symmetry

MDPI

Article

A General and Comprehensive Subclass of Univalent Functions Associated with Certain Geometric Functions

Tariq Al-Hawary [1,2], Basem Frasin [3] and Ibtisam Aldawish [4,*]

[1] Department of Applied Science, Ajloun College, Al-Balqa Applied University, Ajloun 26816, Jordan; tariq_amh@bau.edu.jo
[2] Jadara Research Center, Jadara University, Irbid 21110, Jordan
[3] Faculty of Science, Department of Mathematics, Al al-Bayt University, Mafraq 25113, Jordan; bafrasin@yahoo.com
[4] Mathematics and Statistics Department, College of Science, IMSIU (Imam Mohammad Ibn Saud Islamic University), Riyadh 11623, Saudi Arabia
* Correspondence: imaldawish@imamu.edu.sa

Abstract: In this paper, taking into account the intriguing recent results of Rabotnov functions, Poisson functions, Bessel functions and Wright functions, we consider a new comprehensive subclass $\mathcal{O}_\mu(\Delta_1, \Delta_2, \Delta_3, \Delta_4)$ of univalent functions defined in the unit disk $\Lambda = \{\tau \in \mathbb{C} : |\tau| < 1\}$. More specifically, we investigate some sufficient conditions for Rabotnov functions, Poisson functions, Bessel functions and Wright functions to be in this subclass. Some corollaries of our main results are given. The novelty and the advantage of this research could inspire researchers of further studies to find new sufficient conditions to be in the subclass $\mathcal{O}_\mu(\Delta_1, \Delta_2, \Delta_3, \Delta_4)$ not only for the aforementioned special functions but for different types of special functions, especially for hypergeometric functions, Dini functions, Sturve functions and others.

Keywords: analytic functions; Rabotnov function; Poisson distribution; Bessel function; Wright function

MSC: 30C45

check for updates

Citation: Al-Hawary, T.; Frasin, B.; Aldawish, I. A General and Comprehensive Subclass of Univalent Functions Associated with Certain Geometric Functions. *Symmetry* **2024**, *16*, 982. https://doi.org/10.3390/sym16080982

Academic Editor: Junesang Choi

Received: 27 June 2024
Revised: 21 July 2024
Accepted: 24 July 2024
Published: 2 August 2024

1. Introduction and Preliminaries

In mathematics, hypergeometric functions are a basic family of special functions with wide-ranging applications in number theory, probability theory, mathematical physics, combinatorics and differential equations. In addition to serving as building blocks for creating mathematical models and solving issues in a variety of scientific and engineering disciplines, they offer solutions to differential equations that describe physical phenomena. Their applicability makes them essential tools in mathematical analysis and scientific research. These functions show many mathematical features and linkages as recurrence relations and solutions to differential equations.

Specific instances of hypergeometric functions can be used to represent a wide range of classical special functions, including the Gaussian hypergeometric function, confluent hypergeometric functions and Legendre functions. Moreover, many correlations exist between hypergeometric functions and other mathematical functions, including gamma and Bessel functions.

In particular, in this paper we will consider Rabotnov functions, Poisson functions, Bessel functions and Wright functions, respectively.

At first, let $\mathcal{F}$ the class of analytic and univalent functions $\mathcal{L}$ defined in the unit disk $\Lambda = \{\tau \in \mathbb{C} : |\tau| < 1\}$ and meet $\mathcal{L}'(0) - 1 = \mathcal{L}(0) = 0$. Thus, each $\mathcal{L} \in \mathcal{F}$ has the following series:

$$\mathcal{L}(\tau) = \tau + a_2\tau^2 + a_3\tau^3 + \cdots = \tau + \sum_{u=2}^{\infty} a_u \tau^u, \ \ (\tau \in \Lambda). \tag{1}$$

The Rabotnov function as a special function was introduced by Rabotnov [1] in 1948, as follows:

$$\Phi_{\ell,\beta}(\tau) = \tau^{\ell} \sum_{u=0}^{\infty} \frac{(\beta)^u \tau^{u(\ell+1)}}{\Gamma((u+1)(\ell+1))}, \quad (\ell, \beta, \tau \in \mathbb{C}\ ).$$

The well-known Mittag–Leffler function, which is frequently used to solve integral equations, is represented by the Rabotnov function in particular cases.

The connection between the Mittag–Leffler function M and the Rabotnov function $\Phi_{\ell,\beta}(\tau)$ is shown in the recurrence relation

$$\Phi_{\ell,\beta}(\tau) = \tau^{\ell} M_{\ell+1,\ell+1}(\beta\tau^{\ell+1}), \ \ \ell, \beta, \tau \in \mathbb{C}.$$

For additional details regarding Mittag–Leffler, see [2–5].

The Rabotnov function $\Phi_{\ell,\beta}(\tau)$ is evidently not in class $\mathcal{F}$. Consequently, it makes sense to take into account the normalizing of the Rabotnov function that follows:

$$\mathcal{R}_{\ell,\beta}(\tau) = \tau^{\frac{1}{\ell+1}} \Gamma(\ell+1) \Phi_{\ell,\beta}(\tau^{\frac{1}{\ell+1}}) = \tau + \sum_{u=2}^{\infty} \frac{\beta^{u-1}\Gamma(\ell+1)}{\Gamma((\ell+1)u)} \tau^u, \ \tau \in \Lambda \tag{2}$$

where $\ell > -1$ and $\beta \in \mathbb{C}$.

A variable Y is considered Poisson-distributed if the values are taken to be $0, 1, 2, 3, \cdots$ for the parameter ν with probabilities $e^{-\nu}, \nu\frac{e^{-\nu}}{1!}, \nu^2\frac{e^{-\nu}}{2!}, \nu^3\frac{e^{-\nu}}{3!}, \cdots$, respectively. Thus,

$$P(Y = s) = \frac{\nu^s e^{-\nu}}{s!}, \ s = 0, 1, 2, 3, \cdots .$$

In 2014, the Poisson function was introduced by Porwal [6] (see also [7,8]), as follows:

$$\mathcal{K}(\nu, \tau) = \tau + \sum_{u=2}^{\infty} \frac{\nu^{u-1}}{(u-1)!} e^{-\nu} \tau^u, \qquad (\nu > 0, \tau \in \Lambda). \tag{3}$$

The radius of convergence for $\mathcal{K}(\nu, \tau)$ is infinite.

The generalized Bessel function of the first kind of $\chi(\tau)$, mentioned as:

$$\chi(\tau) = \chi_{p,b,c}(\tau) = \sum_{u=0}^{\infty} \frac{(-1)^u c^u}{u!\Gamma(p + \frac{b+1}{2} + u)} \left(\frac{\tau}{2}\right)^{2u+p}, \ \ \tau \in \mathbb{C} \tag{4}$$

is one specific solution to the linear differential equation of second order

$$\tau^2\chi''(\tau) + b\tau\chi'(\tau) + [c\tau^2 - p^2 + (1-b)p]\chi(\tau) = 0, \quad p, b, c \in \mathbb{C}. \tag{5}$$

The function $\chi(\tau)$ is not univalent in symmetric domain Λ. Now, we examine the function $B_{p,b,c}$ as follows:

$$B_{p,b,c}(\tau) = \epsilon_p = 2^p \Gamma(p + \frac{b+1}{2}) \tau^{\frac{-p}{2}}_{p,b,c} \chi(\tau^{\frac{1}{2}}).$$

Using the Pochhammer symbol, defined for $a \neq 0, -1, -2, \cdots$ by

$$(\zeta)_u = \frac{\Gamma(\zeta + u)}{\Gamma(\zeta)} = \begin{cases} 1, & \text{if } u = 0 \text{ and } \zeta \in \mathbb{C} - \{0\}; \\ \zeta(\zeta+1)...(\zeta+u-1), & \text{if } u \in \mathbb{N} \text{ and } \zeta \in \mathbb{C} \end{cases}$$

we can rewrite the function $B_{p,b,c}$ as follows:

$$B_{p,b,c}(\tau) = \sum_{u=0}^{\infty} \frac{\left(\frac{-c}{4}\right)^u}{(p+\frac{b+1}{2})_u} \frac{\tau^u}{u!}$$

and define the function $\mathcal{L}_{p,b}(u,c;\tau)$ as

$$\mathcal{L}_{p,b}(u,c;\tau) = \tau B_{p,b,c}(\tau) = \tau + \sum_{u=2}^{\infty} \frac{\left(\frac{-c}{4}\right)^{u-1}}{(\delta)_{u-1}(u-1)!}\tau^u \equiv \sum_{u=0}^{\infty} \frac{\left(\frac{-c}{4}\right)^u}{(\delta)_u u!}\tau^u \tag{6}$$

where $c < 0, -1, -2, \cdots$ and $\delta = p + \frac{b+1}{2} \neq 0$. The function $\mathcal{L}_{p,b}(u,c;\tau) = \mathcal{L}_p(\tau)$ is entire and satisfies the equation

$$4\tau^2 \mathcal{L}_p''(\tau) + 2(2p+b+1)\tau\mathcal{L}_p'(\tau) + c\tau\mathcal{L}_p(\tau) = 0. \tag{7}$$

The Bessel function is essential in many areas of mathematical physics and applied mathematics, including as signal processing, hydrodynamics, radio physics and acoustics. Thus, a great deal of research has been conducted on Bessel functions. For example, Baricz et al. [9] gave sufficient conditions for Bessel functions. Frasin et al. in [10] determined some conditions for the function $z(2-u_p(z))$, where u_p denotes the Bessel function of order p to be in various subclasses of analytic functions. Saiful et al. in [11] determined various conditions in which Bessel functions have certain geometric properties in the unit disc.

Finally, we will examine the Wright special function, which is defined as

$$\sigma(\varphi,\varepsilon;\tau) = \sum_{u=0}^{\infty} \frac{1}{\Gamma(\varphi u+\varepsilon)} \frac{\tau^u}{u!}, \varphi > -1, \varepsilon, \tau \in \mathbb{C}. \tag{8}$$

Wright functions have been mentioned in papers pertaining to partial differential equations and in other applications; see, for example [12–15].

Remark 1. *For $\varphi = 1$, $\varepsilon = p+1$ and the functions $\sigma(1,p+1;-\tau^2/4)$, it is possible to write the following using the Bessel functions $J_p(\tau)$:*

$$J_p(\tau) = \left(\frac{\tau}{2}\right)^p \sigma(1,p+1;\frac{-\tau^2}{4}) = \sum_{u=0}^{\infty} \frac{1}{\Gamma(u+p+1)} \frac{(\tau/2)^{2u+p}}{u!}.$$

Furthermore, for $\varphi > 0$ and $p > -1$, the function $\sigma(\varphi, p+1; -\tau) \equiv J_p^{\varphi}(\tau)$ is generalized Bessel function.

Observe that the Wright function $\sigma(\varphi,\varepsilon,\tau)$ is not in class $\mathcal{F}$. Thus, we define the next two Wright functions:

$$\begin{aligned}\mathcal{W}^{(1)}(\varphi,\varepsilon;\tau) &:= \Gamma(\varepsilon)\tau\sigma(\varphi,\varepsilon;\tau) = \sum_{u=0}^{\infty} \frac{\Gamma(\varepsilon)}{\Gamma(\varphi u+\varepsilon)} \frac{\tau^{u+1}}{u!} \\ &= \tau + \sum_{u=2}^{\infty} \frac{\Gamma(\varepsilon)}{\Gamma(\varphi(u-1)+\varepsilon)} \frac{\tau^u}{(u-1)!}, \quad \varphi > -1,\ \varepsilon > 0,\ \tau \in \Lambda\end{aligned} \tag{9}$$

and

$$\begin{aligned}\mathcal{W}^{(2)}(\varphi,\varepsilon;\tau) &:= \Gamma(\varphi+\varepsilon)\left(\sigma(\varphi,\varepsilon;\tau) - \frac{1}{\Gamma(\varepsilon)}\right) = \sum_{u=0}^{\infty} \frac{\Gamma(\varphi+\varepsilon)}{\Gamma(\varphi u+\varphi+\varepsilon)} \frac{\tau^{u+1}}{(u+1)!} \\ &= \tau + \sum_{u=2}^{\infty} \frac{\Gamma(\varphi+\varepsilon)}{\Gamma(\varphi(u-1)+\varphi+\varepsilon)} \frac{\tau^u}{u!}, \quad \varphi > -1,\ \varphi+\varepsilon > 0,\ \tau \in \Lambda.\end{aligned} \tag{10}$$

Furthermore, observe that $\mathcal{W}^{(1)}(\varphi,\varepsilon;\tau)$, $\mathcal{W}^{(2)}(\varphi,\varepsilon;\tau)$ and $H_{\varphi,\varepsilon}(\tau) = \frac{\mathcal{W}^{(1)}(\varphi,\varepsilon;\tau)}{\tau}$ are satisfied by the following relations, respectively

$$\varphi\tau(\mathcal{W}^{(1)}(\varphi,\varepsilon;\tau))' = (\varepsilon-1)\mathcal{W}^{(1)}(\varphi,\varepsilon-1;\tau) + (\varphi+\varepsilon+1)\mathcal{W}^{(1)}(\varphi,\varepsilon;\tau),$$

$$\tau(\mathcal{W}^{(2)}(\varphi,\varepsilon;\tau))' = \mathcal{W}^{(1)}(\varphi,\varphi+\varepsilon;\tau)$$

and

$$H'_{\varphi,\varepsilon}(\tau) = \frac{\Gamma(\varepsilon)}{\Gamma(\varphi+\varepsilon)} H_{\varphi,\varphi+\varepsilon}(\tau).$$

Recently, numerous scholars have examined classes of analytic functions that involve special functions in order to determine certain conditions in Λ. Many features, generalizations and applications of many kinds of geometric functions have been covered widely in the literature. For example, Miller et al. in [16] determined conditions for the univalence of Gaussian. Ponnusamy et al. in [17] found conditions for function f to be starlike or convex. Ponnusamy et al. in [18] studed sufficient and necessary conditions, in terms of the coefficient A_n, for a function $f \in \mathcal{A}$ to be in subclasses of univalent functions. Ponnusamy et al. in [19] determined conditions for convexity and starlikeness. Yagmur et al. in [20] presented some applications of convexity involving Struve functions. In [21], Frasin et al. provided conditions for the Struve functions to be in two classes of analytic functions.

In this paper, we introduce a new subclass of analytic functions. $\mathcal{O}_\mu(\Delta_1,\Delta_2,\Delta_3,\Delta_4)$ of the class $\digamma$, which generalizes many of the previous classes of analytic functions defined in Λ.

Definition 1. *A function $\mathcal{L} \in \digamma$ is said to be in class $\mathcal{O}_\mu(\Delta_1,\Delta_2,\Delta_3,\Delta_4)$ if*

$$\sum_{u=2}^{\infty}\Big(\Delta_1 u^3 + \Delta_2 u^2 + \Delta_3 u + \Delta_4\Big)|a_u| \le \mu,$$

where $\Delta_1,\Delta_2,\Delta_3$ and Δ_4 are real numbers and $\mu > 0$.

By precisely specializing for the coefficients Δ_1, Δ_2, Δ_3, Δ_4 and μ in Definition 1, we note the following:

(1) A function $\mathcal{L} \in \wp(\lambda,\gamma) \equiv \mathcal{O}_{1-\gamma}(0,\lambda,1-\lambda,-\gamma)$ (which is due to [22]) if

$$\sum_{u=2}^{\infty}(u+\lambda u(u-1)-\gamma)|a_u| \le 1-\gamma.$$

(2) A function $\mathcal{L} \in \mathcal{K}(\lambda,\gamma) \equiv \mathcal{O}_{1-\gamma}(\lambda,1-\lambda,-\gamma,0)$ (which is due to [23]) if

$$\sum_{u=2}^{\infty}u(u+\lambda u(u-1)-\gamma)|a_u| \le 1-\gamma.$$

(3) A function $\mathcal{L} \in \wp^*_\lambda(\gamma,\beta) \equiv \mathcal{O}_{1-\gamma}(0,0,\beta+1,-\lambda(\gamma+\beta))$ (which is due to [24]) if

$$\sum_{u=2}^{\infty}(u(\beta+1)-\lambda(\gamma+\beta))|a_u| \le 1-\gamma.$$

(4) A function $\mathcal{L} \in \ominus^*_\lambda(\gamma,\beta) \equiv \mathcal{O}_{1-\gamma}(0,\beta+1,-\lambda(\gamma+\beta),0)$ (which is due to [24]) if

$$\sum_{u=2}^{\infty}u(u(\beta+1)-\lambda(\gamma+\beta))|a_u| \le 1-\gamma.$$

(5) A function $\mathcal{L} \in \mathcal{T}(\lambda, \alpha) \equiv \mathcal{O}_{1-\alpha}(0, 0, 1-\alpha\lambda, -\alpha(1-\lambda))$ (which is due to [25]) if

$$\sum_{u=2}^{\infty} (u(1-\alpha\lambda) - \alpha(1-\lambda))|a_u| \leq 1-\alpha.$$

(6) A function $\mathcal{L} \in \mathcal{C}(\lambda, \alpha) \equiv \mathcal{O}_{1-\alpha}(0, 1-\alpha\lambda, -\alpha(1-\lambda), 0)$ (which is due to [25]) if

$$\sum_{u=2}^{\infty} u(u(1-\alpha\lambda) - \alpha(1-\lambda))|a_u| \leq 1-\alpha.$$

(7) A function $\mathcal{L} \in \mathcal{SP}_p(\sigma, v) \equiv \mathcal{O}_{\cos\sigma - v}(0, 0, 2, -\cos\sigma - v)$ (which is due to [26]) if

$$\sum_{u=2}^{\infty} (2u - \cos\sigma - v)|a_u| \leq \cos\sigma - v.$$

(8) A function $\mathcal{L} \in \mathcal{UCSP}_p(\sigma, v) \equiv \mathcal{O}_{\cos\sigma - v}(0, 2, -\cos\sigma - v, 0)$ (which is due to [26]) if

$$\sum_{u=2}^{\infty} u(2u - \cos\sigma - v)|a_u| \leq \cos\sigma - v.$$

To give sufficient conditions for the Rabotnov function, Poisson distribution function, Bessel function and Wright functions to be in the comprehensive subclass $\mathcal{O}_\mu(\Delta_1, \Delta_2, \Delta_3, \Delta_4)$, we need the next Lemma given by Sümer Eker [27]:

Lemma 1. *For $u \in \mathbb{N}$ and $\hbar \geq 0$, then*

$$(\hbar+1)^{u-1}(u-1)!\Gamma(\hbar+1) \leq \Gamma(u(\hbar+1)).$$

Furthermore, from this Lemma, we can write

$$\frac{\Gamma(\hbar+1)}{\Gamma(u(\hbar+1))} \leq \frac{1}{(\hbar+1)^{u-1}(u-1)!}. \tag{11}$$

Furthermore, since $\Gamma(u-1+\daleth) \leq \Gamma(\varphi(u-1)+\daleth)$ for $\daleth > 0$ and $u \in \mathbb{C}$, the Lemma holds. We can write

$$\frac{\Gamma(\daleth)}{\Gamma(\varphi(u-1)+\daleth)} \leq \frac{1}{(\daleth)_{u-1}}, \quad u \in \mathbb{C} \tag{12}$$

Further, since

$$(\daleth)_{u-1} = \daleth(\daleth+1)(\daleth+2)\cdots(\daleth+u-2) \geq \daleth(\daleth+1)^{u-2}, \quad u \in \mathbb{C}$$

and using (12), then

$$\frac{\Gamma(\daleth)}{\Gamma(\varphi(u-1)+\daleth)} \leq \frac{1}{\daleth(\daleth+1)^{u-2}}. \tag{13}$$

2. Main Results

In this part, we examine a few prerequisites that must be met for Rabotnov functions, Poisson functions, Bessel functions and Wright functions to be in the comprehensive subclass $\mathcal{O}_\mu(\Delta_1, \Delta_2, \Delta_3, \Delta_4)$.

Theorem 1. *If the following inequality is valid,*

$$\left(\frac{\Delta_1\beta^3}{(\ell+1)^3} + \frac{(6\Delta_1+\Delta_2)\beta^2}{(\ell+1)^2} + \frac{(7\Delta_1+3\Delta_2+\Delta_3)\beta}{\ell+1} + \Delta_1+\Delta_2+\Delta_3+\Delta_4\right)e^{\frac{\beta}{\ell+1}}$$
$$\leq \mu+\Delta_1+\Delta_2+\Delta_3+\Delta_4.$$

for $\mu > 0$, *then the Rabotnov function* $\mathcal{R}_{\ell,\beta}(\tau)$ *is in the class* $\mathcal{O}_{\mu}(\Delta_1, \Delta_2, \Delta_3, \Delta_4)$.

Proof. By Definition 1 and the Rabotnov function given by (2), it suffices to show that

$$\sum_{u=2}^{\infty}\left(\Delta_1 u^3 + \Delta_2 u^2 + \Delta_3 u + \Delta_4\right)\frac{\beta^{u-1}\Gamma(\ell+1)}{\Gamma((\ell+1)u)} \leq \mu.$$

Let

$$\hbar_1(\lambda, \beta; \mu) = \sum_{u=2}^{\infty}\left(\Delta_1 u^3 + \Delta_2 u^2 + \Delta_3 u + \Delta_4\right)\frac{\beta^{u-1}\Gamma(\ell+1)}{\Gamma((\ell+1)u)}.$$

Setting

$$\begin{cases} u = (u-1) + 1; \\ u^2 = (u-1)(u-2) + 3(u-1) + 1; \\ u^3 = (u-1)(u-2)(u-3) + 6(u-1)(u-2) + 7(u-1) + 1, \end{cases} \tag{14}$$

we obtain

$$\begin{aligned} \hbar_1(\lambda, \beta; \mu) = &\sum_{u=2}^{\infty}\{\Delta_1(u-1)(u-2)(u-3) + (6\Delta_1 + \Delta_2)(u-1)(u-2) \\ &+ (7\Delta_1 + 3\Delta_2 + \Delta_3)(u-1) + \Delta_1 + \Delta_2 + \Delta_3 + \Delta_4\}\frac{\beta^{u-1}\Gamma(\ell+1)}{\Gamma((\ell+1)u)}. \\ &= \sum_{u=2}^{\infty}\frac{\Delta_1(u-1)(u-2)(u-3)\beta^{u-1}\Gamma(\ell+1)}{\Gamma((\ell+1)u)} \\ &+ \sum_{u=2}^{\infty}\frac{(6\Delta_1 + \Delta_2)(u-1)(u-2)\beta^{u-1}\Gamma(\ell+1)}{\Gamma((\ell+1)u)} \\ &+ \sum_{u=2}^{\infty}\frac{(7\Delta_1 + 3\Delta_2 + \Delta_3)(u-1)\beta^{u-1}\Gamma(\ell+1)}{\Gamma((\ell+1)u)} \\ &+ \sum_{u=2}^{\infty}\frac{(\Delta_1 + \Delta_2 + \Delta_3 + \Delta_4)\beta^{u-1}\Gamma(\ell+1)}{\Gamma((\ell+1)u)}. \end{aligned}$$

Under hypothesis (11), we have

$$\hbar_1(\lambda,\beta;\mu) \leq \Delta_1 \sum_{u=2}^{\infty} \frac{(u-1)(u-2)(u-3)\beta^{u-1}}{(\ell+1)^{u-1}(u-1)!} + (6\Delta_1+\Delta_2)\sum_{u=2}^{\infty}\frac{(u-1)(u-2)\beta^{u-1}}{(\ell+1)^{u-1}(u-1)!}$$
$$+ (7\Delta_3+3\Delta_2+\Delta_3)\sum_{u=2}^{\infty}\frac{(u-1)\beta^{u-1}}{(\ell+1)^{u-1}(u-1)!}$$
$$+ (\Delta_1+\Delta_2+\Delta_3+\Delta_4)\sum_{u=2}^{\infty}\frac{\beta^{u-1}}{(\ell+1)^{u-1}(u-1)!}$$
$$= \Delta_1\sum_{u=4}^{\infty}\frac{\beta^{u-1}}{(\ell+1)^{u-1}(u-4)!} + (6\Delta_1+\Delta_2)\sum_{u=3}^{\infty}\frac{\beta^{u-1}}{(\ell+1)^{u-1}(u-3)!}$$
$$+ (7\Delta_1+3\Delta_2+\Delta_3)\sum_{u=2}^{\infty}\frac{\beta^{u-1}}{(\ell+1)^{u-1}(u-2)!}$$
$$+ (\Delta_1+\Delta_2+\Delta_3+\Delta_4)\sum_{u=2}^{\infty}\frac{\beta^{u-1}}{(\ell+1)^{u-1}(u-1)!}$$
$$= \frac{\Delta_1\beta^3}{(\ell+1)^3}e^{\frac{\beta}{\ell+1}} + \frac{(6\Delta_1+\Delta_2)\beta^2}{(\ell+1)^2}e^{\frac{\beta}{\ell+1}} + \frac{(7\Delta_1+3\Delta_2+\Delta_3)\beta}{\ell+1}e^{\frac{\beta}{\ell+1}}$$
$$+ (\Delta_1+\Delta_2+\Delta_3+\Delta_4)\left(e^{\frac{\beta}{\ell+1}}-1\right). \tag{15}$$

The right-hand term in (15) is bounded above by μ; thus,

$$\left(\frac{\Delta_1\beta^3}{(\ell+1)^3} + \frac{(6\Delta_1+\Delta_2)\beta^2}{(\ell+1)^2} + \frac{(7\Delta_1+3\Delta_2+\Delta_3)\beta}{\ell+1} + \Delta_1+\Delta_2+\Delta_3+\Delta_4\right)e^{\frac{\beta}{\ell+1}}$$
$$\leq \mu+\Delta_1+\Delta_2+\Delta_3+\Delta_4.$$

This concludes the proof of Theorem 1. □

Theorem 2. *If the following inequality is valid,*

$$\Delta_1\nu^3 + (6\Delta_1+\Delta_2)\nu^2 + (7\Delta_1+3\Delta_2+\Delta_3)\nu + (\Delta_1+\Delta_2+\Delta_3+\Delta_4)\left(1-e^{-\nu}\right) \leq \mu.$$

for $\mu > 0$, *then the Poisson distribution function* $\mathcal{K}(\nu,\tau)$ *belongs to* $\mathcal{O}_\mu(\Delta_1,\Delta_2,\Delta_3,\Delta_4)$.

Proof. In view of Definition 1 and and Equality (3) for the Poisson function, it suffices to show that

$$\sum_{u=2}^{\infty}\left(\Delta_1u^3+\Delta_2u^2+\Delta_3u+\Delta_4\right)\frac{\nu^{u-1}}{(u-1)!}e^{-\nu} \leq \mu.$$

Let

$$\hbar_2(\nu,\mu) = \sum_{u=2}^{\infty}\left(\Delta_1u^3+\Delta_2u^2+\Delta_3u+\Delta_4\right)\frac{\nu^{u-1}}{(u-1)!}e^{-\nu}.$$

Similar to Theorem 1, we obtain

$$\hbar_2(\nu,\mu) = \Delta_1\sum_{u=4}^{\infty}\frac{\nu^{u-1}}{(u-4)!}e^{-\nu} + (6\Delta_1+\Delta_2)\sum_{u=3}^{\infty}\frac{\nu^{u-1}}{(u-3)!}e^{-\nu}$$
$$+ (7\Delta_1+3\Delta_2+\Delta_3)\sum_{u=2}^{\infty}\frac{\nu^{u-1}}{(u-2)!}e^{-\nu} + (\Delta_1+\Delta_2+\Delta_3+\Delta_4)\sum_{u=2}^{\infty}\frac{\nu^{u-1}}{(u-1)!}e^{-\nu}.$$
$$= \Delta_1\nu^3 + (6\Delta_1+\Delta_2)\nu^2 + (7\Delta_1+3\Delta_2+\Delta_3)\nu + (\Delta_1+\Delta_2+\Delta_3+\Delta_4)\left(1-e^{-\nu}\right) \leq \mu$$

This concludes the proof of Theorem 2. □

Theorem 3. *If the following inequality is valid,*

$$\Delta_1\mathcal{L}_p'''(1)+(6\Delta_1+\Delta_2)\mathcal{L}_p''(1)+(7\Delta_1+3\Delta_2+\Delta_3)\mathcal{L}_p'(1)+(\Delta_1+\Delta_2+\Delta_3+\Delta_4)\mathcal{L}_p(1)$$
$$\leq \mu+\Delta_1+\Delta_2+\Delta_3+\Delta_4.$$

for $\mu>0$, then the Bessel function $\mathcal{L}_{p,b}(u,c;\tau)$ belongs to $\mathcal{O}_\mu(\Delta_1,\Delta_2,\Delta_3,\Delta_4)$.

Proof. By Definition 1 and the Bessel function given by (6), it suffices to show that

$$\sum_{u=2}^{\infty}\left(\Delta_1u^3+\Delta_2u^2+\Delta_3u+\Delta_4\right)\frac{\left(\frac{-c}{4}\right)^{u-1}}{(\delta)_{u-1}(u-1)!}\leq\mu.$$

Let

$$\hbar_3(c,\delta;\mu)=\sum_{u=2}^{\infty}\left(\Delta_1u^3+\Delta_2u^2+\Delta_3u+\Delta_4\right)\frac{\left(\frac{-c}{4}\right)^{u-1}}{(\delta)_{u-1}(u-1)!}.$$

Similar to Theorem 1 , we obtain

$$\begin{aligned}\hbar_3(c,\delta;\mu)&=\Delta_1\sum_{u=4}^{\infty}\frac{\left(\frac{-c}{4}\right)^{u-1}}{(\delta)_{u-1}(u-4)!}+(6\Delta_1+\Delta_2)\sum_{u=3}^{\infty}\frac{\left(\frac{-c}{4}\right)^{u-1}}{(\delta)_{u-1}(u-3)!}\\&+(7\Delta_1+3\Delta_2+\Delta_3)\sum_{u=2}^{\infty}\frac{\left(\frac{-c}{4}\right)^{u-1}}{(\delta)_{u-1}(u-2)!}+(\Delta_1+\Delta_2+\Delta_3+\Delta_4)\sum_{u=2}^{\infty}\frac{\left(\frac{-c}{4}\right)^{u-1}}{(\delta)_{u-1}(u-1)!}\\&=\frac{\Delta_1\left(\frac{-c}{4}\right)^3}{\delta(\delta+1)(\delta+2)}\sum_{u=0}^{\infty}\frac{\left(\frac{-c}{4}\right)^{u}}{(\delta+3)_{u-2}u!}+\frac{(6\Delta_1+\Delta_2)\left(\frac{-c}{4}\right)^2}{\delta(\delta+1)}\sum_{u=0}^{\infty}\frac{\left(\frac{-c}{4}\right)^{u}}{(\delta+2)_{u-1}u!}\\&+\frac{(7\Delta_1+3\Delta_2+\Delta_3)\left(\frac{-c}{4}\right)}{\delta}\sum_{u=0}^{\infty}\frac{\left(\frac{-c}{4}\right)^{u}}{(\delta+1)_{u}u!}+(\Delta_1+\Delta_2+\Delta_3+\Delta_4)\sum_{u=0}^{\infty}\frac{\left(\frac{-c}{4}\right)^{u+1}}{(\delta)_{u+1}(u+1)!}\\&=\frac{\Delta_1\left(\frac{-c}{4}\right)^3}{\delta(\delta+1)(\delta+2)}\mathcal{L}_{p+3}(1)+\frac{(6\Delta_1+\Delta_2)\left(\frac{-c}{4}\right)^2}{\delta(\delta+1)}\mathcal{L}_{p+2}(1)\\&+\frac{(7\Delta_1+3\Delta_2+\Delta_3)\left(\frac{-c}{4}\right)}{\delta}\mathcal{L}_{p+1}(1)+(\Delta_1+\Delta_2+\Delta_3+\Delta_4)(\mathcal{L}_p(1)-1)\\&=\Delta_1\mathcal{L}_p'''(1)+(6\Delta_1+\Delta_2)\mathcal{L}_p''(1)+(7\Delta_1+3\Delta_2+\Delta_3)\mathcal{L}_p'(1)\\&+(\Delta_1+\Delta_2+\Delta_3+\Delta_4)(\mathcal{L}_p(1)-1)\\&\leq\mu.\end{aligned}$$

□

Theorem 4. *If the following inequality is valid,*

$$e^{\frac{1}{\varepsilon+1}}\left(\frac{\Delta_1}{\varepsilon(\varepsilon+1)^2}+\frac{6\Delta_1+\Delta_2}{\varepsilon(\varepsilon+1)}+\frac{7\Delta_1+3\Delta_2+\Delta_3}{\varepsilon}+\frac{(\varepsilon+1)(\Delta_1+\Delta_2+\Delta_3+\Delta_4)}{\varepsilon}\right)$$
$$\leq\mu+\frac{(\varepsilon+1)(\Delta_1+\Delta_2+\Delta_3+\Delta_4)}{\varepsilon}$$

for $\mu>0$, then $\mathcal{W}^{(1)}(\varphi,\varepsilon;\tau)\in\mathcal{O}_\mu(\Delta_1,\Delta_2,\Delta_3,\Delta_4)$.

Proof. By Definition 1 and the Wright function given by (9), it suffices to show that

$$\sum_{u=2}^{\infty}\left(\Delta_1u^3+\Delta_2u^2+\Delta_3u+\Delta_4\right)\frac{\Gamma(\varepsilon)}{\Gamma(\varphi(u-1)+\varepsilon)(u-1)!}\leq\mu.$$

Let

$$\hbar_4(\varepsilon,\varphi;\mu)=\sum_{u=2}^{\infty}\left(\Delta_1u^3+\Delta_2u^2+\Delta_3u+\Delta_4\right)\frac{\Gamma(\varepsilon)}{\Gamma(\varphi(u-1)+\varepsilon)(u-1)!}.$$

By (14), we obtain

$$\begin{aligned}\hbar_4(\varepsilon,\varphi;\mu) &= \Delta_1 \sum_{u=4}^{\infty} \frac{\Gamma(\varepsilon)}{\Gamma(\varphi(u-1)+\varepsilon)(u-4)!} + (6\Delta_1+\Delta_2) \sum_{u=3}^{\infty} \frac{\Gamma(\varepsilon)}{\Gamma(\varphi(u-1)+\varepsilon)(u-3)!} \\ &+ (7\Delta_1+3\Delta_2+\Delta_3) \sum_{u=2}^{\infty} \frac{\Gamma(\varepsilon)}{\Gamma(\varphi(u-1)+\varepsilon)(u-2)!} \\ &+ (\Delta_1+\Delta_2+\Delta_3+\Delta_4) \sum_{u=2}^{\infty} \frac{\Gamma(\varepsilon)}{\Gamma(\varphi(u-1)+\varepsilon)(u-1)!}.\end{aligned}$$

Under hypothesis (13), we obtain

$$\begin{aligned}\hbar_4(\varepsilon,\varphi;\mu) &\leq \Delta_1 \sum_{u=4}^{\infty} \frac{1}{\varepsilon(\varepsilon+1)^{u-2}(u-4)!} + (6\Delta_1+\Delta_2) \sum_{u=3}^{\infty} \frac{1}{\varepsilon(\varepsilon+1)^{u-2}(u-3)!} \\ &+ (7\Delta_1+3\Delta_2+\Delta_3) \sum_{u=2}^{\infty} \frac{1}{\varepsilon(\varepsilon+1)^{u-2}(u-2)!} \\ &+ (\Delta_1+\Delta_2+\Delta_3+\Delta_4) \sum_{u=2}^{\infty} \frac{1}{\varepsilon(\varepsilon+1)^{u-2}(u-1)!} \\ &= \frac{\Delta_1}{\varepsilon(\varepsilon+1)^2} e^{\frac{1}{\varepsilon+1}} + \frac{6\Delta_1+\Delta_2}{\varepsilon(\varepsilon+1)} e^{\frac{1}{\varepsilon+1}} + \frac{7\Delta_1+3\Delta_2+\Delta_3}{\varepsilon} e^{\frac{1}{\varepsilon+1}} \\ &+ \frac{(\varepsilon+1)(\Delta_1+\Delta_2+\Delta_3+\Delta_4)}{\varepsilon}\left(e^{\frac{1}{\varepsilon+1}}-1\right) \leq \mu.\end{aligned}$$

□

Theorem 5. *If the following inequality is valid,*

$$\begin{aligned}&\Delta_1(\varphi+\varepsilon+1)e^{\frac{1}{\varphi+\varepsilon+1}} + (3\Delta_1+\Delta_2)e^{\frac{1}{\varphi+\varepsilon+1}} + (\Delta_1+\Delta_2+\Delta_3)(\varphi+\varepsilon+1)\left(e^{\frac{1}{\varphi+\varepsilon+1}}-1\right) \\ &+ \Delta_4(\varphi+\varepsilon+1)^2\left(e^{\frac{1}{\varphi+\varepsilon+1}} - \frac{1}{\varphi+\varepsilon+1} - 1\right) \leq (\varphi+\varepsilon)\mu\end{aligned}$$

for $\mu > 0$, *then* $\mathcal{W}^{(2)}(\varphi,\varepsilon;\tau) \in \mathcal{O}_\mu(\Delta_1,\Delta_2,\Delta_3,\Delta_4)$.

Proof. By Definition 1 and the Wright function given by (10), it suffices to show that

$$\sum_{u=2}^{\infty}\left(\Delta_1 u^3+\Delta_2 u^2+\Delta_3 u+\Delta_4\right)\frac{\Gamma(\varphi+\varepsilon)}{\Gamma(\varphi(u-1)+\varphi+\varepsilon)u!} \leq \mu.$$

Let

$$\hbar_5(\varepsilon,\varphi;\mu) = \sum_{u=2}^{\infty}\left(\Delta_1 u^3+\Delta_2 u^2+\Delta_3 u+\Delta_4\right)\frac{\Gamma(\varphi+\varepsilon)}{\Gamma(\varphi(u-1)+\varphi+\varepsilon)u!}.$$

Similar to Theorem 1 , we obtain:

$$\begin{aligned}
\hbar_5(\varepsilon,\varphi;\mu) &= \sum_{u=2}^{\infty}\frac{\Delta_1 u^2\Gamma(\varphi+\varepsilon)}{\Gamma(\varphi(u-1)+\varphi+\varepsilon)(u-1)!}+\sum_{u=2}^{\infty}\frac{\Delta_2 u\Gamma(\varphi+\varepsilon)}{\Gamma(\varphi(u-1)+\varphi+\varepsilon)(u-1)!}\\
&+\sum_{u=2}^{\infty}\frac{\Delta_3\Gamma(\varphi+\varepsilon)}{\Gamma(\varphi(u-1)+\varphi+\varepsilon)(u-1)!}+\sum_{u=2}^{\infty}\frac{\Delta_4\Gamma(\varphi+\varepsilon)}{\Gamma(\varphi(u-1)+\varphi+\varepsilon)u!}\\
&=\sum_{u=2}^{\infty}\frac{\Delta_1(u-1)(u-2)\Gamma(\varphi+\varepsilon)}{\Gamma(\varphi(u-1)+\varphi+\varepsilon)(u-1)!}+\sum_{u=2}^{\infty}\frac{(3\Delta_1+\Delta_2)(u-1)\Gamma(\varphi+\varepsilon)}{\Gamma(\varphi(u-1)+\varphi+\varepsilon)(u-1)!}\\
&+\sum_{u=2}^{\infty}\frac{(\Delta_1+\Delta_2+\Delta_3)\Gamma(\varphi+\varepsilon)}{\Gamma(\varphi(u-1)+\varphi+\varepsilon)(u-1)!}+\sum_{u=2}^{\infty}\frac{\Delta_4\Gamma(\varphi+\varepsilon)}{\Gamma(\varphi(u-1)+\varphi+\varepsilon)u!}\\
&=\sum_{u=3}^{\infty}\frac{\Delta_1\Gamma(\varphi+\varepsilon)}{\Gamma(\varphi(u-1)+\varphi+\varepsilon)(u-3)!}+\sum_{u=2}^{\infty}\frac{(3\Delta_1+\Delta_2)\Gamma(\varphi+\varepsilon)}{\Gamma(\varphi(u-1)+\varphi+\varepsilon)(u-2)!}\\
&+\sum_{u=2}^{\infty}\frac{(\Delta_1+\Delta_2+\Delta_3)\Gamma(\varphi+\varepsilon)}{\Gamma(\varphi(u-1)+\varphi+\varepsilon)(u-1)!}+\sum_{u=2}^{\infty}\frac{\Delta_4\Gamma(\varphi+\varepsilon)}{\Gamma(\varphi(u-1)+\varphi+\varepsilon)u!}.
\end{aligned}$$

Using hypothesis (13), with $\varepsilon=\varphi+\varepsilon$, we obtain

$$\begin{aligned}
\hbar_5(\varepsilon,\varphi;\mu) &\le \sum_{u=3}^{\infty}\frac{\Delta_1}{(\varphi+\varepsilon)(\varphi+\varepsilon+1)^{u-2}(u-3)!}+\sum_{u=2}^{\infty}\frac{3\Delta_1+\Delta_2}{(\varphi+\varepsilon)(\varphi+\varepsilon+1)^{u-2}(u-2)!}\\
&+\sum_{u=2}^{\infty}\frac{\Delta_1+\Delta_2+\Delta_3}{(\varphi+\varepsilon)(\varphi+\varepsilon+1)^{u-2}(u-1)!}+\sum_{u=2}^{\infty}\frac{\Delta_4}{(\varphi+\varepsilon)(\varphi+\varepsilon+1)^{u-2}u!}\\
&=\sum_{u=0}^{\infty}\frac{\Delta_1}{(\varphi+\varepsilon)(\varphi+\varepsilon+1)^{u+1}u!}+\sum_{u=0}^{\infty}\frac{3\Delta_1+\Delta_2}{(\varphi+\varepsilon)(\varphi+\varepsilon+1)^{u}u!}\\
&+\sum_{u=0}^{\infty}\frac{\Delta_1+\Delta_2+\Delta_3}{(\varphi+\varepsilon)(\varphi+\varepsilon+1)^{u}(u+1)!}+\sum_{u=0}^{\infty}\frac{\Delta_4}{(\varphi+\varepsilon)(\varphi+\varepsilon+1)^{u}(u+2)!}\\
&=\frac{\Delta_1(\varphi+\varepsilon+1)}{\varphi+\varepsilon}e^{\frac{1}{\varphi+\varepsilon+1}}+\frac{3\Delta_1+\Delta_2}{\varphi+\varepsilon}e^{\frac{1}{\varphi+\varepsilon+1}}+\frac{(\Delta_1+\Delta_2+\Delta_3)(\varphi+\varepsilon+1)}{\varphi+\varepsilon}\left(e^{\frac{1}{\varphi+\varepsilon+1}}-1\right)\\
&+\frac{\Delta_4(\varphi+\varepsilon+1)^2}{\varphi+\varepsilon}\left(e^{\frac{1}{\varphi+\varepsilon+1}}-\frac{1}{\varphi+\varepsilon+1}-1\right)\le\mu.
\end{aligned}$$

□

3. Corollaries

By specializing the parameters Δ_1, Δ_2, Δ_3, Δ_4 and μ in our Theorems, we obtain many results studied by many authors. The following is an illustration:

Let $\Delta_1=0$, $\Delta_2=\lambda$, $\Delta_3=1-\lambda$, $\Delta_4=-\gamma$ and $\mu=1-\gamma$ in Theorem 1. We conclude the subsequent corollary, which is due to [28] in Theorem 1.

Corollary 1. *If $0\le\gamma<1$ and $0\le\lambda<1$, then the Rabotnov function $\mathcal{R}_{\ell,\beta}(\tau)$ belongs to the class $\mathcal{O}_{1-\gamma}(0,\lambda,1-\lambda,-\gamma)$ if*

$$\left(\frac{\lambda\beta^2}{(\ell+1)^2}+\frac{(2\lambda+1)\beta}{\ell+1}+1-\gamma\right)e^{\frac{\beta}{\ell+1}}\le 2(1-\gamma).$$

Let $\Delta_1=\lambda$, $\Delta_2=1-\lambda$, $\Delta_3=-\gamma$, $\Delta_4=0$ and $\mu=1-\gamma$ in Theorem 1. We conclude the subsequent corollary, which is due to [28] in Theorem 2.

Corollary 2. *If $0\le\gamma<1$ and $0\le\lambda<1$, then the Rabotnov function $\mathcal{R}_{\ell,\beta}(\tau)$ belongs to the class $\mathcal{O}_{1-\gamma}(\lambda,1-\lambda,-\gamma,0)$ if*

$$\left(\frac{\lambda\beta^3}{(\ell+1)^3}+\frac{(5\lambda+1)\beta^2}{(\ell+1)^2}+\frac{(4\lambda-\gamma+3)\beta}{\ell+1}+1-\gamma\right)e^{\frac{\beta}{\ell+1}}\le 2(1-\gamma).$$

Let $\Delta_1 = \Delta_2 = 0$, $\Delta_3 = \beta + 1$, $\Delta_4 = -\lambda(\gamma + \beta)$ and $\mu = 1 - \gamma$ in Theorem 2. We conclude the subsequent corollary, which is due to [29] in Theorem 1.

Corollary 3. *If $0 \leq \gamma < 1$, $0 \leq \lambda \leq 1$ and $\beta \geq 0$, then the Poisson distribution function $\mathcal{K}(\nu, \tau)$ belongs to the class $\mathcal{O}_{1-\gamma}(0, 0, \beta + 1, -\lambda(\gamma + \beta))$ if*

$$(\beta + 1)\nu + (\beta + 1 - \lambda(\gamma + \beta))\left(1 - e^{-\nu}\right) \leq 1 - \gamma.$$

Let $\Delta_1 = 0$, $\Delta_2 = \beta + 1$, $\Delta_3 = -\lambda(\gamma + \beta)$, $\Delta_4 = 0$ and $\mu = 1 - \gamma$ in Theorem 2. We conclude the subsequent corollary, which is due to [29] in Theorem 2.

Corollary 4. *If $0 \leq \gamma < 1$, $0 \leq \lambda \leq 1$ and $\beta \geq 0$, then the Poisson distribution function $\mathcal{K}(\nu, \tau)$ belongs to the class $\mathcal{O}_{1-\gamma}(0, \beta + 1, -\lambda(\gamma + \beta), 0)$ if*

$$(\beta + 1)\nu^2 + (3\beta + 3 - \lambda(\gamma + \beta))\nu + (\beta + 1 - \lambda(\gamma + \beta))\left(1 - e^{-\nu}\right) \leq 1 - \gamma.$$

Let $\Delta_1 = \Delta_2 = 0$, $\Delta_3 = 1 - \alpha\lambda$, $\Delta_4 = -\alpha(1 - \lambda)$ and $\mu = \beta|b|$ in Theorem 3. We conclude the subsequent corollary, which is due to [30] in Theorem 5.

Corollary 5. *If $0 < \beta \leq 1$, $0 \leq \lambda \leq 1$ and $b \in \mathbb{C} - \{0\}$, then the Bessel function $\mathcal{L}_{p,b}(u, c; \tau)$ belongs to the class $\mathcal{O}_{1-\alpha}(0, 0, 1 - \alpha\lambda, -\alpha(1 - \lambda))$ if*

$$(1 - \alpha\lambda)\mathcal{L}'_p(1) + (1 - \alpha)\mathcal{L}_p(1) \leq 2(1 - \alpha).$$

Let $\Delta_1 = \Delta_2 = 0$, $\Delta_3 = 1 - \alpha\lambda$, $\Delta_4 = -\alpha(1 - \lambda)$ and $\mu = 1 - \alpha$ in Theorem 3. We conclude the subsequent corollary, which is due to [30] in Theorem 7.

Corollary 6. *If $0 < \beta \leq 1$, $0 \leq \lambda \leq 1$ and $b \in \mathbb{C} - \{0\}$, then the Bessel function $\mathcal{L}_{p,b}(u, c; \tau)$ belongs to the class $\mathcal{O}_{1-\alpha}(0, 1 - \alpha\lambda, -\alpha(1 - \lambda), 0)$ if*

$$(1 - \alpha\lambda)\mathcal{L}''_p(1) + (3 - \alpha(1 + 2\lambda))\mathcal{L}'_p(1) + (1 - \alpha)\mathcal{L}_p(1) \leq 2(1 - \alpha).$$

Let $\Delta_1 = \Delta_2 = 0$, $\Delta_3 = 2$, $\Delta_4 = -\cos\sigma - v$ and $\mu = \cos\sigma - v$ in Theorem 4. We conclude the subsequent corollary, which is due to [31] in Theorem 1.

Corollary 7. *If $|\sigma| < \frac{\pi}{2}$ and $0 \leq v < 1$, then $\mathcal{W}^{(1)}(\varphi, \varepsilon; \tau)$ in $\mathcal{O}_{\cos\sigma - v}(0, 0, 2, -\cos\sigma - v)$ if*

$$\varepsilon(\cos\sigma - v) + (\varepsilon + 1)(\cos\sigma + v - 2)\left(e^{\frac{1}{\varepsilon+1}} - 1\right) - 2e^{\frac{1}{\varepsilon+1}} \geq 0.$$

Let $\Delta_1 = \Delta_2 = 0$, $\Delta_3 = 2$, $\Delta_4 = -\cos\sigma - v$ and $\mu = \cos\sigma - v$ in Theorem 4. We conclude the subsequent corollary, which is due to [31] in Theorem 2.

Corollary 8. *If $|\sigma| < \frac{\pi}{2}$ and $0 \leq v < 1$, then $\mathcal{W}^{(1)}(\varphi, \varepsilon; \tau)$ in $\mathcal{O}_{\cos\sigma - v}(0, 2, -\cos\sigma - v, 0)$ if*

$$\begin{aligned} &\varepsilon(\varepsilon + 1)(\cos\sigma - v) - 2e^{\frac{1}{\varepsilon+1}} + (\varepsilon + 1)(\cos\sigma + v - 6)e^{\frac{1}{\varepsilon+1}} \\ &+ (\varepsilon + 1)^2(\cos\sigma + v - 2)\left(e^{\frac{1}{\varepsilon+1}} - 1\right) \geq 0. \end{aligned}$$

Let $\Delta_1 = \Delta_2 = 0$, $\Delta_3 = 2$, $\Delta_4 = -\cos\sigma - v$ and $\mu = \cos\sigma - v$ in Theorem 5. We conclude the subsequent corollary, which is due to [31] in Theorem 3.

Corollary 9. *If* $|\sigma| < \frac{\pi}{2}$ *and* $0 \leq v < 1$, *then* $\mathcal{W}^{(2)}(\varphi, \varepsilon; \tau)$ *in* $\mathcal{O}_{\cos\sigma - v}(0, 0, 2, -\cos\sigma - v)$ *if*

$$(\varphi + \varepsilon)(\cos\sigma - v) + (\cos\sigma + v)(\varphi + \varepsilon + 1)^2\left(e^{\frac{1}{\varphi+\varepsilon+1}} - 1\right)$$
$$+ (\varphi + \varepsilon + 1)\left(2 - 2e^{\frac{1}{\varphi+\varepsilon+1}} - \cos\sigma - v\right) \geq 0.$$

Let $\Delta_1 = \Delta_2 = 0$, $\Delta_3 = 2$, $\Delta_4 = -\cos\sigma - v$ and $\mu = \cos\sigma - v$ in Theorem 5. We conclude the subsequent corollary, which is due to [31] in Theorem 4.

Corollary 10. *If* $|\sigma| < \frac{\pi}{2}$ *and* $0 \leq v < 1$, *then* $\mathcal{W}^{(2)}(\varphi, \varepsilon; \tau)$ *in* $\mathcal{O}_{\cos\sigma - v}(0, 2, -\cos\sigma - v, 0)$ *if*

$$(\varphi + \varepsilon)(\cos\sigma - v) + (\cos\sigma + v - 2)(\varphi + \varepsilon + 1)\left(e^{\frac{1}{\varphi+\varepsilon+1}} - 1\right) - 2e^{\frac{1}{\varphi+\varepsilon+1}} \geq 0.$$

4. Conclusions

In this investigation, we determine sufficient conditions for the functions $\mathcal{R}_{\ell,\beta}(\tau)$, $\mathcal{K}(\nu, \tau)$, $\mathcal{L}_{p,b}(u, c; \tau)$, $\mathcal{W}^{(1)}(\varphi, \varepsilon; \tau)$ and $\mathcal{W}^{(2)}(\varphi, \varepsilon; \tau)$ through the general comprehensive subclass $\mathcal{O}_{\mu}(\Delta_1, \Delta_2, \Delta_3, \Delta_4)$ of univalent functions introduced in Definition 1. Furthermore, some corollaries of the results are also discussed. By suitably specializing the real constants $\Delta_1, \Delta_2, \Delta_3, \Delta_4$ and μ in our main results, we can determine new sufficient conditions for the functions $\mathcal{R}_{\ell,\beta}(\tau)$, $\mathcal{K}(\nu, \tau)$, $\mathcal{L}_{p,b}(u, c; \tau)$, $\mathcal{W}^{(1)}(\varphi, \varepsilon; \tau)$ and $\mathcal{W}^{(2)}(\varphi, \varepsilon; \tau)$ in other subclasses of analytic functions, which are new and have not been studied so far.

Author Contributions: Conceptualization, T.A.-H. and I.A.; methodology, I.A.; validation, B.F.; formal analysis, T.A.-H. and I.A.; investigation and resources, T.A.-H. and I.A.; data curation, T.A.-H.; writing—review and editing, I.A.; visualization and supervision, T.A.-H. and I.A. All authors have read and agreed to the published version of the manuscript.

Funding: This work was supported and funded by the Deanship of Scientific Research at Imam Mohammad Ibn Saud Islamic University (IMSIU) (grant number IMSIU-RPP2023093).

Data Availability Statement: Data is contained within the article.

Acknowledgments: The authors extend their appreciation to the Deanship of Scientific Research at Imam Mohammad Ibn Saud Islamic University (IMSIU) for funding this research through the project number IMSIU-RPP2023093.

Conflicts of Interest: The authors declare that they have no conflicts of interest.

References

1. Rabotnov, Y. Equilibrium of an Elastic Medium with After-Effect. *Prikladnaya Matematika i Mekhanika* **1948**, *12*, 53–62 (In Russian); reprinted in *Fract. Calc. Appl. Anal.* **2014**, *17*, 684–696. [CrossRef]
2. Attiya, A.A. Some applications of Mittag-Leffler function in the unit disk. *Filomat* **2016**, *30*, 2075–2081. [CrossRef]
3. Bansal, D.; Prajapat, J.K. Certain geometric properties of the Mittag-Leffler functions. *Complex Var. Elliptic Equ.* **2016**, *61*, 338–350. [CrossRef]
4. Frasin, B.A.; Al-Hawary, T.; Yousef, F. Some properties of a linear operator involving generalized Mittag-Leffler function. *Stud. Univ. Babeş-Bolyai Math.* **2020**, *65*, 67–75. [CrossRef]
5. Garg, M.; Manohar, P.; Kalla, S.L. A Mittag-Leffler-type function of two variables. *Integral Transform. Spec. Funct.* **2013**, *24*, 934–944. [CrossRef]
6. Porwal, S. An application of a Poisson distribution series on certain analytic functions. *J. Complex Anal.* **2014**, *2014*, 984135. [CrossRef]
7. Murugusundaramoorthy, G. Subclasses of starlike and convex functions involving Poisson distribution series. *Afr. Mat.* **2017**, *28*, 1357–1366. [CrossRef]
8. Murugusundaramoorthy, G.; Vijaya, K.; Porwal, S. Some inclusion results of certain subclass of analytic functions associated with Poisson distribution series. *Hacettepe J. Math. Stat.* **2016**, *45*, 1101–1107. [CrossRef]
9. Baricz, A.; Ponnusamy, S. Starlikeness and convexity of generalized Bessel functions. *Integral Transform. Spec. Funct.* **2010**, *21*, 641–653. [CrossRef]

10. Frasin, B.A.; Yousef, F.; Al-Hawary, T.; Aldawish, I. Application of generalized Bessel functions to classes of analytic functions. *Afr. Mat.* **2021**, *32*, 431–439. [CrossRef]
11. Saiful R.; Mondal, S.R.; Swaminathan, A. Geometric properties of Generalized Bessel functions. *Bull. Malays. Math. Sci. Soc.* **2012**, *35*, 179–194.
12. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. Theory and applications of fractional differential equations. In *North-Holland Mathematical Studies*; Elsevier (North-Holland) Science Publishers: Amsterdam, The Netherlands, 2006; Volume 204.
13. Mainardi, F. *Fractional Calculus: Some Basic Problems in Continuum and Statistical Mechanics*; Springer: Wien, Austria, 1971.
14. Podlubny, I. *Fractional Differential Equations*; Academic Press: San Diego, CA, USA, 1999.
15. Samko, S.G.; Kilbas, A.A.; Marichev, O.I. *Fractional Integrals and Derivatives: Theory and Applications*; Gordon and Breach: New York, NY, USA, 1993.
16. Miller, S.S.; Mocanu, P.T. *Differential Subordinations: Theory and Applications*; Series on Monographs and Textbooks in Pure and Applied Mathematics; Marcel Dekker Incorporated: New York, NY, USA; Basel, Switzerland, 2000; Volume 225.
17. Ponnusamy, S.; Singh, V.; Vasundhra, P. Starlikeness and convexity of an integral transform. *Integral Transform. Spec. Funct.* **2004**, *15*, 267–280. [CrossRef]
18. Ponnusamy, S.; Vuorinen, M. Univalence and convexity properties for Gaussian hypergeometric functions. *Rocky Mt. J. Math.* **2001**, *31*, 327–353. [CrossRef]
19. Ponnusamy, S.; Vuorinen, M. Univalence and convexity properties for confluent hypergeometric functions. *Complex Var. Elliptic Equ.* **1998**, *36*, 73–97. [CrossRef]
20. Yagmur, N.; Orhan, H. Hardy space of generalized Struve functions. *Complex Var. Elliptic Equ.* **2014**, *59*, 929–936. [CrossRef]
21. Frasin, B.A.; Al-Hawary, T.; Yousef, F.; Aldawish, I. On Subclasses of Analytic Functions Associated with Struve Functions. *Nonlinear Func. Anal. App.* **2022**, 27, 99–110.
22. Thulasiram, T.; Suchithra, K.; Sudharsan, T.; Murugusundaramoorthy, G.G. Some inclusion results associated with certain subclass of analytic functions inolving Hohlov operator. *Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. Math.* **2014**, *108*, 711–720. [CrossRef]
23. Şeker, B.; Sümer Eker, S. On Certain Subclasses of Starlike and Convex Functions Associated with Pascal Distribution Series. *Turk. J. Math.* **2020**, *44*, 1982–1989. [CrossRef]
24. Murugusundaramoorthy, G.; Vijaya; K.; Uma, K. Subordination results for a class of analytic functions involving the Hurwitz-Lerch zeta function. *Int. J. Nonlinear Sci.* **2010**, *10*, 430–437.
25. Altintas, O.; Owa, S. On subclasses of univalent functions with negative coefficients. *East Asian Math. J.* **1988**, *4*, 41–56.
26. Selvaraj, C.; Geetha, R. On subclasses of uniformly convex spirallike functions and corresponding class of spirallike functions. *Int. J. Contemp. Math. Sci.* **2010**, *5*, 37–40.
27. Sümer Eker, S.; Ece, S. Geometric Properties Of Normalized Rabotnov Function. *Hacet. J. Math. Stat.* **2022**, *51*, 1248–1259. [CrossRef]
28. Sümer Eker, S.; Seker, B.; Ece, S. On normalized Rabotnov function associated with certain subclasses of analytic functions. *Probl. Anal. Issues Anal.* **2023**, *2*, 97–106. [CrossRef]
29. Murugusundaramoorthy, G. Parabolic Starlike and Uniformly Convex Functions Associated with Poisson Distribution Series. *Bull. Transilv. Univ. Brasov. Ser. Math. Inform. Phys.* **2017**, *10*, 91–100.
30. Saurabh Porwal; Dixit, K.K. An application of generalized Bessel functions on certain analytic functions. *Acta Univ. Matthiae Belii Ser. Math.* **2013**, *21*, 55–61.
31. Al-Hawary, T.; Aldawish, I.; Frasin, B.A.; Alkam, O.; Yousef, F. Necessary and Sufficient Conditions for Normalized Wright Functions to Be in Certain Classes of Analytic Functions. *Mathematics* **2022**, *10*, 4693. [CrossRef]