boundary conditions. Tiwana et al. [45] explored MHD convection flow of Aldroid-B fluids under simultaneous boundary conditions.

In 2006, Daftardar-Gejji and Jaffari [46] introduced the Numerical Iterative Method (NIM), a novel approach for solving both linear and nonlinear equations. This technique simplifies the solution process by avoiding complex computations such as Adomian polynomials, Lagrange coefficients, and differential numerical methods, which are often required in other methods like the Adomian Decomposition Method (ADM) or Variational Iteration Method (VIM). Unlike traditional perturbation methods, NIM does not rely on small parameter assumptions, making it versatile for a wide range of problems. However, it requires an initial condition to start the iterative process. Since its introduction, NIM has been applied to various problems including algebraic equations, evolutionary equations, and nonlinear dynamic systems. For example, Daftardar-Gejji et al. [47] utilized NIM for linear and nonlinear fractional diffraction wave equations with Dirichlet boundary conditions. Bhalekar et al. [48,49]employed NIM to develop fractional versions of logistic equations and compare it with other decomposition approaches. Hemeda [50] applied NIM to solve sequence linear and nonlinear co-differential equations, and Yaseen et al. [51] used it for the Seagull-Newell-Whitehead equation. The iterative approach for solving nonlinear equations includes just the function's first derivative as proposed by Patade et al. [52]. Iterative methods have been proposed to solve the nonlinear equation system by Noor et al. [53]Authors in the 60s proposed and analyzed some of the iterative relationships that generate different classes of iteration techniques to investigate non-linear problems using the equation method. The non-linear Baranai and Roberts models were tested using NIM by Noor et al. [54] In comparison to other analytical technologies, NIM offers solutions that are more quickly converged. H. Waqas et al. [55] investigated the heat transfer characteristics of hybrid nanofluid flow with thermal radiation over a stretching sheet. H. Waqas et al. [56] conducted a numerical simulation to analyze entropy generation in nanofluids, considering the effects of thermal radiation and the Cattaneo-Christov heat flux model. Liu, Dong et al. [57] examined the heat transfer performance and conducted an entropy generation analysis of Taylor-Couette flow with a helical slit wall. Ji, Chenhui et al. [58] explored the enhancement of melting performance in phase change materials within a thermal energy storage unit using nanoparticles. Hafeez, Abdul et al. [59] studied the melting heat transfer characteristics in a Bodewadt flow of a hybrid nanofluid ($Cu-Al_2O_3$/water) over a stationary stretching disk. Waqas, Hassan et al. [60] investigated the melting performance of phase change materials (PCM) enhanced with MoS_2 and Fe_3O_4 nanoparticles, utilizing leaf-shaped fins with various orientations in a shell-and-tube thermal energy storage (TES) system. Waqas, Hassan et al. [61] examined the impact of MHD radiative flow in a hybrid nanofluid over a rotating disk.

The novelty of the Numerical Iterative Method (NIM), introduced by Daftardar-Gejji and Jaffari in 2006, lies in its ability to solve both linear and nonlinear equations without relying on complex computations or small parameter assumptions. Unlike traditional methods such as the Adomian Decomposition Method (ADM) or Variational Iteration Method (VIM), NIM simplifies the solution process and achieves faster convergence. Its versatility allows it to handle a wide range of problems, from algebraic to nonlinear dynamic systems, with notable applications in fractional differential equations and various other nonlinear models. In Table 1 these properties are essential for evaluating the suitability of these nanoparticles in enhancing the thermal and electrical performance of nanofluids in various engineering applications.

2. Problem formulation

Consider a three-dimensional, incompressible, second-grade hybrid nanofluid flow moving past a rotating disc, influenced by several forces. An externally applied magnetic field B_0 is present on the disc's surface. The flow is affected by nonlinear thermal radiation, heat generation and absorption, and Joule heating.

- The disk revolves at angular velocity α_1 at $z = 0$.
- The temperature and concentration of the disk are T_w *and* C_w.
- whereas far away temperature and concentration are T_∞ *and* C_∞. Here $(\hat{u}, \hat{v}, \hat{w})$ are along the directions of the velocity components $(\hat{r}, \varnothing, \hat{z})$. Fig. 1 shows Hybrid nanofluid geomatery on rotating disk.

The mass equation is,

$$\hat{u}r + \frac{\hat{u}}{r} + \hat{w}_z = 0, \tag{1}$$

The momentum equations along $(\hat{u}, \hat{v}, \hat{w})$ are,

$$\rho_{hnf}\left(\hat{u}\hat{u}_r - \frac{\hat{v}^2}{r} + \hat{w}\hat{u}_z\right) + \rho_r = \mu_{hnf}\left(\hat{u}_{rr} + \frac{1}{r}\hat{u}_r - \frac{\hat{u}}{r^2} + \hat{u}_{zz}\right) - \sigma_{hnf}B_0^2\hat{u}, \tag{2}$$

$$\rho_{hnf}\left(\hat{u}\hat{v}_r + \frac{\hat{u}\hat{v}}{r} + \hat{w}\hat{v}_z\right) = \mu_{hnf}\left(\hat{v}_{rr} + \frac{1}{r}\hat{v}_r - \frac{\hat{v}}{r^2}\right) - \sigma_{hnf}B_0^2\hat{v}, \tag{3}$$

$$\rho_{hnf}(\hat{u}\hat{w}_r + \hat{w}\hat{w}_z) + p_r = \mu_{hnf}\left(\hat{w}_{rr} + \frac{1}{r}\hat{w}_r + \hat{w}_{zz}\right) \tag{4}$$

Energy equation with thermal radiation is

$$(\rho c_p)_{hnf}(\hat{u}\hat{T}_r + \hat{w}\hat{T}_z) = k_{hnf}\left(\hat{T}_{rr} + \frac{1}{r}\hat{T}_r + \hat{T}_{zz}\right) - (\hat{q}_r)_z \tag{5}$$

The thermal radiation heat flux is expressed as:

$$q_r = \frac{4\sigma^*}{3k^*}\hat{T}_z^4 = -\frac{16\sigma^*}{3k^*}\hat{T}^3\hat{T}_z \tag{6}$$

Here $T = T_\infty[1 + (\theta_w - 1)\theta]$ *here* $\theta_{w = \frac{Tw}{T\infty}}$.

The applicable boundary conditions are,

$$\begin{cases} \hat{u} = 0, \\ r = \Omega r, \\ \hat{w} = \hat{w}_0, \\ \hat{T} - \hat{T}_w \end{cases} at\ z = 0, \begin{cases} \hat{u} \to 0, \\ \hat{v} \to 0, \\ \hat{T} \to \hat{T}_\infty \\ \hat{P} \to \hat{P}_\infty \end{cases} at\ z \to \infty \tag{7}$$

Conversion of Naviour stokes equations in ODEs system.

The suitable similarity transformation are,

Table 1
Thermophysical properties of water and selected nanoparticles.

Properties & Particle	H2 O	SWCNT	TiO2	MWCNT	CoFe2 O4
c_p	4180	425	686.2	2000	700
P	997	2600	4250	2100	4907
K	0.6071	6600	8.9538	710	3.7
Σ	5.5 × 106	10^6–10^7	2.38 × 106	2.1 × 105	5.51 × 109

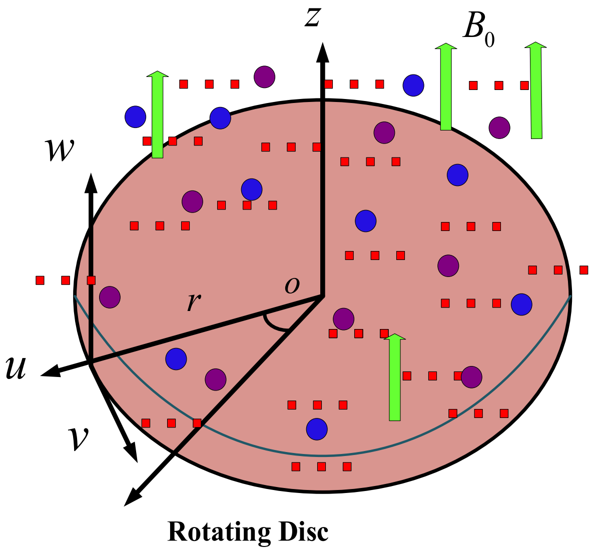

Fig. 1. Hybrid nanofluid geomatery on rotating disk.

$$\eta = \left(\frac{2\Omega}{\nu_f}\right)^{\frac{1}{2}}\hat{z}, \hat{u} = r\Omega f'(\eta), \hat{v} = r\Omega g(\eta),$$
$$\hat{w} = -\left(2\Omega\hat{\nu}_f\right)^{\frac{1}{2}} f(\eta), \vartheta(\eta) = \frac{\hat{T} - \hat{T}_\infty}{\hat{T}_w - \hat{T}_\infty}, \hat{P} = \hat{P}_\infty + 2\Omega\mu\hat{P}(\eta), \tag{8}$$

Substitues equation (8) into (1–7) give the following nonlinear ODEs system

$$f''' + \frac{A_1}{2A_4}\left(2ff'' - f'^2 + g^2\right) - \frac{A_4}{2A_4}Mf' = 0, \tag{9}$$

$$2g'' + \frac{A_1}{2A_u}\left(2fg' - 2f'g\right) - \frac{A_5}{2A_u}Mg = 0, \tag{10}$$

$$\vartheta'' + P_r\frac{A_2}{A_3}f\vartheta' + \frac{Rd}{A_3}\ \vartheta''(1 + (\vartheta_w - 1)\vartheta)^3 + 3(\vartheta_w - 1)$$
$$\left.(1 + (\vartheta_w - 1)\vartheta)^2\vartheta'^2\right) - P_r\lambda\frac{A_2}{A_3}\left(f^2\vartheta'' + \vartheta' ff'\right) = 0, \tag{11}$$

$$\begin{cases} f = -f_w, \\ f' = 0, \\ g = 1 \\ \vartheta = 1 \end{cases} at\ \eta = 0, and\ \begin{matrix} f' = 0 \\ g \to 0, \\ \vartheta \to 0, \end{matrix} \quad at\ \eta \to \infty \tag{12}$$

Where nondimensional quantities are

$$M = \frac{B_0^2}{\Omega\rho_f}, Pr = \frac{(\rho C_p)}{k_f}\nu_f, Rd = \frac{16\sigma T\infty^3}{3k_f k}, \lambda = 2\gamma\Omega$$

M signifies the magnetic parameter, Rd represents the thermal radiation parameter, Pr is the Prandtl number, w describes the temperature ratio parameter, and fw denotes the suction and injection parameter.

Here

$$A_1 = \frac{\rho_{hnf}}{\rho_f}, A_2 = \frac{(\rho C_p)_{hnf}}{(\rho C_p)_f}, A_3 = \frac{k_{hnf}}{k_f}, A_4 = \frac{\mu_{hnf}}{\mu_f}, A_5 = \frac{\sigma_{hnf}}{\sigma_f} \tag{13}$$

3. Basic principles of the NIM

The New Iterative Method will be employed to derive the nonlinear regression variance PDE. We'll present these correlations in the forthcoming poster. let us consider the following differential equation [64–65].

$$.\widetilde{u}(x,t) = \widetilde{f}(x,t) + \widetilde{N}(\widetilde{u}(x,t)), \widetilde{u}(x,t) + \widetilde{N}(\widetilde{u}(x,t)) \tag{14}$$

Here the function $f(x)$ consists of a series of variables as, $= x_1, x_2 \ldots\ldots. x_n$, and N represents a nonlinear functional operator within a collection of space. B → B,

Applying the fundamental concept of NIM, the solution to equation (14) can be constructed in a series

$$\widetilde{u}(x,t) = \sum_{k=0}^{\infty} \widetilde{u}_k(x,t), \tag{15}$$

The operator N can be expanded as such:

$$\widetilde{N}\left(\sum_{k=0}^{\infty} \widetilde{u}_k\right) = \widetilde{N}(\widetilde{u}_0) + \sum_{k=1}^{\infty}\left\{\widetilde{N}\left(\sum_{j=0}^{k} \widetilde{u}_j\right) - \widetilde{N}\left(\sum_{j=0}^{k-1} \widetilde{u}_j\right)\right\} \tag{16}$$

Hence general equation can be written as

$$\widetilde{u}(x) = \sum_{k=0}^{\infty} \widetilde{u}_k(x) \tag{17}$$

$$= f + \widetilde{N}(\widetilde{u}_0) + \sum_{k=1}^{\infty}\left\{\widetilde{N}\left(\sum_{j=0}^{k} \widetilde{u}_j\right) - \widetilde{N}\left(\sum_{j=0}^{k-1} \widetilde{u}_j\right)\right\} \tag{18}$$

we have

$$\widetilde{u}_0 = f,$$

$$G_0 = \widetilde{N}(\widetilde{u}_0),$$

$$G_1 = \widetilde{N}(\widetilde{u}_0 + \widetilde{u}_1) - \widetilde{N}(\widetilde{u}_0),$$

$$G_2 = \widetilde{N}(\widetilde{u}_0 + \widetilde{u}_1 + \widetilde{u}_2) - \widetilde{N}(\widetilde{u} + \widetilde{u}_1)$$

And in general

$$G_m = \widetilde{N}(\widetilde{u} + \widetilde{u}_1 + \ldots\ldots + \widetilde{u}_m) - \widetilde{N}(\widetilde{u}_0 + \widetilde{u}_1 + \ldots\ldots \widetilde{u}_{m-1}) m = 1, 2, 3 \ldots \tag{19}$$

The approximate solution for the kth term is …

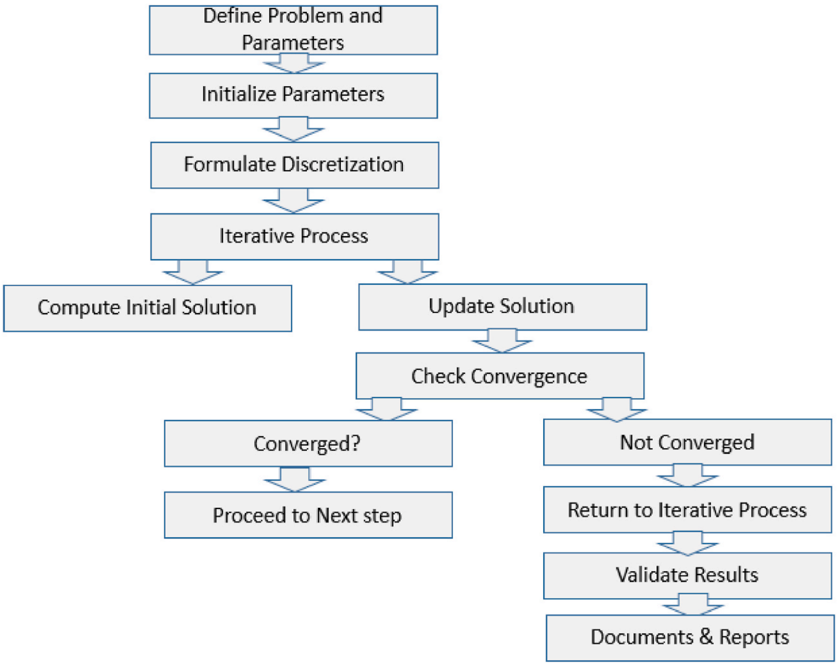

Fig. 2. Block diagram of problem formulation.

$$\tilde{u} = \tilde{u}_0 + \tilde{u}_1 + \ldots\ldots + \tilde{u}_{-1k}$$

The New Iterative Method offers several advantages: it provides high accuracy and convergence for solving complex, nonlinear differential equations by iteratively refining solutions. Its flexibility allows it to handle a wide range of parameter values and boundary conditions effectively. Additionally, the method's efficiency in computational resource usage makes it suitable for large-scale problems, ensuring reliable and robust results for the MHD flow analysis. In Fig. 2 show the block diagram of the above problem formulation.

4. Results and discussion

The parameters for SWCNT-TiO_2 and MWCNT-$Cofe_2O_4$ are thoroughly analyzed and illustrated in Figs. 3–14. Two scenarios for hybrid nanofluids are examined: SWCNT-TiO_2 and MWCNT-CoFe2O4. The effects of suction and injection parameters (fw) and the magnetic parameter (M) on the radial flow of the non-hybrid variants of SWCNT-TiO_2 and MWCNT-$Cofe_2O_4$ are demonstrated in Fig. 3a and b. The magnetic parameter (M) creates a resistive Lorentz force, reducing radial velocity further. Together, these parameters control the flow dynamics around the rotating disc.

Lorentz forces generated by a strong magnetic field reduce the hybrid current, leading to a decrease in the radial velocity of the hybrid nanofluid. The influence of suction and injection parameters on the radial profile, as well as the magnetic parameter on the radial disc profile, is illustrated in Fig. 4a and b. These figures show that an increase in the magnetic parameter results in a decrease in the velocity of the hybrid nanofluid, confirming that larger magnetic parameters enhance Lorentz forces, which in turn reduce fluid velocity.

Fig. 5a and b examine the effect of nonlinear thermal radiation parameters on the friction profiles of solid particles, demonstrating that changes in temperature fields are exaggerated by varying the sphere's shape. Changes in the shape of the particles further enhance this effect, altering heat distribution and friction profiles. This highlights the significant role of particle geometry and thermal radiation in controlling surface friction and heat transfer. Fig. 6a and b reveal that for large values of suction/injection parameters, A lower Prandtl number indicates a higher rate of thermal diffusion relative to momentum diffusion, causing the fluid to transfer heat more efficiently, the Prandtl number decreases in SWCNT-TiO_2 and MWCNT-$CoFe_2O$ hybrid nanofluids, leading to a temperature increase as shown in Fig. 6a His suggests that higher suction/injection enhances heat transfer, raising the overall temperature in the system.

Fig. 7a depict how the thermal field of non-hybrid materials increases with a decrease in the thermal relaxation parameter and solid particles parameter. Similarly, a decrease in the solid particles parameter reduces resistance to heat transfer, allowing the thermal field to expand. This results in an overall increase in temperature distribution within the fluid. Fig. 8a and b highlight the effects of thermal radiation on magnetic profiles, showing that changes in the magnetic field and suction/injection profiles affect skin friction. Increased thermal radiation enhances the interaction between the magnetic field and fluid, altering flow dynamics and thereby affecting the resistance or friction at the surface. This indicates that both magnetic effects and thermal radiation play significant roles in determining the frictional characteristics of the fluid flow. Fig. 9a and b demonstrate that an increase in nanoparticle volume fraction leads to enhanced velocity, as solid particle friction reduces the basic fluid's viscosity. This is because the presence of solid particles reduces the viscosity of the base fluid, making it less resistant to flow. As a result, the enhanced flow properties allow for higher velocity within the fluid system.

Finally, Fig. 10a and b focus on the analysis of volume friction for SWCNT TiO_2 and MWCNT-$CoFe_2O$ nanofluids, indicating that an increase in nanoparticle volume enhances friction. This occurs because a higher concentration of nanoparticles increases the interactions between particles and the fluid, raising the resistance to flow. Consequently, the fluid exhibits greater frictional forces as the volume fraction of nanoparticles increases. Fig. 11a and b illustrate that the shape factors of nanoparticles affect heat distribution, Different shapes affect the thermal conductivity and dispersion of nanoparticles, thereby altering the overall heat transfer characteristics. while Fig. 12 shows an increase in the Nusselt number due to nonlinear thermal radiation. as evidenced by an increased

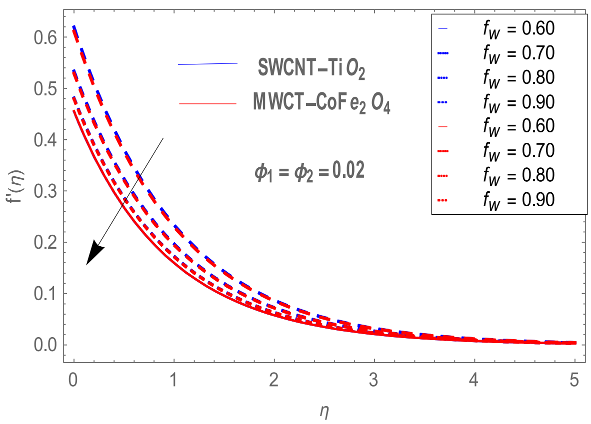

Fig. 3a. Effect of suction and injection parameter on velocity profile.

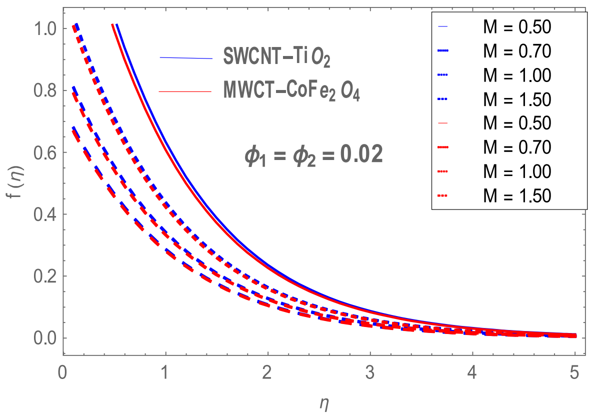

Fig. 3b. Impact of magnetic parameter on velocity profile.

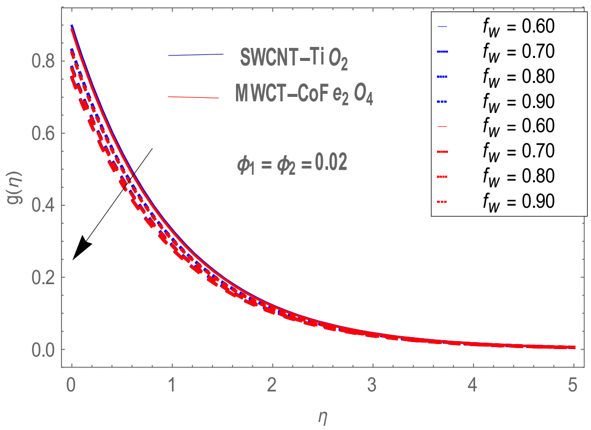

Fig. 4a. Suction and injection parameter on Radial velocity profile.

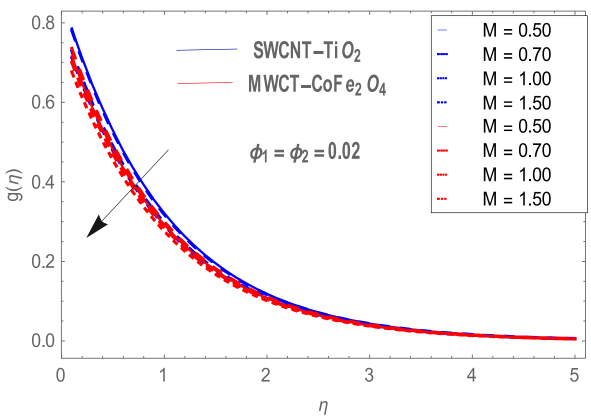

Fig. 4b. Magnatic parameter on Radial velocity profile.

Nusselt number, which signifies better convective heat transfer. Fig. 13 highlights that the magnetic parameter decreases on the suction and injection profiles. affecting fluid flow and heat transfer by altering the magnetic field's interaction with the fluid (see Table 2).

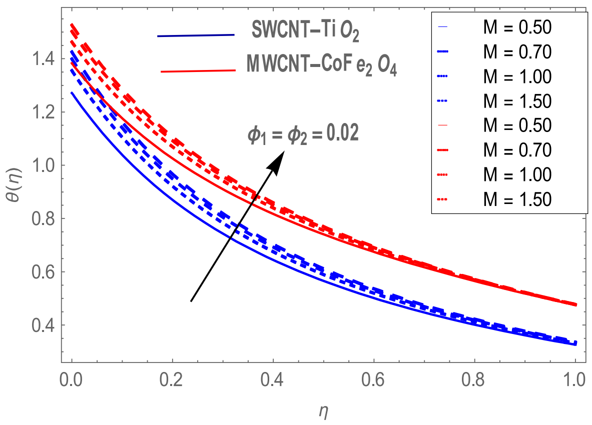

Fig. 5a. Magnetic parameter on energy profile.

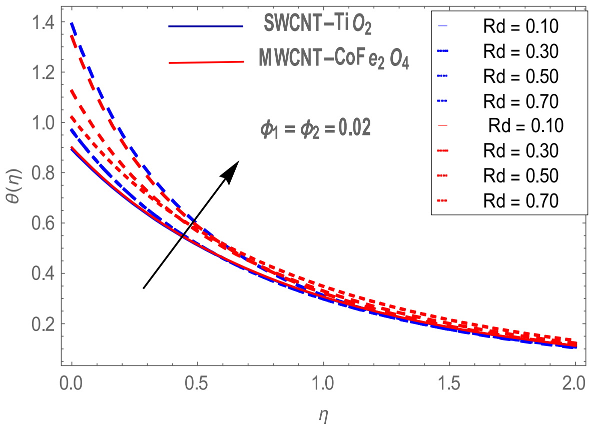

Fig. 5b. Nonlinear thermal radiation parameter energy distribution.

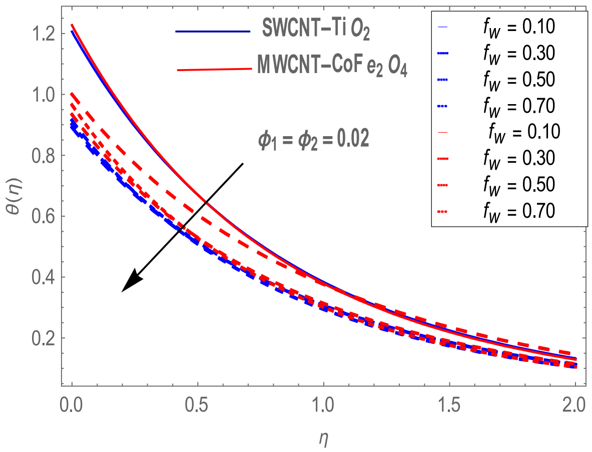

Fig. 6a. Suction and injection parameter on temperature profile.

5. Conclusions

In this study, the MHD flow of a hybrid nanofluid over a rotating disc, with the influence of nonlinear thermal radiation, has been analyzed. The hybrid nanofluid, composed of SWCNT-TiO2 and MWCNT-CoFe2O4 nanoparticles in a water base fluid, is modeled

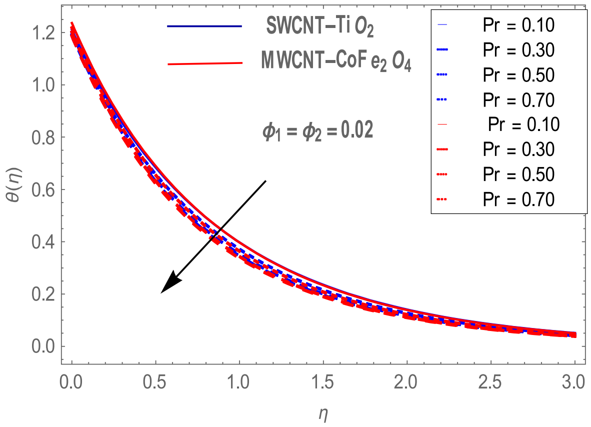

Fig. 6b. Prandtl number parameter on temperature profile.

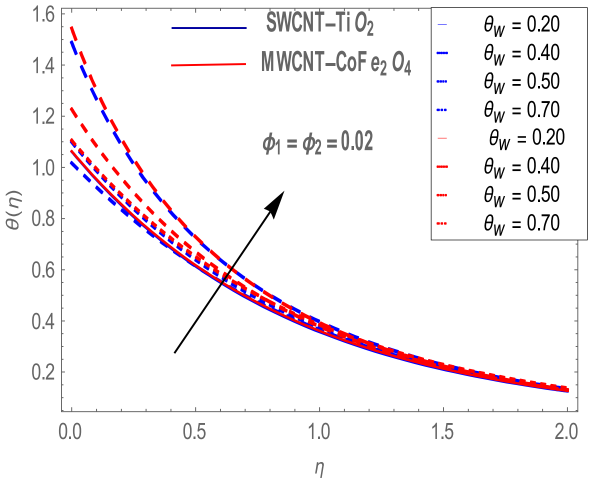

Fig. 7a. Temperature ratio parameter on temperature profile.

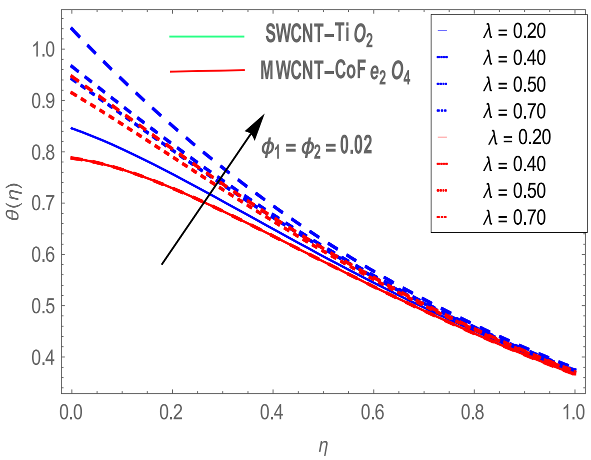

Fig. 8a. Thermal relaxtation parameter on temperature profile.

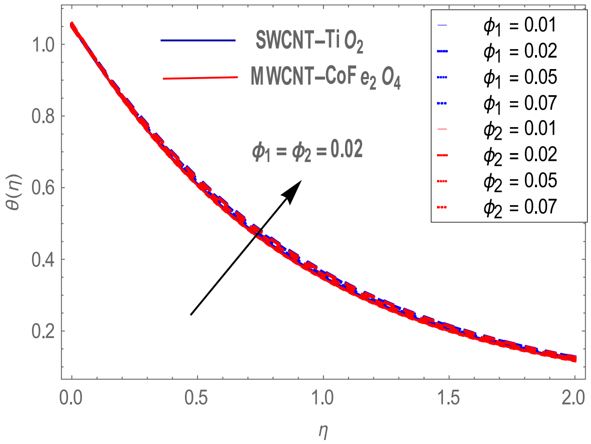

Fig. 8b. Solid particles parameter on temperature.

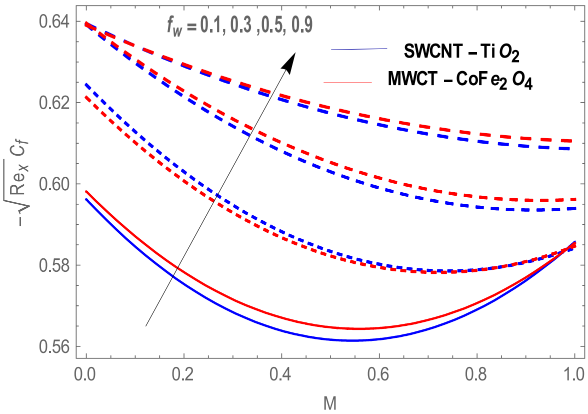

Fig. 9a. Suction injection parameter against magnetic field in Skin friction field.

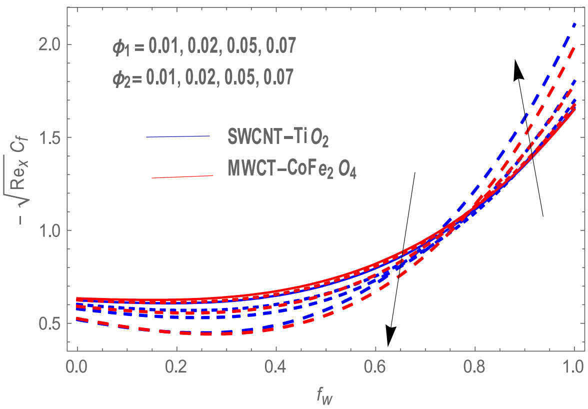

Fig. 9b. Thermal radiation Skin friction.

using the Tiwari-Das approach. The mathematical model is solved numerically using the New Iterative Method (NIM) implemented in Mathematica. The research explores how various flow parameters affect velocity components and thermal fields. The following is a summary of current research.

a) Decreasing the suction and injection parameters affects the velocity profile, radial velocity profile, temperature profile, and magnetic force in the Nusselt number.

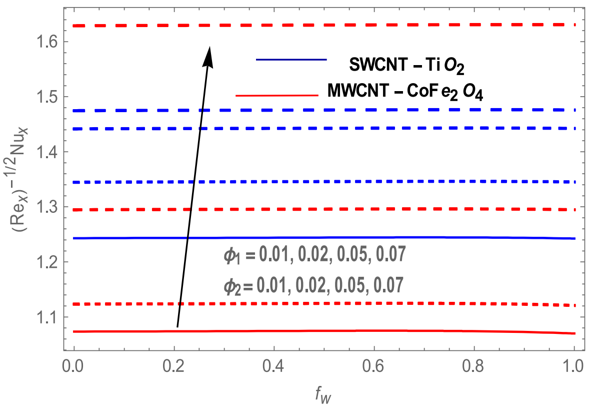

Fig. 10a. Suction injection parameter against magnetic field in Nusselt number $\varphi_1 = \varphi_2$.

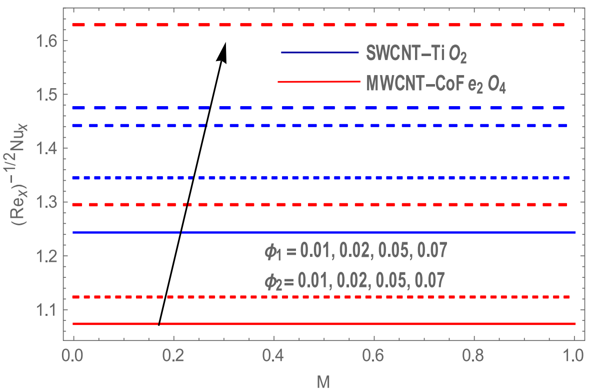

Fig. 10b. Magnetic parameter against magnetic field in Nusselt number $\varphi_1 = \varphi_2$.

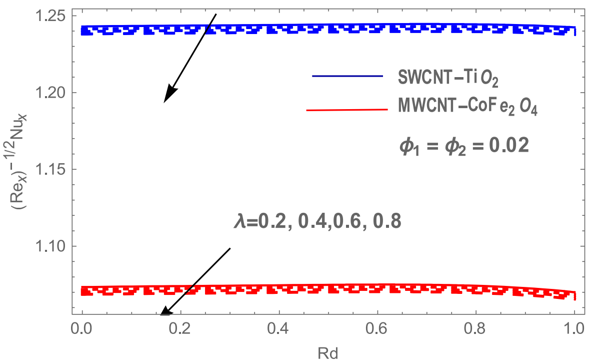

Fig. 11a. Thermal radiation against thermal rexation in Skin friction profile$\varphi_1 = \varphi_2$.

b) Increasing the suction and injection parameters impacts the magnetic field, skin friction, and Nusselt number.
c) Higher magnetic parameters influence the energy profile and magnetic field in the Nusselt number, while reducing the radial velocity profile, suction and injection profile, and Nusselt number.
d) Non-linear thermal radiation parameters increase energy distribution.
e) On the temperature profile, the Prandtl number parameter decreases.
f) Temperature ratio, thermal relaxation, and solid particles parameters all rise as the temperature rises.
g) Thermal radiation increases and decreases radiation skin friction and suction/injection in skin friction.
h) This sort of challenge may be used in the biomedical sector as well as in industrial technologies to increase heat transfer rates.

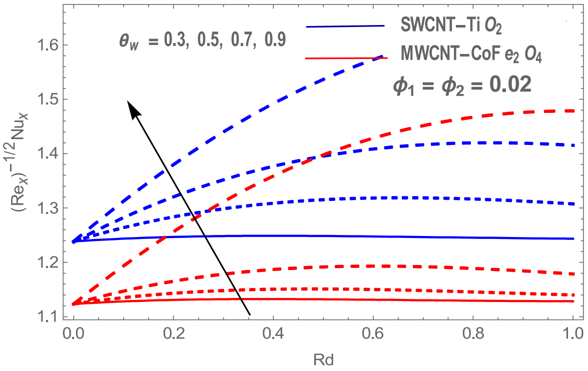

Fig. 11b. Thermal radiation against suction/injection in Skin friction profile $\varphi_1 = \varphi_2$.

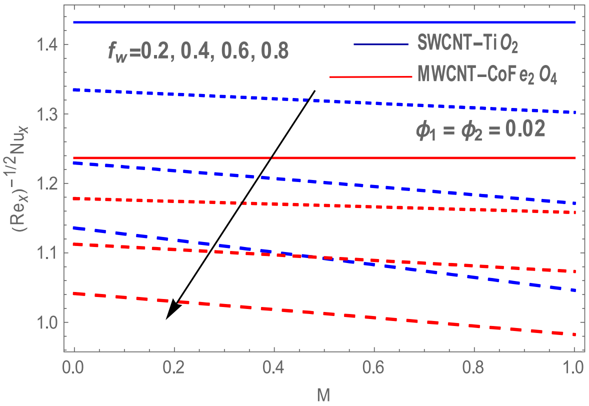

Fig. 12a. suction/injection parameter against Magnetic force in Nusselt number.

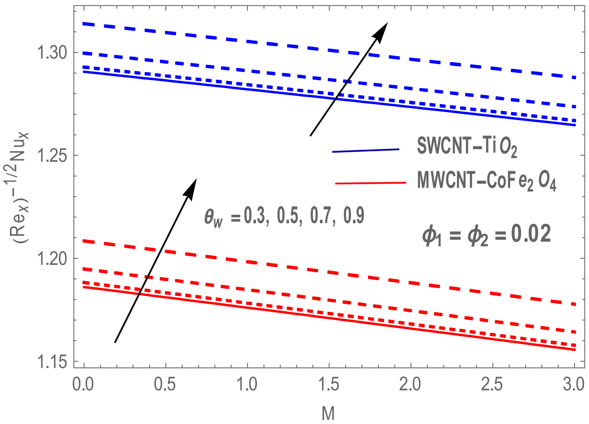

Fig. 12b. Temperature ratio parameters for Nusselt number.

6. Future work

Extension to 3D Flow and Complex Geometries: Future research could extend the current analysis to three-dimensional flows over more complex geometries, such as curved or irregular surfaces, to study the effects of non-linear thermal radiation and magnetic fields in more realistic industrial applications.

Investigation with Different Nanoparticle Combinations**:** The study could be expanded by experimenting with other hybrid nanoparticles or base fluids to analyze their impact on the thermal and flow characteristics. This could help identify combinations that enhance heat transfer in different industrial applications.

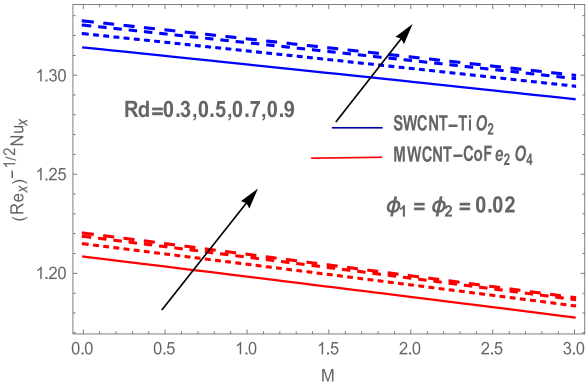

Fig. 13a. Nonlinear thermal radiation for Nusselt number.

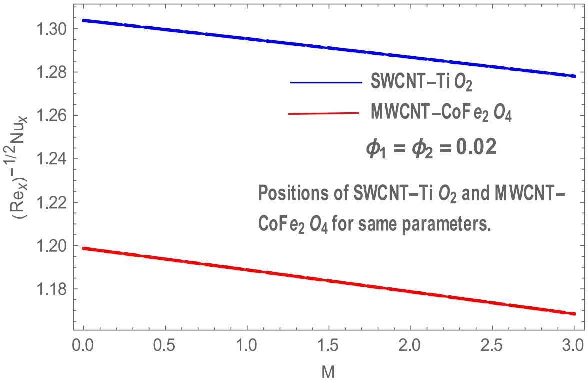

Fig. 13b. Nano particle position for Nusselt number.

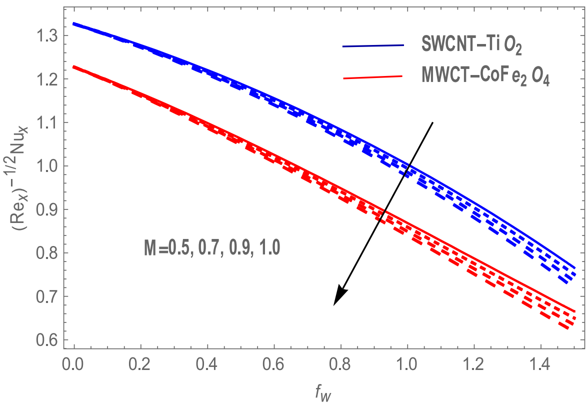

Fig. 14. Magnatic parameter against Suction and injection profile in Nusselt number.

Non-Newtonian Fluid Models: Introducing non-Newtonian fluid behavior into the mathematical model could better mimic biological fluids, providing more insight into biomedical applications like drug delivery or cooling mechanisms in medical devices.

Inclusion of Time-Dependent or Transient Effects: Future work could focus on time-dependent (transient) MHD flows and how the system evolves dynamically, especially under varying magnetic fields or fluctuating thermal conditions. This would be useful for applications that involve time-varying processes.

Optimization Studies: Applying optimization algorithms (such as machine learning models) to determine the optimal parameters for maximum heat transfer efficiency or minimal skin friction could help in engineering applications, particularly in cooling systems and energy efficiency.

Experimental Validation: Conducting experimental studies to validate the numerical findings and improve the reliability of the

Table 2
Advanced thermophysical and electrical characteristics of hybrid nanofluids.

Properties	Hybrid Nanofluid
Density	$\rho_{hnf} = \rho_f(1 - \varnothing_1 - \varnothing_2) + \varnothing_1\rho_{s1} + \varnothing_2\rho_{s2}$
Viscosity	$\mu_{h\hat{n}f} = \frac{\mu_f}{(1-\varnothing_1)^{2.5}(1-\varnothing_2)^{2.5}}$
Heat Capacity	$(\rho cp)_{h\hat{n}f} = (\rho c_p)_f((1 - \varnothing_1 - \varnothing_2)) + \varnothing_1(\rho c_p)_{\hat{s}1} + \varnothing_2(\rho c_p)_{\hat{s}2}$
Thermal conductivity	$\frac{\widehat{k}_{h\ddot{n}f}}{\widehat{k}_{b\hat{f}}k} = \frac{\widehat{k}_{\hat{s}2} + (\widehat{n}-1)\widehat{k}_f - (\widehat{n}-1)(\widehat{k}_{\hat{b}f} - \widehat{k}_{\hat{s}2})\varnothing_2}{\widehat{k}_{\hat{s}2} + (\widehat{n}-1)\widehat{k}_{bf} + \varnothing_2(\widehat{k}_{bf} - \widehat{k}_{s2})}$ *where* $\frac{\widehat{k}_{bf}}{\widehat{k}_f} = \frac{\widehat{k}_{\hat{s}1} + (\widehat{n}-1)\widehat{k}_f - (\widehat{n}-1)(\widehat{k}_f - \widehat{k}_{\hat{s}1})\varnothing_1}{k_{\hat{s}1} + (\widehat{n}-1)\widehat{k}_f + \varnothing_2(\widehat{k}_f - \widehat{k}_{\hat{s}1})}$
Electrical	$\frac{\sigma_{h\hat{n}f}}{\sigma_{\hat{b}f}} = \frac{\sigma_{\hat{s}2} + 2\sigma_{\hat{b}f} + 2\varnothing_2(\sigma_{\hat{b}f} - \sigma_{\hat{s}2})}{(\sigma_2 + 2\sigma_{\hat{b}f} + \varnothing_1(\sigma_f - \sigma_{\hat{s}2}))}$ *where* $\frac{\sigma_{\hat{b}f}}{\sigma_f} = \frac{\sigma_{\hat{s}1} + 2\sigma_f - 2\varnothing_1(\sigma_f - \sigma_{\hat{s}1})}{\sigma_{\hat{s}1} + 2\sigma_f + \varnothing_2(\sigma_f - \sigma_{\hat{s}1})}$

model in real-world scenarios. This could enhance the practical application of the research, especially in industries like electronics cooling, biomedical devices, and advanced materials manufacturing.

Impact of Other External Forces: Exploring the impact of additional external forces, such as electric fields or varying gravitational effects, could provide deeper insights, especially in space-related technologies or high-tech manufacturing processes.

CRediT authorship contribution statement

Muhammad Jebran Khan: Writing – original draft, Supervision, Methodology, Conceptualization. **Samina Zuhra:** Writing – review & editing. **Zareen A. Khan:** Software. **Mohsin Ali:** Writing – original draft, Supervision, Methodology, Conceptualization. **Li Chen:** Supervision. **Abdul Haq:** Data curation. **Firas Zawaideh:** Formal analysis. **Radwan M. Batyha:** Validation. **Diaa Salama AbdElminaam:** Visualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Data availability

Data will be made available on request.

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