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Nonlinear Neutral Delay Differential Equations: Novel Criteria for Oscillation and Asymptotic Behavior

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Abstract: This research deals with the study of the oscillatory behavior of solutions of second-order differential equations containing neutral conditions, both in sublinear and superlinear terms, with a focus on the noncanonical case. The research provides a careful analysis of the monotonic properties of solutions and their derivatives, paving the way for a deeper understanding of this complex behavior. The research is particularly significant as it extends the scope of previous studies by addressing more complex forms of neutral differential equations. Using the linearization technique, strict conditions are developed that exclude the existence of positive solutions, which allows the formulation of innovative criteria for determining the oscillatory behavior of the studied equations. This research highlights the theoretical and applied aspects of this mathematical phenomenon, which contributes to enhancing the scientific understanding of differential equations with neutral conditions. To demonstrate the effectiveness of the results, the research includes two illustrative examples that prove the validity and importance of the proposed methodology. This work represents a qualitative addition to the mathematical literature, as it lays new foundations and opens horizons for future studies in this vital field.

Keywords: oscillation; non-oscillation; sublinear type; superlinear type; neutral differential equation; second order

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1. Introduction

In this paper, we study the second-order noncanonical neutral differential equation of the following form:

$$\left(\kappa(\mathrm{s})\left[\left(y(\mathrm{s})+g_1(\mathrm{s})y^{\gamma_1}(\mathfrak{r}_1(\mathrm{s}))+g_2(\mathrm{s})y^{\gamma_2}(\mathfrak{r}_2(\mathrm{s}))\right)'\right]^{\alpha}\right)'+q(\mathrm{s})y^{\beta}(\sigma(\mathrm{s}))=0,\ \mathrm{s}\geq \mathrm{s}_0. \quad (1)$$

To ensure a comprehensive study of this equation, we assume the following hypotheses:

H1. $\alpha>1$, $\beta>0$, $\gamma_1>1$ *and* $\gamma_2<1$ *are ratios of odd positive integers;*

H2. $g_1, g_2, q\in C([\mathrm{s}_0,\infty),\mathbb{R}^+)$, *and* $q\neq 0$;

H3. $\mathfrak{r}_1, \mathfrak{r}_2, \sigma \in C^1([s_0,\infty),\mathbb{R})$ *satisfy* $\sigma(s) \leq s$, $\mathfrak{r}_1(s) \leq s$, $\mathfrak{r}_2(s) \leq s$, $\sigma'(s) > 0$ *and* $\lim_{s\to\infty} \mathfrak{r}_1(s) = \lim_{s\to\infty} \mathfrak{r}_2(s) = \lim_{s\to\infty} \sigma(s) = \infty$;

H4. $\kappa \in C^1([s_0,\infty),\mathbb{R}^+)$. *We define*

$$\pi(s) = \int_s^\infty \frac{1}{\kappa^{1/\alpha}(v)} dv,$$

and

$$\pi(s, s_0) := \int_{s_0}^s \frac{1}{\kappa^{1/\alpha}(v)} dv < \infty \text{ as } s \to \infty. \tag{2}$$

Below, we provide some basic definitions:

(1) A function $y(s) \in C([s_y,\infty),\mathbb{R})$, $s_y \geqslant s_0$, is said to be a solution of (1) if it has the property $\kappa(s)\left[\left(y(s) + g_1(s)y^{\gamma_1}(\mathfrak{r}_1(s)) + g_2(s)y^{\gamma_2}(\mathfrak{r}_2(s))\right)'\right]^\alpha \in C^1[s_y,\infty)$, and it satisfies the Equation (1) for all $s \in [s_y,\infty)$. We consider only those solutions $y(s)$ of (1) that exist on some half-line $[s_y,\infty)$ and satisfy the condition

$$\sup\{|y(s)| : s \geqslant S\} > 0, \text{ for all } S \geq s_y.$$

(2) The solution of (1) is said to be oscillatory if it is neither eventually positive nor eventually negative.

(3) If the solution of (1) is eventually positive or negative, it is non-oscillatory.

(4) The equation is considered oscillatory if all its solutions are oscillatory.

Since the time of Newton, differential equations (DEs) have been one of the basic tools for understanding dynamic systems and modeling natural phenomena, as they are used to describe changes in physical, chemical, and biological systems. With the continuous development of science and the expansion of its applications, the need for more accurate and comprehensive models has emerged. Among these models are delay differential equations (DDEs), which are characterized by taking into account the effect of the temporal memory of systems, making them more efficient in representing many natural phenomena, see [1,2].

However, finding accurate solutions to these equations represents a major challenge that hinders a deep understanding of these phenomena. Therefore, qualitative theories are an essential tool that allows studying the properties of equations without the need to find their detailed solutions. Among these theories, Oscillation Theory stands out, which focuses on studying the oscillatory and non-oscillatory behavior of solutions, in addition to the infinite analysis of the distribution of roots. Recent advancements have further enriched this foundational theory, see Chuanxi and Ladas [3], Kiguradze and Chanturia [4], Bazighifan [5,6], and Masood et al. [7], which provide innovative criteria and methodologies to analyze these complex equations.

The study of oscillation criteria for second-order DEs has been a cornerstone of mathematical analysis due to its wide applicability in physical and engineering systems. Agarwal et al. [8] laid the foundation for oscillation control criteria, and subsequent research [9] extended to include linear and nonlinear equations. Džurina et al. [10] developed further criteria specific to delay differential equations, while Erbe et al. [11] provided criteria for nonlinear equations. Later, Hassan [12] and Grace et al. [13] improved these neutral equations. In recent years, Zhang et al. [14] and Baculíková [15] contributed significantly to the understanding of second-order equations with noncanonical operators and deviating arguments, respectively. Jadlovská [16] extended these findings to include Kneser-type criteria and sublinear neutral terms. Li et al. [17] and Grace et al. [18] introduced advanced differential equations with diverse terms. Finally, Muhib [19] presented important developments in the study of noncanonical neutral equations. For more details, see Moaaz et al. [20], Masood et al. [21], Alsharidi and Muhib [22], and Alemam et al. [23].

Various oscillation criteria for second order NDEs impose specific constraints on their coefficients:

Agarwal et al. [24] and Han et al. [25] studied the oscillation of second-order linear NDEs

$$\left(\kappa(\mathrm{s})(y(\mathrm{s})+\mathrm{g}(\mathrm{s})y(\mathfrak{r}(\mathrm{s})))'\right)'+q(\mathrm{s})y(\sigma(\mathrm{s}))=0,$$

introducing new criteria under the condition $0 \leq g(\mathrm{s}) \leq g_0 < \infty$.

Agarwal et al. [26] considered NDEs with a nonlinear term

$$\left(\kappa(\mathrm{s})(y(\mathrm{s})+g_1(\mathrm{s})y^{\gamma_1}(\mathfrak{r}_1(\mathrm{s})))'\right)'+q(\mathrm{s})y(\sigma(\mathrm{s}))=0,$$

where $0<\gamma_1\leq 1$. They introduced conditions to ensure oscillation in the cases

$$\int_{\mathrm{s}}^{\infty}\frac{1}{\kappa(v)}\mathrm{d}v=\infty,$$

and

$$\int_{\mathrm{s}}^{\infty}\frac{1}{\kappa(v)}\mathrm{d}v<\infty.$$

Wang et al. [27] investigated the asymptotic properties of second-order nonlinear delay equations

$$\left(\kappa(\mathrm{s})\left[(y(\mathrm{s})-g_1(\mathrm{s})y(\mathfrak{r}_1(\mathrm{s})))'\right]^{\alpha}\right)'+q(\mathrm{s})f(y(\sigma(\mathrm{s})))=0,$$

with a non-positive neutral coefficient, and $f(u)\geq ky^{\alpha}(u)$.

Tamilvanan et al. [28] addressed the oscillatory nature of a similar DEs with a nonlinear neutral term.

$$\left(\kappa(\mathrm{s})(y(\mathrm{s})+g_1(\mathrm{s})y^{\gamma_1}(\mathfrak{r}_1(\mathrm{s})))'\right)'+q(\mathrm{s})y^{\beta}(\sigma(\mathrm{s}))=0.$$

Džurina and Jadlovská [29] introduced criteria to ensure the oscillation of nonlinear equations

$$\left(\kappa(\mathrm{s})\left[y'(\mathrm{s})\right]^{\alpha}\right)'+q(\mathrm{s})y^{\beta}(\sigma(\mathrm{s}))=0.$$

in a noncanonical form.

Džurina et al. [30] and Wu et al. [31] used the Riccati method to study the oscillation in the nonlinear NDEs

$$\left(\kappa(\mathrm{s})\left[(y(\mathrm{s})+\mathrm{g}(\mathrm{s})y^{\gamma_1}(\mathfrak{r}(\mathrm{s})))'\right]^{\alpha}\right)'+q(\mathrm{s})y^{\beta}(\sigma(\mathrm{s}))=0.$$

Despite the importance of these models, understanding the oscillatory behavior of their solutions, especially in nonlinear and noncanonical cases, remains a major challenge. Most previous research has focused on linear or quasi-linear forms of these equations, creating a knowledge gap regarding more complex cases.

This research aims to fill this gap by establishing novel and generalized criteria for analyzing the oscillatory behavior of neutral second-order differential equations. Previous studies primarily considered neutral terms of the form

$$\mathfrak{u}(\mathrm{s}):=y(\mathrm{s})+g_1(\mathrm{s})y(\mathfrak{r}_1(\mathrm{s})),$$

as presented in [25], with some extending to sublinear cases such as

$$\mathfrak{u}(\mathrm{s}) := y(\mathrm{s}) + g_1(\mathrm{s}) y^{\gamma_1}(\mathfrak{r}_1(\mathrm{s})),\ \gamma_1 < 1,$$

as presented in [26,28,31]. In contrast, this study introduces a more comprehensive framework that incorporates both sublinear $\gamma_1 < 1$ and superlinear $\gamma_2 > 1$ terms within the generalized relationship:

$$\mathfrak{u}(\mathrm{s}) := y(\mathrm{s}) + g_1(\mathrm{s}) y^{\gamma_1}(\mathfrak{r}_1(\mathrm{s})) + g_2(\mathrm{s}) y^{\gamma_2}(\mathfrak{r}_2(\mathrm{s})).$$

This dual consideration of multiple nonlinear terms significantly broadens the scope of oscillation theory for neutral differential equations. Through advanced techniques, including linearization and rigorous analytical methods, new criteria are derived and validated with illustrative examples, demonstrating their effectiveness and applicability. This research contributes to the existing literature by expanding the understanding of oscillatory and asymptotic properties of neutral differential equations, providing a solid foundation for future investigations into more complex systems.

2. Preliminary Results

Let us define

$$\mathfrak{u}(\mathrm{s}) := y(\mathrm{s}) + g_1(\mathrm{s}) y^{\gamma_1}(\mathfrak{r}_1(\mathrm{s})) + g_2(\mathrm{s}) y^{\gamma_2}(\mathfrak{r}_2(\mathrm{s})),$$

$$\mu(\mathrm{s},\mathrm{s}_0) := \begin{cases} 1, & \text{if } \alpha = \beta, \\ M_1^{\beta-\alpha}, & \text{if } \alpha < \beta, \\ M_2^{\beta-\alpha} \pi^{\beta-\alpha}(\mathrm{s},\mathrm{s}_1), & \text{if } \alpha > \beta, \end{cases}$$

$$\mu_1(\mathrm{s},\mathrm{s}_0) := \begin{cases} 1, & \text{if } \alpha = \beta, \\ M_3^{\beta-\alpha}, & \text{if } \alpha > \beta, \\ M_4^{\beta-\alpha} \pi^{\beta-\alpha}(\mathrm{s}), & \text{if } \alpha < \beta, \end{cases}$$

$$\mathrm{H}(\mathrm{s}) := \frac{1}{\alpha} C^{\beta} \pi^{\alpha-1}(\sigma(\mathrm{s}),\mathrm{s}_0) q(\mathrm{s}),$$

and

$$\widetilde{\mathrm{H}}(\mathrm{s}) := \rho(\mathrm{s}) \int_{\mathrm{s}}^{\infty} \mu(\sigma(v),\mathrm{s}_0) \mathrm{H}(v) \mathrm{d}v.$$

Lemma 1 ([32])**.** *Let α be a ratio of two odd positive integers, $A > 0$ and B are constants. Then,*

$$Bu - Au^{(\alpha+1)/\alpha} \leq \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}} \frac{B^{\alpha+1}}{A^{\alpha}},\ A > 0. \tag{3}$$

Lemma 2. *Assume that $y(\mathrm{s})$ is an eventually positive solution of (1). Then, the corresponding function $\mathfrak{u}(\mathrm{s})$ satisfies one of two cases eventually:*

$$(\mathrm{N}_1) \quad : \quad \mathfrak{u}(\mathrm{s}) > 0, \kappa(\mathrm{s})(\mathfrak{u}'(\mathrm{s}))^{\alpha} > 0, \left(\kappa(\mathrm{s})(\mathfrak{u}'(\mathrm{s}))^{\alpha}\right)' < 0,$$

$$(\mathrm{N}_2) \quad : \quad \mathfrak{u}(\mathrm{s}) > 0, \kappa(\mathrm{s})(\mathfrak{u}'(\mathrm{s}))^{\alpha} < 0, \left(\kappa(\mathrm{s})(\mathfrak{u}'(\mathrm{s}))^{\alpha}\right)' < 0,$$

for $\mathrm{s} \geqslant \mathrm{s}_1 \geqslant \mathrm{s}_0$.

Proof. Assume that $y(\mathrm{s})$ is a positive solution of (1). In view of (H_1)–(H_4), there exists $\mathrm{s}_1 \geqslant \mathrm{s}_0$ such that $y(\sigma(\mathrm{s})) > 0$, $y(\mathfrak{r}_1(\mathrm{s})) > 0$ and $y(\mathfrak{r}_2(\mathrm{s})) > 0$, for all $\mathrm{s} \geqslant \mathrm{s}_1$, then,

$$\mathfrak{u}(\mathrm{s}) = y(\mathrm{s}) + g_1(\mathrm{s}) y^{\gamma_1}(\mathfrak{r}_1(\mathrm{s})) + g_2(\mathrm{s}) y^{\gamma_2}(\mathfrak{r}_2(\mathrm{s})) > 0,$$

for all $\mathrm{s} \geqslant \mathrm{s}_1$. Then, from (1), we obtain

$$\left(\kappa(\mathrm{s})(\mathfrak{u}'(\mathrm{s}))^{\alpha}\right)' = -q(\mathrm{s})(y(\sigma(\mathrm{s})))^{\beta} < 0.$$

From the above inequality, we can obtain that $\kappa(\mathrm{s})(\mathfrak{u}'(\mathrm{s}))^{\alpha}$ is decreasing. Then, $\kappa(\mathrm{s})(\mathfrak{u}'(\mathrm{s}))^{\alpha} > 0$ or $\kappa(\mathrm{s})(\mathfrak{u}'(\mathrm{s}))^{\alpha} < 0$. Hence, the proof is complete. □

Notation 1. *The notation* N_i *denotes the set comprising all solutions that eventually become positive and satisfy condition* (N_i) *for* $i = 1, 2$.

3. Main Results

In this section, we provide key results regarding the monotonic behavior of solutions to Equation (1) and its derivatives. Additionally, we establish conditions that rule out the existence of positive solutions, addressing the cases (N_1) and (N_2) separately.

3.1. Category N_1

In this subsection, we introduce a collection of lemmas focused on the asymptotic properties of solutions belonging to the (N_1) class.

Lemma 3. *Let* $y(\mathrm{s}) \in \mathrm{N}_1$. *Assume that*

$$\lim_{\mathrm{s}\to\infty} g_1(\mathrm{s})\pi^{\gamma_1 - 1}(\mathrm{s}, \mathrm{s}_1) = 0, \text{ and } \lim_{\mathrm{s}\to\infty} g_2(\mathrm{s}) = 0. \tag{4}$$

Then, eventually

(A_{11}) $\mathfrak{u}(\mathrm{s}) \geqslant \kappa^{1/\alpha}(\mathrm{s})\mathfrak{u}'(\mathrm{s})\pi(\mathrm{s}, \mathrm{s}_0)$;
(A_{12}) $\mathfrak{u}(\mathrm{s})/\pi(\mathrm{s}, \mathrm{s}_0)$ *is decreasing*;
(A_{13}) $\mathfrak{u}^{\beta-\alpha}(\mathrm{s}) \geq \mu(\mathrm{s}, \mathrm{s}_0)$;
(A_{14}) $y(\mathrm{s}) \geq C\mathfrak{u}(\mathrm{s})$, *where* $C \in (0, 1)$.

Proof. Let $y(\mathrm{s}) \in \mathrm{N}_1$. Then, there exists an $\mathrm{s}_1 \geq \mathrm{s}_0$, such that $y(\mathrm{s}) > 0$, $y(\mathfrak{r}_1(\mathrm{s})) > 0$, $y(\mathfrak{r}_2(\mathrm{s})) > 0$ and $y(\sigma(\mathrm{s})) > 0$ for $\mathrm{s} \geq \mathrm{s}_1$.

(A_{11}) With the monotonicity property of $\kappa^{1/\alpha}(\mathrm{s})\mathfrak{u}'(\mathrm{s})$, we obtain

$$\begin{aligned}
\mathfrak{u}(\mathrm{s}) &= \mathfrak{u}(\mathrm{s}_1) + \int_{\mathrm{s}_1}^{\mathrm{s}} \frac{\kappa^{1/\alpha}(v)\mathfrak{u}'(v)}{\kappa^{1/\alpha}(v)} \mathrm{d}v \\
&\geq \mathfrak{u}(\mathrm{s}_1) + \kappa^{1/\alpha}(\mathrm{s})\mathfrak{u}'(\mathrm{s}) \int_{\mathrm{s}_1}^{\mathrm{s}} \frac{1}{\kappa^{1/\alpha}(v)} \mathrm{d}v \\
&\geq \kappa^{1/\alpha}(\mathrm{s})\mathfrak{u}'(\mathrm{s})\pi(\mathrm{s}, \mathrm{s}_1), \ \mathrm{s} \geq \mathrm{s}_1.
\end{aligned}$$

(A_{12}) Using the above inequality, we deduce that

$$\left(\frac{\mathfrak{u}(\mathrm{s})}{\pi(\mathrm{s}, \mathrm{s}_1)}\right)' = \frac{\kappa^{1/\alpha}(\mathrm{s})\mathfrak{u}'(\mathrm{s})\pi(\mathrm{s}, \mathrm{s}_1) - \mathfrak{u}(\mathrm{s})}{\kappa^{1/\alpha}(\mathrm{s})\pi^2(\mathrm{s}, \mathrm{s}_1)} \leq 0.$$

(A_{13}) We have the following cases:

Case $\alpha \le \beta$: Since $\mathfrak{u}'(s) > 0$, there exists a constant $M_1 > 0$, such that

$$\mathfrak{u}(s) \ge M_1,$$

and therefore,

$$\mathfrak{u}^{\beta-\alpha}(s) \ge M_1^{\beta-\alpha}.$$

Case $\alpha > \beta$: Since $\mathfrak{u}(s)/\pi(s,s_1)$ is a decreasing function for $s \ge s_1$ we can find $s_2 \ge s_1$ and a constant $M_2 > 0$ such that

$$\mathfrak{u}(s) \le M_2\pi(s,s_1).$$

Thus

$$\mathfrak{u}^{\beta-\alpha}(s) \ge M_2^{\beta-\alpha}\pi^{\beta-\alpha}(s,s_1).$$

(A_{14}) From the definition of $\mathfrak{u}(s)$, we have $\mathfrak{u}(s) \ge y(s)$. Then, we express $y(s)$ as

$$\begin{aligned} y(s) &= \mathfrak{u}(s) - g_1(s)y^{\gamma_1}(\mathfrak{r}_1(s)) - g_2(s)y^{\gamma_2}(\mathfrak{r}_2(s)) \\ &\ge \mathfrak{u}(s) - g_1(s)\mathfrak{u}^{\gamma_1}(\mathfrak{r}_1(s)) - g_2(s)\mathfrak{u}^{\gamma_2}(\mathfrak{r}_2(s)). \end{aligned}$$

Now, because $\gamma_1 > 1$, and $\gamma_2 < 1$, and noting that $\mathfrak{u}' > 0$, we find

$$\begin{aligned} y(s) &\ge \mathfrak{u}(s) - g_1(s)\mathfrak{u}^{\gamma_1}(s) - g_2(s)\mathfrak{u}^{\gamma_2}(s) \\ &= \left[1 - g_1(s)\pi^{\gamma_1-1}(s,s_1)\left(\frac{\mathfrak{u}(s)}{\pi(s,s_1)}\right)^{\gamma_1-1} - g_2(s)\mathfrak{u}^{\gamma_2-1}(s)\right]\mathfrak{u}(s). \end{aligned} \tag{5}$$

Since $\mathfrak{u}(s)/\pi(s,s_1)$ is decreasing and positive, and $\mathfrak{u}(s)$ is increasing, there exist two constants, c_1 and c_2 such that

$$\left(\frac{\mathfrak{u}(s)}{\pi(s,s_1)}\right)^{\gamma_1-1} \le c_1,$$

and

$$\mathfrak{u}^{\gamma_2-1}(s) \le c_2.$$

So (5) it becomes

$$y(s) \ge \left[1 - c_1g_1(s)\pi^{\gamma_1-1}(s,s_1) - c_2g_2(s)\right]\mathfrak{u}(s).$$

By (4), we can choose $C \in (0,1)$ such that

$$y(s) \ge C\mathfrak{u}(s).$$

Accordingly, the proof is finished. □

Lemma 4. *Assume that (4) holds. If*

$$\int_{s_0}^{\infty} \frac{1}{\kappa^{1/\alpha}(t)} \int_{s_0}^{t} \mu(\sigma(v),s_0)\pi^{\alpha-1}(\sigma(v),s_1)q(v)\mathrm{d}v\mathrm{d}t = \infty, \tag{6}$$

then $N_1 = \varnothing$.

Proof. Let $y \in N_1$. Then, there exists an $s_1 \ge s_0$, such that

$$y(\mathfrak{r}_1(s)) > 0,\ y(\mathfrak{r}_2(s)) \text{ and } y(\sigma(s)) > 0 \text{ for } s \geq s_1 \geq s_0.$$

We begin with the relation

$$\left(\kappa(s)\left(u'(s)\right)^{\alpha}\right)' = \alpha\left(\kappa^{1/\alpha}(s)u'(s)\right)'\left(\kappa^{1/\alpha}(s)u'(s)\right)^{\alpha-1},$$

we can derive

$$\left(\kappa^{1/\alpha}(s)u'(s)\right)' = \frac{1}{\alpha}\left(\kappa(s)\left(u'(s)\right)^{\alpha}\right)'\left(\kappa^{1/\alpha}(s)u'(s)\right)^{1-\alpha}.$$

Using (1) and (A_{14}) gives

$$\begin{aligned}\left(\kappa^{1/\alpha}(s)u'(s)\right)' &= -\frac{1}{\alpha}\left(\kappa^{1/\alpha}(s)u'(s)\right)^{1-\alpha}q(s)y^{\beta}(\sigma(s)) \\ &\leq -\frac{1}{\alpha}C^{\beta}\left(\kappa^{1/\alpha}(s)u'(s)\right)^{1-\alpha}q(s)u^{\beta}(\sigma(s)).\end{aligned}$$

Applying Lemma 3, we deduce

$$\begin{aligned}\left(\kappa^{1/\alpha}(s)u'(s)\right)' &\leq -\frac{1}{\alpha}C^{\beta}\left(\kappa^{1/\alpha}(s)u'(s)\right)^{1-\alpha}q(s)u^{\beta-\alpha}(\sigma(s))u^{\alpha}(\sigma(s)) \\ &\leq -\frac{1}{\alpha}C^{\beta}\left(\kappa^{1/\alpha}(s)u'(s)\right)^{1-\alpha}q(s)\mu(\sigma(s),s_0)u^{\alpha}(\sigma(s)) \\ &\leq -\frac{1}{\alpha}C^{\beta}\left(\frac{u(s)}{\pi(s,s_1)}\right)^{1-\alpha}q(s)\mu(\sigma(s),s_0)u^{\alpha}(\sigma(s)) \\ &\leq -\frac{1}{\alpha}C^{\beta}\left(\frac{u(\sigma(s))}{\pi(\sigma(s),s_1)}\right)^{1-\alpha}q(s)\mu(\sigma(s),s_0)u^{\alpha}(\sigma(s)) \\ &= -\frac{1}{\alpha}C^{\beta}\mu(\sigma(s),s_0)\pi^{\alpha-1}(\sigma(s),s_1)q(s)u(\sigma(s)).\end{aligned}$$

Thus,

$$\left(\kappa^{1/\alpha}(s)u'(s)\right)' \leq -\frac{1}{\alpha}C^{\beta}\mu(\sigma(s),s_0)\pi^{\alpha-1}(\sigma(s),s_1)q(s)u(\sigma(s)). \tag{7}$$

Since $u(s)$ is an increasing function, there exists a constant $M_2 > 0$ such that $u(s) \geq M_2$. Using this in inequality (7), we obtain

$$\left(\kappa^{1/\alpha}(s)u'(s)\right)' \leq -\frac{1}{\alpha}M_2C^{\beta}\mu(\sigma(s),s_0)\pi^{\alpha-1}(\sigma(s),s_1)q(s). \tag{8}$$

Integrating this inequality from s_1 to s, we have

$$\kappa^{1/\alpha}(s)u'(s) \leq \kappa^{1/\alpha}(s)u'(s) - \kappa^{1/\alpha}(s_1)u'(s_1) \leq -\frac{1}{\alpha}C^{\beta}M_2\int_{s_1}^{s}\mu(\sigma(v),s_0)\pi^{\alpha-1}(\sigma(v),s_1)q(v)dv,$$

which implies that

$$u'(s) \leq -\frac{1}{\alpha}\frac{C^{\beta}M_2}{\kappa^{1/\alpha}(s)}\int_{s_1}^{s}\mu(\sigma(v),s_0)\pi^{\alpha-1}(\sigma(v),s_1)q(v)dv.$$

Integrating this inequality from s_1 to s, we have

$$u(s) \leq -\frac{1}{\alpha}C^{\beta}M_2\int_{s_1}^{s}\frac{1}{\kappa^{1/\alpha}(t)}\int_{s_1}^{t}\mu(\sigma(v),s_0)\pi^{\alpha-1}(\sigma(v),s_1)q(v)dvdt.$$

As $s \to \infty$, this leads to a contradiction with the assumption that $u(s)$ is positive. This concludes the proof. □

Lemma 5. *Let (4) hold. Assume that there exists a non-decreasing function* $\rho(s) \in C^1([s_0,\infty,),(0,\infty))$*, and* $\sigma'(s) > 0$*. If*

$$\limsup_{s\to\infty}\left[\widetilde{H}(s) + \int_{s_0}^{s}\left(\rho(v)\mu(\sigma(v),s_0)H(v) - \frac{\kappa^{1/\alpha}(v)[\rho'(v)]^2}{4\sigma'(v)\rho(v)}\right)dv\right] = \infty, \tag{9}$$

then $N_1 = \varnothing$.

Proof. Let $y \in N_1$. Then, there exists an $s_1 \geq s_0$, such that

$$y(s) > 0,\ y(\mathfrak{r}_1(s)) > 0,\ y(\mathfrak{r}_2(s)) \text{ and } y(\sigma(s)) > 0 \text{ for } s \geq s_1 \geq s_0.$$

Using (7), we obtain

$$\left(\kappa^{1/\alpha}(s)u'(s)\right)' \leq -\frac{1}{\alpha}C^{\beta}\mu(\sigma(s),s_0)\pi^{\alpha-1}(\sigma(s),s_1)q(s)u(\sigma(s)).$$

Integrating the above inequality from s to ∞, we have

$$\begin{aligned}
\kappa^{1/\alpha}(s)u'(s) &\geq \frac{1}{\alpha}C^{\beta}\int_s^{\infty}\mu(\sigma(v),s_0)\pi^{\alpha-1}(\sigma(v),s_1)q(v)u(\sigma(v))dv \\
&\geq u(\sigma(s))\int_s^{\infty}\mu(\sigma(v),s_0)H(v)dv.
\end{aligned} \tag{10}$$

Define

$$\varphi(s) = \rho(s)\frac{\kappa^{1/\alpha}(s)u'(s)}{u(\sigma(s))} \geq \rho(s)\int_s^{\infty}\mu(\sigma(v),s_0)H(v)dv > 0. \tag{11}$$

Then

$$\begin{aligned}
\varphi'(s) &= \rho'(s)\frac{\kappa^{1/\alpha}(s)u'(s)}{u(\sigma(s))} - \rho(s)\mu(\sigma(s),s_0)H(s) - \rho(s)\sigma'(s)\frac{\kappa^{1/\alpha}(s)u'(s)u'(\sigma(s))}{u^2(\sigma(s))} \\
&\leq \frac{\rho'(s)}{\rho(s)}\varphi(s) - \rho(s)\mu(\sigma(s),s_0)H(s) - \rho(s)\sigma'(s)\frac{\kappa^{1/\alpha}(s)[u'(s)]^2}{u^2(\sigma(s))} \\
&= \frac{\rho'(s)}{\rho(s)}\varphi(s) - \rho(s)\mu(\sigma(s),s_0)H(s) - \frac{\sigma'(s)}{\kappa^{1/\alpha}(s)\rho(s)}\varphi^2(s).
\end{aligned} \tag{12}$$

Using Lemma 1, we see that

$$\frac{\rho'(s)}{\rho(s)}w(s) - \frac{\sigma'(s)}{\kappa^{1/\alpha}(s)\rho(s)}w^2(s) \leq \frac{1}{4}\frac{\kappa^{1/\alpha}(s)[\rho'(s)]^2}{\sigma'(s)\rho(s)}.$$

By substituting the above inequality in (13), we obtain

$$\varphi'(s) \leq -\rho(s)\mu(\sigma(s),s_0)H(s) + \frac{1}{4}\frac{\kappa^{1/\alpha}(s)[\rho'(s)]^2}{\sigma'(s)\rho(s)}.$$

Integrating the last inequality from s_2 to s and then using (11), we deduce that

$$\rho(s)\int_{s}^{\infty}\mu(\sigma(v),s_0)H(v)dv+\int_{s_2}^{s}\left[\rho(v)\mu(\sigma(v),s_0)H(v)-\frac{1}{4}\frac{\kappa^{1/\alpha}(v)[\rho'(v)]^2}{\sigma'(v)\rho(v)}\right]dv\leq\varphi(s_2).$$

By applying the lim sup to both sides of this inequality as $s\to\infty$, we reach a contradiction. This concludes the proof. □

By setting $\rho(s)=1$, and considering Lemma 5, we immediately derive the following result.

Corollary 1. *Let (4) hold. If*

$$\limsup_{s\to\infty}\int_{s_0}^{s}\mu(\sigma(v),s_0)H(v)dv=\infty, \tag{13}$$

then $N_1=\varnothing$.

Lemma 6. *Let (4) hold. If*

$$\int_{s_0}^{\infty}\pi^{\beta}(\sigma(v))q(v)dv=\infty, \tag{14}$$

then $N_1=\varnothing$.

Proof. Let $y\in N_1$. Then, there exists an $s_1\geq s_0$, such that

$$y(s)>0,\ y(\mathfrak{r}_1(s))>0,\ y(\mathfrak{r}_2(s))\text{ and }y(\sigma(s))>0\text{ for }s\geq s_1\geq s_0.$$

Using Lemma 3 (A_{14}), the expression in (1) can be transformed into the following inequality

$$\left(\kappa(s)\left(u'(s)\right)^{\alpha}\right)'+C^{\beta}q(s)u^{\beta}(\sigma(s))\leq 0. \tag{15}$$

Integrating the last inequality from s_2 to s, we have

$$\begin{aligned}\kappa(s)\left(u'(s)\right)^{\alpha} &\leq \kappa(s_2)\left(u'(s_2)\right)^{\alpha}-C^{\beta}\int_{s_2}^{s}q(v)u^{\beta}(\sigma(v))dv\\ &\leq \kappa(s_2)\left(u'(s_2)\right)^{\alpha}-C^{\beta}u^{\beta}(\sigma(s_2))\int_{s_2}^{s}q(v)dv. \end{aligned}\tag{16}$$

Since $\pi'<0$, we deduce that

$$\int_{s_2}^{s}\pi^{\beta}(\sigma(v))q(v)dv\leq\pi^{\beta}(\sigma(s_2))\int_{s_2}^{s}q(v)dv.$$

From (14), it follows that

$$\int_{s_2}^{s}q(v)dv\to\infty\text{ as }s\to\infty.$$

Consequently, from Equation (16), we deduce that $u'(s)\to-\infty$ as $s\to\infty$, which leads to a contradiction. □

3.2. Category N_2

In this subsection, we introduce a collection of lemmas focused on the asymptotic properties of solutions belonging to the (N_2) class.

Lemma 7. *Let* $y(\mathrm{s}) \in \mathrm{N}_2$. *Assume that*

$$\lim_{\mathrm{s}\to\infty} g_1(\mathrm{s})\left(\frac{\pi(\mathfrak{r}_1(\mathrm{s}))}{\pi(\mathrm{s})}\right)^{\gamma_1} = 0, \text{ and } \lim_{\mathrm{s}\to\infty} g_2(\mathrm{s})\frac{\pi^{\gamma_2}(\mathfrak{r}_2(\mathrm{s}))}{\pi(\mathrm{s})} = 0. \tag{17}$$

Then, eventually

(A_{21}) $\mathrm{u}(\mathrm{s}) \geqslant -\kappa^{1/\alpha}(\mathrm{s})\mathrm{u}'(\mathrm{s})\pi(\mathrm{s})$;
(A_{22}) $\mathrm{u}(\mathrm{s})/\pi(\mathrm{s})$ *is increasing;*
(A_{23}) $\mathrm{u}^{\beta-\alpha}(\mathrm{s}) \geq \mu_1(\mathrm{s},\mathrm{s}_0)$;
(A_{24}) $y(\mathrm{s}) \geq \widetilde{C}\mathrm{u}(\mathrm{s})$, *where* $\widetilde{C} \in (0,1)$.

Proof. Let $y(\mathrm{s}) \in \mathrm{N}_2$. Then, there exists an $\mathrm{s}_1 \geq \mathrm{s}_0$, such that $y(\mathrm{s}) > 0$, $y(\mathfrak{r}_1(\mathrm{s})) > 0$, $y(\mathfrak{r}_2(\mathrm{s})) > 0$ and $y(\sigma(\mathrm{s})) > 0$ for $\mathrm{s} \geq \mathrm{s}_1$.

(A_{21}) Since $\left(\kappa(v)(\mathrm{u}'(\mathrm{s}))^{\alpha}\right)' < 0$, we get

$$\kappa(v)\left(\mathrm{u}'(v)\right)^{\alpha} \leq \kappa(\mathrm{s})\left(\mathrm{u}'(\mathrm{s})\right)^{\alpha} \text{ for } v \geq \mathrm{s} \geq \mathrm{s}_1,$$

or equivalently

$$\mathrm{u}'(v) \leq \frac{1}{\kappa^{1/\alpha}(v)}\kappa^{1/\alpha}(\mathrm{s})\mathrm{u}'(\mathrm{s}).$$

Integrating this inequality from s to ∞, we deduce that

$$-\mathrm{u}(\mathrm{s}) \leq \kappa^{1/\alpha}(\mathrm{s})\mathrm{u}'(\mathrm{s})\int_{\mathrm{s}}^{\infty}\frac{1}{\kappa^{1/\alpha}(v)}\mathrm{d}v = \kappa^{1/\alpha}(\mathrm{s})\mathrm{u}'(\mathrm{s})\pi(\mathrm{s}).$$

That is,

$$\mathrm{u}(\mathrm{s}) \geq -\kappa^{1/\alpha}(\mathrm{s})\mathrm{u}'(\mathrm{s})\pi(\mathrm{s}).$$

(A_{22}) Using the above inequality, we deduce that

$$\left(\frac{\mathrm{u}(\mathrm{s})}{\pi(\mathrm{s})}\right)' = \frac{\kappa^{1/\alpha}(\mathrm{s})\mathrm{u}'(\mathrm{s})\pi(\mathrm{s}) + \mathrm{u}(\mathrm{s})}{\kappa^{1/\alpha}(\mathrm{s})\pi^2(\mathrm{s})} \geq 0.$$

(A_{23}) we have the following cases:

Case $\alpha \geq \beta$: Since $\mathrm{u}'(\mathrm{s}) < 0$, there exists a constant $M_3 > 0$, such that

$$\mathrm{u}(\mathrm{s}) \leq M_3,$$

and therefore,

$$\mathrm{u}^{\beta-\alpha}(\mathrm{s}) \geq M_3^{\beta-\alpha}.$$

Case $\alpha < \beta$: Since $\mathrm{u}(\mathrm{s})/\pi(\mathrm{s})$ is an increasing function for $\mathrm{s} \geq \mathrm{s}_1$, we can find $\mathrm{s}_2 \geq \mathrm{s}_1$ and a constant $M_4 > 0$ such that

$$\mathrm{u}(\mathrm{s}) \geq M_4\pi(\mathrm{s}).$$

Thus

$$\mathrm{u}^{\beta-\alpha}(\mathrm{s}) \geq M_4^{\beta-\alpha}\pi^{\beta-\alpha}(\mathrm{s}).$$

(A_{24}) From the definition of $\mathrm{u}(\mathrm{s})$, we have $\mathrm{u}(\mathrm{s}) \geq y(\mathrm{s})$. Then, we express $y(\mathrm{s})$ as

$$\begin{aligned} y(\mathrm{s}) &= \mathrm{u}(\mathrm{s}) - g_1(\mathrm{s})y^{\gamma_1}(\mathfrak{r}_1(\mathrm{s})) - g_2(\mathrm{s})y^{\gamma_2}(\mathfrak{r}_2(\mathrm{s})) \\ &\geq \mathrm{u}(\mathrm{s}) - g_1(\mathrm{s})\mathrm{u}^{\gamma_1}(\mathfrak{r}_1(\mathrm{s})) - g_2(\mathrm{s})\mathrm{u}^{\gamma_2}(\mathfrak{r}_2(\mathrm{s})). \end{aligned}$$

Now, because $\gamma_1 > 1$, and $\gamma_2 < 1$, and noting that $\mathrm{u}' < 0$, we find

$$\begin{aligned} y(\mathrm{s}) &\geq \mathrm{u}(\mathrm{s}) - g_1(\mathrm{s})\mathrm{u}^{\gamma_1}(\mathrm{s}) - g_2(\mathrm{s})\mathrm{u}^{\gamma_2}(\mathrm{s}) \\ &= \mathrm{u}(\mathrm{s}) - g_1(\mathrm{s})\pi^{\gamma_1}(\mathfrak{r}_1(\mathrm{s}))\frac{\mathrm{u}^{\gamma_1}(\mathfrak{r}_1(\mathrm{s}))}{\pi^{\gamma_1}(\mathfrak{r}_1(\mathrm{s}))} - g_2(\mathrm{s})\pi^{\gamma_2}(\mathfrak{r}_2(\mathrm{s}))\frac{\mathrm{u}^{\gamma_2}(\mathfrak{r}_2(\mathrm{s}))}{\pi^{\gamma_2}(\mathfrak{r}_2(\mathrm{s}))} \\ &\geq \mathrm{u}(\mathrm{s}) - g_1(\mathrm{s})\pi^{\gamma_1}(\mathfrak{r}_1(\mathrm{s}))\frac{\mathrm{u}^{\gamma_1}(\mathrm{s})}{\pi^{\gamma_1}(\mathrm{s})} - g_2(\mathrm{s})\pi^{\gamma_2}(\mathfrak{r}_2(\mathrm{s}))\frac{\mathrm{u}^{\gamma_2}(\mathrm{s})}{\pi^{\gamma_2}(\mathrm{s})} \\ &= \left[1 - g_1(\mathrm{s})\left(\frac{\pi(\mathfrak{r}_1(\mathrm{s}))}{\pi(\mathrm{s})}\right)^{\gamma_1}\mathrm{u}^{\gamma_1-1}(\mathrm{s}) - g_2(\mathrm{s})\frac{\pi^{\gamma_2}(\mathfrak{r}_2(\mathrm{s}))}{\pi(\mathrm{s})}\left(\frac{\mathrm{u}(\mathrm{s})}{\pi(\mathrm{s})}\right)^{\gamma_2-1}\right]\mathrm{u}(\mathrm{s}). \end{aligned}$$

Since $\mathrm{u}(\mathrm{s})/\pi(\mathrm{s})$ is positive and increasing, and $\mathrm{u}(\mathrm{s})$ is decreasing, there exist two constants, c_3 and c_4 such that

$$y(\mathrm{s}) \geq \left[1 - c_3 g_1(\mathrm{s})\left(\frac{\pi(\mathfrak{r}_1(\mathrm{s}))}{\pi(\mathrm{s})}\right)^{\gamma_1} - c_4 g_2(\mathrm{s})\frac{\pi^{\gamma_2}(\mathfrak{r}_2(\mathrm{s}))}{\pi(\mathrm{s})}\right]\mathrm{u}(\mathrm{s}).$$

By (17), we can choose $\widetilde{C} \in (0,1)$ such that

$$y(\mathrm{s}) \geq \widetilde{C}\mathrm{u}(\mathrm{s}).$$

Accordingly, the proof is finished. □

Lemma 8. *Assume that (17) holds. If*

$$\int_{\mathrm{s}_1}^{\infty} \frac{1}{\kappa^{1/\alpha}(t)} \int_{\mathrm{s}_0}^{t} \mu_1(\sigma(v), \mathrm{s}_0)\pi^{\alpha}(v)q(v)\mathrm{d}v\mathrm{d}t = \infty, \tag{18}$$

then $\mathrm{N}_2 = \varnothing$.

Proof. Let $y \in \mathrm{N}_2$. Then, there exists an $\mathrm{s}_1 \geq \mathrm{s}_0$, such that

$$y(\mathrm{s}) > 0,\ y(\mathfrak{r}_1(\mathrm{s})) > 0,\ y(\mathfrak{r}_2(\mathrm{s})) \text{ and } y(\sigma(\mathrm{s})) > 0 \text{ for } \mathrm{s} \geq \mathrm{s}_1 \geq \mathrm{s}_0.$$

We know that

$$\left(\kappa^{1/\alpha}(\mathrm{s})\mathrm{u}'(\mathrm{s})\right)' = \frac{1}{\alpha}\left(\kappa(\mathrm{s})(\mathrm{u}'(\mathrm{s}))^{\alpha}\right)'\left(\kappa^{1/\alpha}(\mathrm{s})\mathrm{u}'(\mathrm{s})\right)^{1-\alpha}.$$

Using (1) and (A_{24}) gives

$$\begin{aligned} \left(\kappa^{1/\alpha}(\mathrm{s})\mathrm{u}'(\mathrm{s})\right)' &= -\frac{1}{\alpha}\left(\kappa^{1/\alpha}(\mathrm{s})\mathrm{u}'(\mathrm{s})\right)^{1-\alpha}q(\mathrm{s})y^{\beta}(\sigma(\mathrm{s})) \\ &\leq -\frac{1}{\alpha}\widetilde{C}^{\beta}\left(\kappa^{1/\alpha}(\mathrm{s})\mathrm{u}'(\mathrm{s})\right)^{1-\alpha}q(\mathrm{s})\mathrm{u}^{\beta}(\sigma(\mathrm{s})). \end{aligned}$$

Applying Lemma 7, we deduce

$$\begin{aligned}\left(\kappa^{1/\alpha}(\mathrm{s})\mathrm{u}'(\mathrm{s})\right)' &\leq -\frac{1}{\alpha}\widetilde{C}^{\beta}\left(\kappa^{1/\alpha}(\mathrm{s})\mathrm{u}'(\mathrm{s})\right)^{1-\alpha}q(\mathrm{s})\mathrm{u}^{\beta-\alpha}(\sigma(\mathrm{s}))\mathrm{u}^{\alpha}(\sigma(\mathrm{s}))\\ &\leq -\frac{1}{\alpha}\widetilde{C}^{\beta}\left(\kappa^{1/\alpha}(\mathrm{s})\mathrm{u}'(\mathrm{s})\right)^{1-\alpha}q(\mathrm{s})\mu_1(\sigma(\mathrm{s}),\mathrm{s}_0)\mathrm{u}^{\alpha}(\sigma(\mathrm{s}))\\ &\leq -\frac{1}{\alpha}\widetilde{C}^{\beta}\left(\frac{\mathrm{u}(\mathrm{s})}{\pi(\mathrm{s})}\right)^{1-\alpha}q(\mathrm{s})\mu_1(\sigma(\mathrm{s}),\mathrm{s}_0)\mathrm{u}^{\alpha}(\sigma(\mathrm{s}))\\ &\leq -\frac{1}{\alpha}\widetilde{C}^{\beta}\left(\frac{\mathrm{u}(\sigma(\mathrm{s}))}{\pi(\sigma(\mathrm{s}))}\right)^{1-\alpha}q(\mathrm{s})\mu_1(\sigma(\mathrm{s}),\mathrm{s}_0)\mathrm{u}^{\alpha}(\sigma(\mathrm{s}))\\ &= -\frac{1}{\alpha}\widetilde{C}^{\beta}\mu(\sigma(\mathrm{s}),\mathrm{s}_0)\pi^{\alpha-1}(\sigma(\mathrm{s}),\mathrm{s}_1)q(\mathrm{s})\mathrm{u}(\sigma(\mathrm{s})).\end{aligned}$$

Since $\mathrm{u}' < 0$, then, $\mathrm{u}(\sigma(\mathrm{s})) \geq \mathrm{u}(\mathrm{s})$, so that

$$\begin{aligned}\left(\kappa^{1/\alpha}(\mathrm{s})\mathrm{u}'(\mathrm{s})\right)' &\leq -\frac{1}{\alpha}\widetilde{C}^{\beta}\left(\frac{\mathrm{u}(\mathrm{s})}{\pi(\mathrm{s})}\right)^{1-\alpha}q(\mathrm{s})\mu_1(\sigma(\mathrm{s}),\mathrm{s}_0)\mathrm{u}^{\alpha}(\mathrm{s})\\ &= -\frac{1}{\alpha}\widetilde{C}^{\beta}\pi^{\alpha-1}(\mathrm{s})q(\mathrm{s})\mu_1(\sigma(\mathrm{s}),\mathrm{s}_0)\mathrm{u}(\mathrm{s})).\end{aligned}$$

Thus,

$$\left(\kappa^{1/\alpha}(\mathrm{s})\mathrm{u}'(\mathrm{s})\right)' \leq -\frac{1}{\alpha}\widetilde{C}^{\beta}\pi^{\alpha-1}(\mathrm{s})q(\mathrm{s})\mu_1(\sigma(\mathrm{s}),\mathrm{s}_0)\mathrm{u}(\mathrm{s}). \tag{19}$$

Since $\mathrm{u}(\mathrm{s})/\pi(\mathrm{s})$ is an increasing function, there exists a constant $M_5 > 0$ such that $\mathrm{u}(\mathrm{s}) \geq M_5\pi(\mathrm{s})$. Using this in inequality (19), we obtain

$$\left(\kappa^{1/\alpha}(\mathrm{s})\mathrm{u}'(\mathrm{s})\right)' \leq -\frac{1}{\alpha}M_5\widetilde{C}^{\beta}\mu_1(\sigma(\mathrm{s}),\mathrm{s}_0)\pi^{\alpha}(\mathrm{s})q(\mathrm{s}).$$

Integrating this inequality from s_1 to s, we have

$$\kappa^{1/\alpha}(\mathrm{s})\mathrm{u}'(\mathrm{s}) \leq -\frac{1}{\alpha}M_5\widetilde{C}^{\beta}\int_{\mathrm{s}_1}^{\mathrm{s}}\mu_1(\sigma(v),\mathrm{s}_0)\pi^{\alpha}(v)q(v)\mathrm{d}v,$$

which implies that

$$\mathrm{u}'(\mathrm{s}) \leq -\frac{1}{\alpha}\frac{M_5\widetilde{C}^{\beta}}{\kappa^{1/\alpha}(\mathrm{s})}\int_{\mathrm{s}_1}^{\mathrm{s}}\mu_1(\sigma(v),\mathrm{s}_0)\pi^{\alpha}(v)q(v)\mathrm{d}v.$$

Integrating this inequality from s_1 to s, we have

$$\mathrm{u}(\mathrm{s}) \leq -\frac{1}{\alpha}M_5\widetilde{C}^{\beta}\int_{\mathrm{s}_1}^{\mathrm{s}}\frac{1}{\kappa^{1/\alpha}(t)}\int_{\mathrm{s}_1}^{t}\mu_1(\sigma(v),\mathrm{s}_0)\pi^{\alpha}(v)q(v)\mathrm{d}v\mathrm{d}t.$$

As $\mathrm{s} \to \infty$, this leads to a contradiction with the assumption that $\mathrm{u}(\mathrm{s})$ is positive. This concludes the proof. □

Lemma 9. *Assume that (17) hold. If*

$$\limsup_{\mathrm{s}\to\infty}\pi(\mathrm{s})\int_{\mathrm{s}_0}^{\mathrm{s}}\mu_1(\sigma(v),\mathrm{s}_0)\pi^{\alpha-1}(v)q(v)\mathrm{d}v = \infty, \tag{20}$$

then $\mathrm{N}_2 = \varnothing$.

Proof. From (19), we know that

$$\left(\kappa^{1/\alpha}(\mathrm{s})\mathrm{u}'(\mathrm{s})\right)' \leq -\frac{1}{\alpha}\widetilde{C}^{\beta}\pi^{\alpha-1}(\mathrm{s})q(\mathrm{s})\mu_1(\sigma(\mathrm{s}),\mathrm{s}_0)\mathrm{u}(\mathrm{s}).$$

Integrating the above inequality from s_1 to s, we have

$$\begin{aligned}\kappa^{1/\alpha}(s)u'(s) &\leq -\frac{1}{\alpha}\widetilde{C}^{\beta}\int_{s_1}^{s}\mu_1(\sigma(v),s_0)\pi^{\alpha-1}(v)q(v)u(v)dv \\ &\leq -\frac{1}{\alpha}\widetilde{C}^{\beta}u(s)\int_{s_1}^{s}\mu_1(\sigma(v),s_0)\pi^{\alpha-1}(v)q(v)dv.\end{aligned}$$

Using Lemma 7 (A_{21}), we infer that

$$\kappa^{1/\alpha}(s)u'(s) \leq \frac{1}{\alpha}\widetilde{C}^{\beta}\kappa^{1/\alpha}(s)u'(s)\pi(s)\int_{s_1}^{s}\mu_1(\sigma(v),s_0)\pi^{\alpha-1}(v)q(v)dv,$$

which leads to

$$1 \geq \frac{1}{\alpha}\widetilde{C}^{\beta}\pi(s)\int_{s_1}^{s}\mu_1(\sigma(v),s_0)\pi^{\alpha-1}(v)q(v)dv.$$

This leads to a contradiction with the condition (20).

This concludes the proof. □

Lemma 10. *Let (4) hold. If (14) holds, then,* $N_2 = \varnothing$.

Proof. Let $y \in N_2$. Then, there exists an $s_1 \geq s_0$, such that

$$y(s) > 0,\ y(\mathfrak{r}_1(s)) > 0,\ y(\mathfrak{r}_2(s)) \text{ and } y(\sigma(s)) > 0 \text{ for } s \geq s_1 \geq s_0.$$

Using Lemma 7 (A_{24}), the expression in (1) can be transformed into the following inequality

$$\left(\kappa(s)(u'(s))^{\alpha}\right)' + \widetilde{C}^{\beta}q(s)u^{\beta}(\sigma(s)) \leq 0. \tag{21}$$

By applying a similar line of reasoning as in Lemma 6, we arrive at a contradiction. This concludes the proof. □

4. Oscillatory Theorems and Examples

In this section, we present a comprehensive set of theorems that establish oscillation criteria, which are formulated directly by summarizing the results obtained in the main results.

Theorem 1. *Assume that (4) and (17) hold. If both (6) and (18) are satisfied, then, (1) is oscillatory.*

Theorem 2. *Assume that (4) and (17) hold. If both (6) and (20) are satisfied, then, (1) is oscillatory.*

Theorem 3. *Assume that (4) and (17) hold. If both (9) and (20) are satisfied, then, (1) is oscillatory.*

Theorem 4. *Assume that (4) and (17) hold. If both (13) and (20) are satisfied, then, (1) is oscillatory.*

Theorem 5. *Assume that (4) and (17) hold. If (14) is satisfied, then, (1) is oscillatory.*

Example 1. *Consider the equation:*

$$\left(s^6\left[\left(y(s) + \frac{1}{s-1}y^3\left(\frac{1}{4}s\right) + \frac{1}{s-2}y^{1/3}\left(\frac{1}{3}s\right)\right)'\right]^3\right)' + s^5y^3\left(\frac{1}{5}s\right) = 0,\ s \geq 1. \tag{22}$$

Comparing Equation (22) with (1), we deduce the following:

$$\begin{aligned} \alpha &= \beta = \gamma_1 = 3,\ \gamma_2 = \frac{1}{3},\ \kappa(\mathrm{s}) = \mathrm{s}^6,\ \sigma(\mathrm{s}) = \frac{1}{5}\mathrm{s},\ \mathfrak{r}_1(\mathrm{s}) = \frac{1}{4}\mathrm{s}, \\ \mathfrak{r}_2(\mathrm{s}) &= \frac{1}{3}\mathrm{s},\ g_1(\mathrm{s}) = \frac{1}{\mathrm{s}-1},\ g_1(\mathrm{s}) = \frac{1}{\mathrm{s}-2},\ \text{and } q(\mathrm{s}) = \mathrm{s}^5. \end{aligned}$$

Additionally, we have:

$$\mu(\mathrm{s},\mathrm{s}_0) = \mu_1(\mathrm{s}) = 1,\ \pi(\mathrm{s}) = \frac{1}{\mathrm{s}} \text{ and } \pi(\mathrm{s},\mathrm{s}_0) = \frac{\mathrm{s}-1}{\mathrm{s}}.$$

Using (4), we compute:

$$\lim_{\mathrm{s}\to\infty} g_1(\mathrm{s})\pi^{\gamma_1-1}(\mathrm{s},\mathrm{s}_1) = \lim_{\mathrm{s}\to\infty} \frac{(\mathrm{s}-1)^2}{(\mathrm{s}-1)\mathrm{s}^2} = 0, \text{ and } \lim_{\mathrm{s}\to\infty} g_2(\mathrm{s}) = \lim_{\mathrm{s}\to\infty} \frac{1}{\mathrm{s}-2} = 0.$$

From (17), we find:

$$\lim_{\mathrm{s}\to\infty} g_1(\mathrm{s})\left(\frac{\pi(\mathfrak{r}_1(\mathrm{s}))}{\pi(\mathrm{s})}\right)^{\gamma_1} = \lim_{\mathrm{s}\to\infty} \frac{64}{\mathrm{s}-1} = 0, \text{ and } \lim_{\mathrm{s}\to\infty} \frac{g_2(\mathrm{s})\pi^{\gamma_2}(\mathfrak{r}_2(\mathrm{s}))}{\pi(\mathrm{s})} = \lim_{\mathrm{s}\to\infty} \frac{3^{1/3}\mathrm{s}^{2/3}}{\mathrm{s}-2} = 0.$$

Several conditions are verified as follows:
Condition (6):

$$\int_{\mathrm{s}_0}^{\infty} \frac{1}{\kappa^{1/\alpha}(t)} \int_{\mathrm{s}_0}^{t} \mu(\sigma(v),\mathrm{s}_0)\pi^{\alpha-1}(\sigma(v),\mathrm{s}_1)q(v)\mathrm{d}v\mathrm{d}t = \int_1^{\infty} \frac{1}{t^{1/3}} \int_1^t \frac{(v-5)^2}{v^2} v^3 \mathrm{d}v\mathrm{d}t = \infty.$$

Condition (13):

$$\limsup_{\mathrm{s}\to\infty} \int_{\mathrm{s}_0}^{\mathrm{s}} \pi^{\alpha-1}(\sigma(v),\mathrm{s}_1)q(v)\mathrm{d}v = \limsup_{\mathrm{s}\to\infty} \int_1^{\mathrm{s}} \frac{(v-5)^2}{v^2} v^3 \mathrm{d}v = \infty.$$

Condition (14):

$$\int_{\mathrm{s}_0}^{\infty} \pi^{\beta}(\sigma(v))q(v)\mathrm{d}v = \int_1^{\infty} \frac{125}{v^3} v^3 \mathrm{d}v = \infty.$$

Condition (18) leads to

$$\int_{\mathrm{s}_0}^{\infty} \frac{1}{\kappa^{1/\alpha}(t)} \int_{\mathrm{s}_0}^{t} \mu_1(\sigma(v),\mathrm{s}_0)\pi^{\alpha}(v)q(v)\mathrm{d}v\mathrm{d}t = \int_1^{\infty} \frac{1}{u^{1/3}} \int_1^t \frac{1}{v^3} v^3 \mathrm{d}v\mathrm{d}u = \infty.$$

Condition (20):

$$\limsup_{\mathrm{s}\to\infty} \pi(\mathrm{s}) \int_{\mathrm{s}_0}^{\mathrm{s}} \mu_1(\sigma(v),\mathrm{s}_0)\pi^{\alpha-1}(v)q(v)\mathfrak{u}(v)\mathrm{d}v = \limsup_{\mathrm{s}\to\infty} \frac{1}{\mathrm{s}} \int_1^{\mathrm{s}} \frac{1}{v^2} v^3 \mathrm{d}v = \limsup_{\mathrm{s}\to\infty} \frac{\mathrm{s}}{2} = \infty.$$

Since the conditions of Theorems 1–5 are satisfied, it follows that every solution of (22) oscillates.

Example 2. *Consider the equation:*

$$\left(\mathrm{s}^{10}\left[\left(y(\mathrm{s}) + \frac{1}{\mathrm{s}}y^5\left(\frac{1}{3}\mathrm{s}\right) + \frac{1}{\mathrm{s}^2}y^{1/5}\left(\frac{1}{2}\mathrm{s}\right)\right)'\right]^5\right)' + \mathrm{s}^5 y^5\left(\frac{1}{4}\mathrm{s}\right) = 0,\ \mathrm{s} \geq 1. \tag{23}$$

Clearly

$$\begin{aligned} \alpha &= \beta = \gamma_1 = 5,\ \gamma_2 = \frac{1}{5},\ \kappa(s) = s^{10},\ \sigma(s) = \frac{1}{4}s,\ \mathfrak{r}_1(s) = \frac{1}{3}s, \\ \mathfrak{r}_2(s) &= \frac{1}{2}s,\ g_1(s) = \frac{1}{s},\ g_1(s) = \frac{1}{s^2},\ \textit{and}\ q(s) = s^5. \end{aligned}$$

Additionally, we have:

$$\mu(s, s_0) = \mu_1(s) = 1,\ \pi(s) = \frac{1}{s}\ \textit{and}\ \pi(s, s_0) = \frac{s-1}{s}.$$

Using (4), we compute:

$$\lim_{s\to\infty} g_1(s)\pi^{\gamma_1 - 1}(s, s_1) = \lim_{s\to\infty} \frac{(s-1)^4}{s^5} = 0,\ \textit{and}\ \lim_{s\to\infty} g_2(s) = \lim_{s\to\infty} \frac{1}{s^2} = 0.$$

From (17), we find:

$$\lim_{s\to\infty} g_1(s)\left(\frac{\pi(\mathfrak{r}_1(s))}{\pi(s)}\right)^{\gamma_1} = \lim_{s\to\infty} \frac{3^5}{s} = 0,\ \textit{and}\ \lim_{s\to\infty} \frac{g_2(s)\pi^{\gamma_2}(\mathfrak{r}_2(s))}{\pi(s)} = \lim_{s\to\infty} \frac{1}{s^2}\frac{2^{1/5}s}{s^{1/5}} = 0.$$

Verification of conditions (6), (13), (14), (18) and (20) proceeds as follows:
Condition (6):

$$\int_1^\infty \frac{1}{t^{1/5}} \int_1^t \frac{(v-4)^4}{v^4} v^5 \mathrm{d}v \mathrm{d}t = \infty.$$

Condition (13):

$$\limsup_{s\to\infty} \int_1^s \frac{(v-4)^4}{v^4} v^5, \mathrm{d}v = \infty.$$

Condition (14):

$$\int_1^\infty \frac{4^5}{v^5} v^5 \mathrm{d}v = \infty.$$

Condition (18) leads to

$$\int_1^\infty \frac{1}{t^{1/5}} \int_1^t \frac{1}{v^5} v^5 \mathrm{d}v \mathrm{d}t = \infty.$$

Condition (20):

$$\limsup_{s\to\infty} \frac{1}{s} \int_1^s \frac{1}{v^4} v^5 \mathrm{d}v = \limsup_{s\to\infty} \frac{s^2}{2s} = \infty.$$

Since the conditions of Theorems 1–5 are satisfied, it follows that every solution of (23) oscillates.

5. Conclusions

The study of the oscillatory behavior of solutions of second-order DEs containing neutrality conditions, whether sublinear or superlinear, is a rich and exciting field in applied and theoretical mathematics. This research has made new contributions in this context by developing analytical criteria that highlight the dynamics of these equations and explain the associated oscillation patterns. The results obtained here contribute to a deeper understanding of the mathematical properties of this class of equations, which enhances the ability of researchers to address similar problems in multiple contexts. What distinguishes this work is that it highlights the dual effect of neutrality conditions in shaping the behavior of solutions, which opens up prospects for extending current models to include more complex real-world applications. This research also adds to the existing literature with precise criteria that contribute to assessing the nature of oscillatory solutions, which lays a strong foundation for future studies. It is worth noting that the application of the approach developed in this study to higher-order DEs represents a promising and interesting direction. Exploring oscillatory effects in the context of higher-order equations may reveal new patterns and add further understanding to the mathematical structure of these systems. One of the exciting directions for future studies is to extend these investigations without the constraints $\gamma_1 > 1$ and $\gamma_2 < 1$, as well as without relying on conditions (4) and (17), which would allow for broader applications and deeper insights into more general systems. This research advances the understanding of oscillatory phenomena in NDEs and lays a foundation for expanding these insights into more complex studies.

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