axioms

Article

Decision-Making with Fermatean Neutrosophic Vague Soft Sets Using a Technique for Order of Preference by Similarity to Ideal Solution

Najla Althuniyan [1], Abedallah Al-shboul [2], Sarah Aljohani [3], Kah Lun Wang [2,*], Kok Bin Wong [2], Khaleed Alhazaymeh [4,5] and Suhad Subhi Aiady [3]

1 College of Computer and Information Sciences, Department of Computer Sciences, Prince Sultan University, Riyadh 11586, Saudi Arabia; nthuniyan@psu.edu.sa
2 Institute of Mathematical Sciences, Faculty of Science, Universiti Malaya, Kuala Lumpur 50603, Malaysia; 23087452@siswa.um.edu.my (A.A.-s.); kbwong@um.edu.my (K.B.W.)
3 Department of Mathematics and Sciences, Prince Sultan University, Riyadh 11586, Saudi Arabia; sjohani@psu.edu.sa (S.A.); ssubhi@psu.edu.sa (S.S.A.)
4 Department of Mathematics, Faculty of Science and Information Technology, Irbid National University, Irbid 21110, Jordan; kh.hazaymeh@inu.edu.jo
5 Jadara Research Center, Jadara University, Irbid 21110, Jordan
* Correspondence: wangkl@um.edu.my

Abstract: This study addresses the challenge of effectively modeling uncertainty and hesitation in complex decision-making environments, where traditional fuzzy and vague set models often fall short. To overcome these limitations, we propose the Fermatean neutrosophic vague soft set (FNVSS), an advanced extension that integrates the concepts of neutrosophic sets with Fermatean membership functions into the framework of vague sets. The FNVSS model enhances the representation of truth, indeterminacy, and falsity degrees, providing greater flexibility and resilience in capturing ambiguous and imprecise information. We systematically develop new operations for the FNVSS, including union, intersection, complementation, the Fermatean neutrosophic vague normalized weighted average (FNVNWA) operator, the generalized Fermatean neutrosophic vague normalized weighted average (GFNVNWA) operator, and an adapted Technique for Order of Preference by Similarity to Ideal Solution (TOPSIS) method. To demonstrate the practicality of the proposed methodology, we apply it to a solar panel selection problem, where managing uncertainty is crucial. Comparative results indicate that the FNVSS significantly outperforms traditional fuzzy and vague set approaches, leading to more reliable and accurate decision outcomes. This work contributes to the advancement of predictive decision-making systems, particularly in fields requiring high precision, adaptability, and robust uncertainty modeling.

Keywords: Fermatean neutrosophic vague soft set; decision-making; TOPSIS; FNVNWA; GFNVNWA

MSC: 91B06; 90B50; 94D05; 03E72

check for updates

Academic Editor: Hsien-Chung Wu

Received: 16 April 2025
Revised: 9 May 2025
Accepted: 12 May 2025
Published: 19 May 2025

Citation: Althuniyan, N.; Al-shboul, A.; Aljohani, S.; Wang, K.L.; Wong, K.B.; Alhazaymeh, K.; Aiady, S.S. Decision-Making with Fermatean Neutrosophic Vague Soft Sets Using a Technique for Order of Preference by Similarity to Ideal Solution. *Axioms* **2025**, *14*, 381. https://doi.org/10.3390/axioms14050381

1. Introduction

In real-life scenarios, data rarely come in a perfect form and are often riddled with gaps, uncertainty, or inconsistencies. These issues arise from various sources, such as randomness, measurement errors, and subjective human judgment. This uncertainty presents a major hurdle in areas such as engineering, economics, healthcare, and environmental science, where decisions must be made despite unclear or incomplete information. To address these challenges, researchers have developed a variety of mathematical models over time, each contributing to a better grasp and representation of uncertainty.

Among the most notable advances is the development of fuzzy set theory by Zadeh [1], who introduced a mathematical approach to represent vague or imprecise information. Fuzzy sets differ from traditional sets by allowing partial membership, offering a more flexible and realistic way to model real-world phenomena. Building on this, Atanassov introduced intuitionistic fuzzy sets, which go a step further by incorporating not only membership but also non-membership values, along with a hesitation margin to reflect uncertainty more fully. Atanassov [2,3] later expanded this framework to include interval-valued intuitionistic fuzzy sets, equipping the model to better deal with intricate and nuanced uncertainty within datasets. Soft sets were proposed by Molodtsov [4] as another commonly used method in handling uncertainties in data. The concept of a fuzzy soft set was introduced by Maji [5].

In parallel, vague set theory was developed by Gau and Buehrer [6], who introduced a concept closely related to intuitionistic fuzzy sets but distinguished by its use of upper and lower bounds for membership degrees. This connection was further elucidated by Bustince and Burillo [7], who showed that vague sets are a derived form of intuitionistic fuzzy sets. Chen [8,9] found applications of vague sets in various fields, such as system reliability analysis, where they are used to model and assess the reliability of systems under fuzzy conditions. Chen defined similarity measures between vague sets, which are essential for comparison and clustering applications in fuzzy environments.

As the landscape of uncertainty modeling evolved, the introduction of neutrosophic set theory by Smarandache [10] opened new possibilities for managing indeterminacy alongside traditional membership and non-membership degrees. This was further expanded by the development of neutrosophic soft set theory [11], a flexible approach that enhances decision-making under uncertainty by accommodating various types of uncertain and imprecise information. Building on this foundation, Alkhazaleh et al. [12] proposed the notation of Fermatean neutrosophic soft sets, which incorporate Fermatean algebra to offer a more refined structure for addressing complex forms of uncertainty. Also, Alkhazaleh et al. [13] introduced the possibility Fermatean neutrosophic soft set, which introduces a probabilistic component from possibility theory and takes this a step further, providing an even more powerful tool to model and resolve uncertainty in dynamic systems. Al-shboul et al. [14] introduced the notation of Fermatean vague soft sets, which provided a new dimension by combining the principles of Fermatean algebra with vague set theory, further enhancing the ability to model complex uncertainties in decision-making scenarios.

The relevance of vague sets in decision-making was further highlighted by Hong and Choi [15], who applied vague set theory to multiattribute decision-making (MADM) problems. Gorzalzany's work [16] on interval-valued fuzzy sets provided a method for inference in approximate reasoning, establishing a foundation for using interval-valued sets in decision-making. In practical applications, Kumar et al. [17,18] used interval-valued vague sets to analyze the reliability of marine power plants and extended this work with arithmetic operations on interval-valued vague sets for system reliability analysis.

Recent research has built on these foundations, incorporating more sophisticated methods such as Einstein hybrid geometric aggregation operators have been applied in MADM, as introduced by Alhazaymeh et al. [19]. Additionally, Einstein operations on vague soft sets have further expanded the theoretical and practical scope of soft set theory, particularly in uncertain environments.

The study of vague soft set relations has also advanced with the introduction of transitive closure operators [20], which refine decision models by improving the way relationships between vague sets are processed. The application of vague sets has been extended through cubic vague sets [21], which provide a more comprehensive decision-making framework by integrating cubic set theory and vague sets. Furthermore, possibility interval-valued vague soft sets [22] and generalized interval-valued vague soft sets [23] offer additional tools for managing uncertainty, particularly in scenarios where membership and non-membership values are best represented as intervals.

Several studies have focused on enhancing decision-making frameworks using advanced fuzzy set theories and their extensions. Alhazaymeh et al. [24] introduced a neutrosophic cubic Einstein hybrid geometric aggregation operator, demonstrating its efficiency in prioritization problems involving multiple attributes. Similarly, Wang et al. [25] explored parameterized OWA operators under vague set theory to strengthen fuzzy multicriteria decision-making (MCDM) strategies. Zhou and Wu [26] extended these approaches through the development of generalized intuitionistic fuzzy rough approximation operators, laying the groundwork for more refined and granular fuzzy decision models. Shahzad et al. [27] contributed to the theoretical foundation by analyzing mappings and stability within fuzzy rough sets. Rahim et al. [28] introduced novel distance measures for Pythagorean cubic fuzzy sets, applying these techniques effectively to the selection of optimal treatments for psychological disorders such as depression and anxiety. In another important advancement, Khan et al. [29] presented covering-based intuitionistic hesitant fuzzy rough set models with specific applications to decision-making problems, highlighting the utility of hybrid hesitant frameworks. Fahmi et al. [30] proposed a group decision-making method based on cubic Fermatean Einstein fuzzy weighted geometric operators, enabling more robust aggregation under uncertainty. Building on this, Fahmi et al. [31] also applied a disaster decision-making strategy using the DDAS method in Fermatean fuzzy environments, incorporating regret theory and philosophy to handle conflicting and uncertain evaluations. Liang et al. [32] described an extended structure that includes cognitive decision-making, path-planning, and motion-control programs using extracted deep Q-networks and inverse reinforcement learning techniques. Wu et al. [33] analyzed the development of contentment prediction success from the integration of societal expertise into AI simulations, continuously exposing additional vital factors for decision-making. Pan et al. [34] presented a decision-level integration strategy to analyze the emotions of the mine worker with an extended Yager rule for insertion. Li et al. [35] proposed an advanced decision-making and scheduling mechanism for an autonomous vehicle that confirms oscillation-free execution.

The Technique for Order of Preference by Similarity to Ideal Solution (TOPSIS) is a multicriteria decision-making approach proposed by Ching et al. [36]. The TOPSIS evaluates options by measuring their geometric distance to an ideal solution, hence promoting objective decision-making in several fields. Chen and Hwang [37] extended the TOPSIS method and introduced a new method of TOPSIS.

Due to the complexity of solar panel selection, a Fermatean neutrosophic vague TOPSIS method offers a systematic and computationally efficient approach for rating many choices. The integration of Fermatean vague sets enables a more sophisticated depiction of uncertainty, guaranteeing that expert evaluations are precisely represented in the decision-making process. This method provides a more reliable and data-informed selection of solar panels, hence facilitating the implementation of cost-effective photovoltaic solutions.

Consequently, by tackling some of the shortcomings of current theories, especially in relation to computing complexity and parameterization, Fermatean vague sets have great potential to further the area of uncertainty modeling. Traditional fuzzy sets, intuitionistic fuzzy sets, and vague sets all fall short in certain contexts; however, FNVSS can handle a broader range of uncertainty values, making them applicable in more diverse and complex decision-making environments. This opens up new research and practical application avenues.

This study is directed by several main research questions such as the following:

1. In what manner may the conventional VS framework be extended via the Fermatean approach and neutrosophic set theory to encompass supplementary dimensions of uncertainty in decision-making?
2. What are the theoretical properties of the FNVSS, and how do these characteristics augment its efficacy in modeling complex, practical scenarios?
3. How can the implementation of the FNVSS framework enhance decision accuracy and robustness relative to conventional methods?

1.1. Motivation and Contribution

In real-world decision-making, information is rarely complete, precise, or crisp. Most often, decision-makers face scenarios where the data are vague, ambiguous, indeterminate, and context-dependent. Classical mathematical tools, such as fuzzy sets, intuitionistic fuzzy sets, and even traditional soft sets, are insufficient to model this multidimensional uncertainty. They either oversimplify the structure of the data or ignore critical components such as conflicting evidence, indeterminacy, and subjectivity.

To address these challenges, the following were proposed:

1. Neutrosophic sets were developed to model truth, indeterminacy, and falsity independently, acknowledging the inherent contradiction and incompleteness of human knowledge.
2. Fermatean fuzzy sets extended traditional fuzzy models by using a nonlinear constraint $T^3 + I^3 + F^3 \leq 1$, allowing higher degrees of truth, indeterminacy, and falsity to be represented simultaneously, something unattainable in earlier fuzzy models.
3. Vague sets introduced linguistic flexibility and interval-based flexibility to handle imprecise and overlapping concepts.
4. Soft sets allowed parameter-driven modeling, which is especially useful in complex multiattribute environments with multiple experts or sources.

However, no single framework before Fermatean neutrosophic vague soft sets (FNVSSs) could simultaneously accomplish the following:

1. Capture nonlinear degrees of truth, falsity, and indeterminacy;
2. Handle vague or linguistic information through interval-based representation;
3. Support the parameterization of context, source, or expert view through soft sets;
4. Enable granular decision modeling in a modular and scalable way.

Therefore, we propose the FNVSS as a unified and highly expressive framework that fills this critical gap. It is designed to bridge the limitations of previous models and to fully capture the multifaceted uncertainty and vagueness encountered in real-life decision-making processes.

In order to operationalize the FNVSS in practical applications, we integrate it with the Technique for Order Preference by Similarity to Ideal Solution (TOPSIS). The TOPSIS method offers a mathematically sound, efficient, and interpretable approach to rank alternatives based on their proximity to ideal conditions. When fused with the expressive power of the FNVSS, this hybrid model becomes a powerful decision-making tool that can outperform traditional models in terms of accuracy, realism, and robustness.

1.2. Structure of Article

The subsequent sections of this work are organized as follows: Section 2 discusses the necessary preliminaries, including definitions of VS, VSS, NSS, FNS, and their associated operations. Section 3 presents the idea of the FNVSS, including basic definitions and mathematical properties. Section 4 discusses basic operations, including union, intersection, complementation, and new type operators of the FNVSS. Section 5 presents an FNV-TOPSIS to solve decision-making issues, which shows the efficacy of the suggested methodology. In Section 6, the FNVSS is compared with other traditional models. Section 7 ultimately summarizes the work by summarizing major findings, prospective applications, and future research avenues.

2. Preliminaries

This section provides some fundamental definitions required in this paper.

2.1. Vague Set

Definition 1 ([6])**.** *Let $X = \{x_1, x_2, \ldots, x_n\}$. A vague set (VS) over X, denoted by Δ_v, is characterized by truth-membership function $\tau_v(x) : X \to [0,1]$ and false-membership function $\phi_v(x) : X \to [0,1]$ such that $\tau_v(x) + \phi_v(x) \leq 1$ for every $x \in X$. The grade of membership μ_v of an element $x \in X$ is not specified precisely but lies within the interval $\tau_v(x) \leq \mu_v(x) \leq 1 - \phi_v(x)$, indicating uncertainty or vagueness in evaluating x's membership.*

The membership interval of $x \in X$ in the vague set Δ_v is given as

$$\Delta_v(x) = [\tau_v(x), 1 - \phi_v(x)].$$

If X is a continuous universe, then a vague set Δ_v on X may be defined as:

$$\Delta_v = \int [\tau_v(x_i), 1 - \phi_v(x_i)]/x_i, \quad x_i \in X.$$

If X is a discrete universe, then a vague set Δ_v on X may be defined as

$$\Delta_v = \sum_{i=1}^{n} [\tau_v(x_i), 1 - \phi_v(x_i)]/x_i, \quad x_i \in X.$$

2.2. Fermatean Neutrosophic Set

Definition 2 ([38])**.** *Let X be a non-empty set. A Fermatean neutrosophic set (FNS) on X is defined as $A = \{(x, T(x), I(x), F(x)) : x \in X\}$, where $T(x), I(x), F(x) \in [0,1]$ satisfying $0 \leq T(x)^3 + I(x)^3 + F(x)^3 \leq 2$ for all $x \in X$.*

2.3. Neutrosophic Soft Set

Definition 3 ([11])**.** *Consider a universal set U, and consider $\mathcal{P}(U)$ as the collection of all neutrosophic sets defined over U. Let E'' be a set of parameters and $A^{\cdots}$ be a subset of E''. A neutrosophic soft set over U is defined by the pair $(\Im, A^{\cdots})$, where $\Im$ is a function that links each parameter in $A^{\cdots}$ to a neutrosophic set in $\mathcal{P}(U)$, formally written as*

$$\Im : A^{\cdots} \to \mathcal{P}(U).$$

2.4. Vague Soft Set

Definition 4 ([39])**.** *Consider a pair $(\Im, A^{\cdots})$, where $A^{\cdots}$ is a collection of parameters and $\Im$ is a function defined as*

$$\Im : A^{\cdots} \to V(U),$$

with $V(U)$ representing the set of vague sets over the universe U. This pair is known as a vague soft set over U. To put it differently, a vague soft set over U is essentially a family of vague sets, each associated with a parameter in $A^{\cdots}$. For a given parameter $\varepsilon \in A^{\cdots}$, the vague set

$$\varrho_{\Im(\varepsilon)} : U \to [0,1]^2$$

describes the degree of truth and the degree of falsity for each element in U. This mapping is considered to define the ε-approximate elements in the vague soft set $(\Im, A^{\cdots})$.

3. Fermatean Neutrosophic Vague Soft Set

This section presents the notation for Fermatean vague sets (FNVSSs).

Definition 5. *Let X be a universal set and E be a set of parameters. Suppose $A \subseteq E$ and $\Im : A \to FNV(X)$, where $FNV(X)$ indicates the collection of all Fermatean neutrosophic vague soft subsets of X. Let $\tilde{\Im} : A \to FNV(X)$ be a function defined as follows:*

$$\tilde{\Im} = \{(x, \overline{T}_v(x), \overline{I}_v(x),\ 1 - \overline{F}_v(x)) : x \in X\},$$

where each membership function is defined as

$$\overline{T}_v(x) = [T^-, T^+], \quad \overline{I}_v(x) = [I^-, I^+], \quad \overline{F}_v(x) = [F^-, F^+]$$

with

$$\overline{T}_v(x) = [T^-, T^+] \to [0,1]$$
$$\overline{I}_v(x) = [I^-, I^+] \to [0,1]$$
$$\overline{F}_v(x) = [F^-, F^+] \to [0,1]$$

representing the truth, indeterminacy, and falsity interval-valued degrees for element $x \in X$, respectively.

To maintain the integrity of the Fermatean neutrosophic condition, the components must satisfy the following conditions:

$$T_v^-(x)^3 + I_v^-(x)^3 + F_v^-(x)^3 \leq 2,$$

$$T_v^+(x)^3 + I_v^+(x)^3 + F_v^+(x)^3 \leq 2.$$

Example 1. *Let $X = \{x_1, x_2, x_3\}$ be a universal set. Suppose $\tilde{\Im}$ is a Fermatean neutrosophic vague soft set in X defined by*

$$\tilde{\Im} = \left\{ \frac{x_1}{[0.3, 0.4], [0.6, 0.5], [0.7, 0.9]}, \frac{x_2}{[0.7, 0.4], [0.1, 0.5], [0.8, 0.9]} \right\}.$$

For element x_1, we have verification for Fermatean and vague conditions:

(i.) $T_v^- = 0.3$ *and* $T_v^+ = 0.4$*;*
(ii.) $I_v^- = 0.6$ *and* $I_v^+ = 0.5$*;*
(iii.) *From* $1 - F_v^- = 0.7$ *and* $1 - F_v^+ = 0.9$*, we obtain* $F_v^- = 0.3$ *and* $F_v^+ = 0.1$*.*
For the vague condition:

$$0.3 + 0.3 = 0.6 \leq 1,$$

$$0.4 + 0.1 = 0.5 \leq 1.$$

For the Fermatean neutrosophic condition:

$$0.3^3 + 0.6^3 + 0.3^3 = 0.27 \le 2,$$

$$0.4^3 + 0.5^3 + 0.1^3 = 0.19 \le 2.$$

Fermatean neutrosophic and vague conditions are satisfied for element x_1. Similarly, Fermatean neutrosophic and vague conditions are satisfied for element x_2.

Definition 6. *Let χ_v be an FNVSS of the universe X, where for each $x \in X$, $\overline{T}_v(x) = [0,0]$, $\overline{I}_v(x) = [1,1]$, and $F_v(x) = [1,1]$. Then, χ_v is said to be the zero FNVSS.*

Example 2. *Let $X = \{x_1, x_2, x_3\}$ be a universal set. Then, the zero FNVSS of the universe X is defined as follows:*

$$\chi_v = \left\{ \frac{x_1}{[0,0],[1,1],[1,1]}, \frac{x_2}{[0,0],[1,1],[1,1]}, \frac{x_3}{[0,0],[1,1],[1,1]} \right\}.$$

Definition 7. *Let Υ_v be an FNVSS of the universe X, where for each $x \in X$, $\overline{T}_v(x) = [1,1]$, $\overline{I}_v(x) = [0,0]$, and $\overline{F}_v(x) = [0,0]$. Then, Υ_v is said to be the unit FNVSS.*

Example 3. *Let $X = \{x_1, x_2, x_3\}$ be a universal set. Then, the unit FNVSS is defined as follows:*

$$\Upsilon_v = \left\{ \frac{x_1}{[1,1],[1,1],[0,0]}, \frac{x_2}{[1,1],[1,1],[0,0]}, \frac{x_3}{[1,1],[1,1],[0,0]} \right\}.$$

Definition 8. *Let X be a universal set. Let $P(e_i)$ and $K(e_i)$ be two FNVSSs of the universe X. If for each $x \in X$, the following hold:*

(i.) $P(e_i)^{\overline{T}} \le K(e_i)^{\overline{T}}$;

(ii.) $P(e_i)^{\overline{I}} \ge K(e_i)^{\overline{I}}$;

(iii.) $P(e_i)^{\overline{F}} \ge K(e_i)^{\overline{F}}$.

then $P(e_i)$ is said to be a subset of $K(e_i)$, and we write $P(e_i) \subseteq K(e_i)$.

Example 4. *Let $X = \{x_1, x_2, x_3\}$ be a universal set. Let $P(e)$ and $K(e)$ be two FNVSSs of the universe X defined by*

$$P(e_1) = \left\{ \frac{x_1}{[0.3,0.4],[0.6,0.5],[0.7,0.9]}, \frac{x_2}{[0.7,0.4],[0.1,0.5],[0.8,0.9]} \right\},$$

$$P(e_2) = \left\{ \frac{x_1}{[0.2,0.6],[0.4,0.4],[0.8,0.8]}, \frac{x_2}{[0.5,0.2],[0.3,0.5],[0.6,0.3]} \right\},$$

$$K(e_1) = \left\{ \frac{x_1}{[0.6,0.5],[0.3,0.4],[0.6,0.8]}, \frac{x_2}{[0.7,0.5],[0.1,0.2],[0.7,0.6]} \right\},$$

$$K(e_2) = \left\{ \frac{x_1}{[0.5,0.8],[0.3,0.3],[0.7,0.8]}, \frac{x_2}{[0.6,0.8],[0.2,0.4],[0.4,0.2]} \right\}.$$

It is clear that $P(e)$ is a Fermatean neutrosophic vague subset of $K(e)$.

4. Union, Intersection, and Complementation of FNVSSs

We will discuss some operations of *FNVSSs* below.

Definition 9. *The complement of an FNVSS $\digamma$ of a universe X is denoted by $(\digamma)^c$, where for each $x \in X$, the following are true:*

(i) $(\overline{T}_v(x))^c = 1 - \overline{F}_v(x)$;
(ii) $(\overline{I}_v(x))^c = 1 - \overline{I}_v(x)$;
(iii) $(1 - \overline{F}_v(x))^c = \overline{T}_v(x)$.

Example 5. *Consider the FNVSS in Example 1 defined as follows.*

$$F = \left\{ \frac{x_1}{[0.3, 0.4], [0.6, 0.5], [0.7, 0.9]}, \frac{x_2}{[0.7, 0.4], [0.1, 0.5], [0.8, 0.9]} \right\}.$$

The complement of F *is*

$$(F)^c = \left\{ \frac{x_1}{[0.7, 0.9], [0.4, 0.5], [0.3, 0.4]}, \frac{x_2}{[0.8, 0.9], [0.9, 0.5], [0.7, 0.4]} \right\}.$$

Definition 10. *Let X be a universal set. Let $P(e)$ and $K(e)$ be two FNVSSs of the universe X. The union of $P(e)$ and $K(e)$ is defined by $P(e) \cup K(e)$, where for each $x \in X$,*

$$P(e) \cup K(e) = \{x, \max[P^T(e), K^T(e)], \min[P^I(e), K^I(e)], \min[P^F(e), K^F(e)]\}$$

Definition 11. *Let X be a universal set. Let $P(e)$ and $K(e)$ be two FNVSSs of the universe X. The intersection of $P(e)$ and $K(e)$ is defined by $P(e) \cap K(e)$, where for each $x \in X$,*

$$P(e) \cap K(e) = \{x, \min[P^T(e), K^T(e)], \max[P^I(e), K^I(e)], \max[P^F(e), K^F(e)]\}$$

Example 6. *Consider Example 4. To find the union and intersection of two FNVSSs, we will use the methods mentioned in Definitions 9 and 10:*

$$P(e_1) = \left\{ \frac{x_1}{[0.3, 0.4], [0.6, 0.5], [0.7, 0.9]}, \frac{x_2}{[0.7, 0.4], [0.1, 0.5], [0.8, 0.9]} \right\},$$

$$P(e_2) = \left\{ \frac{x_1}{[0.2, 0.6], [0.4, 0.4], [0.8, 0.8]}, \frac{x_2}{[0.5, 0.2], [0.3, 0.5], [0.6, 0.3]} \right\},$$

$$K(e_1) = \left\{ \frac{x_1}{[0.6, 0.5], [0.3, 0.4], [0.6, 0.8]}, \frac{x_2}{[0.7, 0.5], [0.1, 0.2], [0.7, 0.6]} \right\},$$

$$K(e_2) = \left\{ \frac{x_1}{[0.5, 0.8], [0.3, 0.3], [0.7, 0.8]}, \frac{x_2}{[0.6, 0.8], [0.2, 0.4], [0.4, 0.2]} \right\}.$$

(i) *To find the union of $P(e)$ and $K(e)$:*

$$P(e_1) \cup K(e_1) = \{x, \max[P^T(e_1), K^T(e_1)], \min[P^I(e_1), K^I(e_1)], \min[P^F(e_1), K^F(e_1)]\}$$

$$P(e_1) \cup K(e_1) = \left\{ \frac{x_1}{[0.6, 0.5], [0.3, 0.4], [0.6, 0.8]}, \frac{x_2}{[0.7, 0.5], [0.1, 0.2], [0.7, 0.6]} \right\}$$

Similarly, for $P(e_2)$ and $K(e_2)$,

$$P(e_2) \cup K(e_2) = \left\{ \frac{x_1}{[0.5, 0.8], [0.3, 0.3], [0.7, 0.8]}, \frac{x_2}{[0.6, 0.8], [0.2, 0.4], [0.4, 0.2]} \right\}.$$

(ii) *To find the intersection of $P(e)$ and $K(e)$:*

$$P(e_1) \cap K(e_1) = \{x, \min[P^T(e_1), K^T(e_1)], \max[P^I(e_1), K^I(e_1)], \max[P^F(e_1), K^F(e_1)]\}$$

$$P(e_1) \cap K(e_1) = \left\{ \frac{x_1}{[0.3, 0.4], [0.6, 0.5], [0.7, 0.9]}, \frac{x_2}{[0.7, 0.4], [0.1, 0.5], [0.8, 0.9]} \right\}$$

Similarly, for $P(e_2)$ and $K(e_2)$,

$$P(e_2) \cap K(e_2) = \left\{ \frac{x_1}{[0.2, 0.6], [0.4, 0.4], [0.8, 0.8]}, \frac{x_2}{[0.5, 0.2], [0.3, 0.5], [0.6, 0.3]} \right\}.$$

Definition 12. *Let X be a universal set. Let $P(e)$ and $K(e)$ be two FNVSSs of the universe X. "$P(e)$ AND $K(e)$", denoted by "$P(e) \wedge K(e)$", is defined by $P(e) \cap K(e)$, where for each $x \in X$,*

$$P(e) \wedge K(e) = \{x, \min[P^T(e), K^T(e)], \max[P^I(e), K^I(e)], \max[P^F(e), K^F(e)]\}$$

Definition 13. *Let X be a universal set. Let $P(e)$ and $K(e)$ be two FNVSSs of the universe X. "$P(e)$ OR $K(e)$", denoted by "$P(e) \vee K(e)$", is defined by $P(e) \cup K(e)$, where for each $x \in X$,*

$$P(e) \vee K(e) = \{x, \max[P^T(e), K^T(e)], \min[P^I(e), K^I(e)], \min[P^F(e), K^F(e)]\}.$$

Definition 14. *For $\nabla_i > 0$ and $\nabla_i \neq 1$, let*

$$\begin{aligned}
\Bbbk &= \left\langle \log_{\nabla_i}[T^-_{\Bbbk}, T^+_{\Bbbk}], \log_{\nabla_i}[I^-_{\Bbbk}, I^+_{\Bbbk}], \log_{\nabla_i}[F^-_{\Bbbk}, F^+_{\Bbbk}] \right\rangle, \\
\Bbbk_1 &= \left\langle \log_{\nabla_i}[T^-_{\Bbbk_1}, T^+_{\Bbbk_1}], \log_{\nabla_i}[I^-_{\Bbbk_1}, I^+_{\Bbbk_1}], \log_{\nabla_i}[F^-_{\Bbbk_1}, F^+_{\Bbbk_1}] \right\rangle, \\
\Bbbk_2 &= \left\langle \log_{\nabla_i}[T^-_{\Bbbk_2}, T^+_{\Bbbk_2}], \log_{\nabla_i}[I^-_{\Bbbk_2}, I^+_{\Bbbk_2}], \log_{\nabla_i}[F^-_{\Bbbk_2}, F^+_{\Bbbk_2}] \right\rangle
\end{aligned}$$

be any three new-type Fermatean neutrosophic vague numbers (FNVNs). Let L_1, L_2, L_3 be positive integers and α be a positive real parameter. Their operations are defined as follows:

1.

$$\Bbbk_1 \oplus \Bbbk_2 = \begin{bmatrix} \Big[\sqrt[L_1]{(\log_{\nabla_i} T^-_{\Bbbk_1})^{L_1} + (\log_{\nabla_i} T^-_{\Bbbk_2})^{L_1} - (\log_{\nabla_i} T_{\Bbbk_1})^{L_1} \cdot (\log_{\nabla_i} T_{\Bbbk_2})^{L_1}}, \\ \sqrt[L_1]{(\log_{\nabla_i} T^+_{\Bbbk_1})^{L_1} + (\log_{\nabla_i} T^+_{\Bbbk_2})^{L_1} - (\log_{\nabla_i} T^+_{\Bbbk_1})^{L_1} \cdot (\log_{\nabla_i} T^+_{\Bbbk_2})^{L_1}} \Big], \\ \Big[\sqrt[L_2]{(\log_{\nabla_i} I^-_{\Bbbk_1})^{L_2} + (\log_{\nabla_i} I^-_{\Bbbk_2})^{L_2} - (\log_{\nabla_i} I^-_{\Bbbk_1})^{L_2} \cdot (\log_{\nabla_i} I^-_{\Bbbk_2})^{L_2}}, \\ \sqrt[L_2]{(\log_{\nabla_i} I^+_{\Bbbk_1})^{L_2} + (\log_{\nabla_i} I^+_{\Bbbk_2})^{L_2} - (\log_{\nabla_i} I^+_{\Bbbk_1})^{L_2} \cdot (\log_{\nabla_i} I^+_{\Bbbk_2})^{L_2}} \Big], \\ \Big[(\log_{\nabla_i} F^-_{\Bbbk_1})^{L_3} \cdot (\log_{\nabla_i} F^-_{\Bbbk_2})^{L_3}, (\log_{\nabla_i} F^+_{\Bbbk_1})^{L_3} \cdot (\log_{\nabla_i} F^+_{\Bbbk_2})^{L_3} \Big] \end{bmatrix}$$

2.

$$\Bbbk_1 \odot \Bbbk_2 = \begin{bmatrix} \Big[(\log_{\nabla_i} T^-_{\Bbbk_1})^{L_1} \cdot (\log_{\nabla_i} T^-_{\Bbbk_2})^{L_1}, (\log_{\nabla_i} T^+_{\Bbbk_1})^{L_1} \cdot (\log_{\nabla_i} T^+_{\Bbbk_2})^{L_1} \Big], \\ \Big[\sqrt[L_2]{(\log_{\nabla_i} I^-_{\Bbbk_1})^{L_2} + (\log_{\nabla_i} I^-_{\Bbbk_2})^{L_2} - (\log_{\nabla_i} I^-_{\Bbbk_1})^{L_2} \cdot (\log_{\nabla_i} I^-_{\Bbbk_2})^{L_2}}, \\ \sqrt[L_2]{(\log_{\nabla_i} I^+_{\Bbbk_1})^{L_2} + (\log_{\nabla_i} I^+_{\Bbbk_2})^{L_2} - (\log_{\nabla_i} I^+_{\Bbbk_1})^{L_2} \cdot (\log_{\nabla_i} I^+_{\Bbbk_2})^{L_2}} \Big], \\ \Big[\sqrt[L_3]{(\log_{\nabla_i} F^-_{\Bbbk_1})^{L_3} + (\log_{\nabla_i} F^-_{\Bbbk_2})^{L_3} - (\log_{\nabla_i} F^-_{\Bbbk_1})^{L_3} \cdot (\log_{\nabla_i} F^-_{\Bbbk_2})^{L_3}}, \\ \sqrt[L_3]{(\log_{\nabla_i} F^+_{\Bbbk_1})^{L_3} + (\log_{\nabla_i} F^+_{\Bbbk_2})^{L_3} - (\log_{\nabla_i} F^+_{\Bbbk_1})^{L_3} \cdot (\log_{\nabla_i} F^+_{\Bbbk_2})^{L_3}} \Big] \end{bmatrix}$$

3.

$$\alpha \cdot \Bbbk = \begin{bmatrix} \Big[\sqrt[L_1]{1 - \Big(1 - (\log_{\nabla_i} T^-_{\Bbbk})^{L_1}\Big)^{\alpha}}, \sqrt[L_1]{1 - \Big(1 - (\log_{\nabla_i} T^+_{\Bbbk})^{L_1}\Big)^{\alpha}} \Big], \\ \Big[\sqrt[L_2]{1 - \Big(1 - (\log_{\nabla_i} I^-_{\Bbbk})^{L_2}\Big)^{\alpha}}, \sqrt[L_2]{1 - \Big(1 - (\log_{\nabla_i} I^+_{\Bbbk})^{L_2}\Big)^{\alpha}} \Big], \\ \Big[\Big(\log_{\nabla_i} F^-_{\Bbbk}\Big)^{L_3\alpha}, \Big(\log_{\nabla_i} F^+_{\Bbbk}\Big)^{L_3\alpha} \Big]. \end{bmatrix}$$

4.

$$\mathbb{k}^{\alpha} = \begin{bmatrix} \left[(\log_{\nabla_i} T_{\mathbb{k}}^{-})^{L_1 \alpha}, (\log_{\nabla_i} T_{\mathbb{k}}^{+})^{L_1 \alpha} \right] \\ \left[\sqrt[L_2]{1 - \left(1 - (\log_{\nabla_i} I_{\mathbb{k}}^{-})^{L_2}\right)^{\alpha}}, \sqrt[L_2]{1 - \left(1 - (\log_{\nabla_i} I_{\mathbb{k}}^{+})^{L_2}\right)^{\alpha}} \right] \\ \left[\sqrt[L_3]{1 - \left(1 - (\log_{\nabla_i} F_{\mathbb{k}}^{-})^{L_3}\right)^{\alpha}}, \sqrt[L_3]{1 - \left(1 - (\log_{\nabla_i} F_{\mathbb{k}}^{+})^{L_3}\right)^{\alpha}} \right]. \end{bmatrix}$$

Example 7. *Let*

$$\mathbb{k}_1 = \langle \log_{10}[0.4, 0.3], \log_{10}[0.3, 0.5], \log_{10}[0.7, 0.8] \rangle,$$
$$\mathbb{k}_2 = \langle \log_{10}[0.5, 0.5], \log_{10}[0.7, 0.2], \log_{10}[0.5, 0.6] \rangle$$

be two FNVNs, $L_1 = L_2 = L_3 = 2$, and $\alpha = 3$. Then,

1.

$$\mathbb{k}_1 \oplus \mathbb{k}_2 = \begin{bmatrix} \left[\sqrt{(\log_{10} 0.4)^2 + (\log_{10} 0.5)^2 - (\log_{10} 0.4)^2 \cdot (\log_{10} 0.5)^2}, \right. \\ \left. \sqrt{(\log_{10} 0.3)^2 + (\log_{10} 0.5)^2 - (\log_{10} 0.3)^2 \cdot (\log_{10} 0.5)^2} \right], \\ \left[\sqrt{(\log_{10} 0.3)^2 + (\log_{10} 0.7)^2 - (\log_{10} 0.3)^2 \cdot (\log_{10} 0.7)^2}, \right. \\ \left. \sqrt{(\log_{10} 0.5)^2 + (\log_{10} 0.2)^2 - (\log_{10} 0.5)^2 \cdot (\log_{10} 0.2)^2} \right], \\ \left[(\log_{10} 0.7)^2 \cdot (\log_{10} 0.5)^2, (\log_{10} 0.8)^2 \cdot (\log_{10} 0.6)^2 \right] \end{bmatrix}$$

$$\mathbb{k}_1 \oplus \mathbb{k}_2 = \begin{bmatrix} \left[\sqrt{(-0.397)^2 + (-0.301)^2 - (0.397)^2 \cdot (-0.301)^2}, \right. \\ \left. \sqrt[2]{(-0.522)^2 + (-0.301)^2 - (-0.522)^2 \cdot (-0.301)^2} \right], \\ \left[\sqrt{(-0.522)^2 + (-0.154)^2 - (-0.522)^2 \cdot (-0.154)^2}, \right. \\ \left. \sqrt{(-0.301)^2 + (-0.698)^2 - (-0.301)^2 \cdot (-0.698)^2} \right], \\ \left[(-0.154)^2 \cdot (-0.301)^2, (-0.096)^2 \cdot (-0.221)^2 \right] \end{bmatrix}$$

Hence,

$$\mathbb{k}_1 \oplus \mathbb{k}_2 = \Big[[0.488, 0.581], [0.538, 0.730], [0.002, 0.004] \Big]$$

2.

$$3 \cdot \mathbb{k}_1 = \begin{bmatrix} \left[\sqrt{1 - (1 - (\log_{10} 0.4)^2)^3}, \sqrt[2]{1 - (1 - (\log_{10} 0.3)^2)^3} \right] \\ \left[\sqrt{1 - (1 - (\log_{10} 0.3)^2)^3}, \sqrt[2]{1 - (1 - (\log_{10} 0.5)^2)^3} \right] \\ \left[(\log_{10} 0.5)^{2 \cdot 3}, (\log_{10} 0.6)^{2 \cdot 3} \right] \end{bmatrix}$$

$$3 \cdot \mathbb{k}_1 = \begin{bmatrix} \left[\sqrt{1 - (1 - (-0.397)^2)^3}, \sqrt{1 - (1 - (-0.522)^2)^3} \right] \\ \left[\sqrt{1 - (1 - (-0.522)^2)^3}, \sqrt{1 - (1 - (-0.301)^2)^3} \right] \\ \left[(-0.301)^6, (-0.221)^6 \right] \end{bmatrix}$$

Hence,

$$3 \cdot \mathbb{k}_1 = \Big[[0.634, 0.784], [0.784, 0.497], [0.0007, 0.0001] \Big].$$

Theorem 1. *For $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, i_j$ let*

$$\mathbb{k}_i = \left\langle \log[T^-_{\mathbb{k}_{p_{ij}}}, T^+_{\mathbb{k}_{p_{ij}}}], \log[I^-_{\mathbb{k}_{p_{ij}}}, I^+_{\mathbb{k}_{p_{ij}}}], \log[F^-_{\mathbb{k}_{p_{ij}}}, F^+_{\mathbb{k}_{p_{ij}}}] \right\rangle,$$
$$\mathcal{W}_i = \left\langle \log[T^-_{\mathbb{k}_{k_{ij}}}, T^+_{\mathbb{k}_{k_{ij}}}], \log[I^-_{\mathbb{k}_{k_{ij}}}, I^+_{\mathbb{k}_{k_{ij}}}], \log[F^-_{\mathbb{k}_{k_{ij}}}, F^+_{\mathbb{k}_{k_{ij}}}] \right\rangle.$$

be FNVN, where p_{ij} and k_{ij} indicate the parameter indexed for the new type and alternative indexed for the new type, respectively. If for each $i = 1, 2, \ldots, n, j = 1, 2, \ldots, i_j$,

$$(\log_{\nabla_i} T^-_{\mathbb{k}_{p_{ij}}} + \log_{\nabla_i} T^+_{\mathbb{k}_{p_{ij}}})^3 \leq (\log_{\nabla_i} T^-_{\mathbb{k}_{k_{ij}}} + \log_{\nabla_i} T^+_{\mathbb{k}_{k_{ij}}})^3,$$
$$(\log_{\nabla_i} I^-_{\mathbb{k}_{p_{ij}}} + \log_{\nabla_i} I^+_{\mathbb{k}_{p_{ij}}})^3 \geq (\log_{\nabla_i} I^-_{\mathbb{k}_{k_{ij}}} + \log_{\nabla_i} I^+_{\mathbb{k}_{k_{ij}}})^3,$$
$$(\log_{\nabla_i} F^-_{\mathbb{k}_{p_{ij}}} + \log_{\nabla_i} F^+_{\mathbb{k}_{p_{ij}}})^3 \geq (\log_{\nabla_i} F^-_{\mathbb{k}_{k_{ij}}} + \log_{\nabla_i} F^+_{\mathbb{k}_{k_{ij}}})^3,$$
$$\textit{then} \quad \mathbb{k}_i \leq \mathcal{W}_i,$$

Hence, the new-type FNVN satisfies:

$$(\mathbb{k}_1, \mathbb{k}_2, \ldots, \mathbb{k}_n) \leq (\mathcal{W}_1, \mathcal{W}_2, \ldots, \mathcal{W}_n).$$

Proof. For any i and j, $(\log_{\nabla_i} T^-_{\mathbb{k}_{p_{ij}}} + \log_{\nabla_i} T^+_{\mathbb{k}_{p_{ij}}})^{L_1} \leq (\log_{\nabla_i} T^-_{\mathbb{k}_{k_{ij}}} + \log_{\nabla_i} T^+_{\mathbb{k}_{k_{ij}}})^{L_1}$. Therefore, $1 - (\log_{\nabla_i} T^-_{\mathbb{k}_{p_{ij}}})^{L_1} + (1 - (\log_{\nabla_i} T^+_{\mathbb{k}_{p_{ij}}})^{L_1} \geq 1 - (\log_{\nabla_i} T^-_{\mathbb{k}_{k_{ij}}})^{L_1} + 1 - (\log_{\nabla_i} T^+_{\mathbb{k}_{k_{ij}}})^{L_1}$.

Hence,
$$\sum_{i=1}^{n}\left(1 - (\log_{\nabla_i} T^-_{\mathbb{k}_{p_{ij}}})^{L_1}\right)^{\chi_i} + \sum_{i=1}^{n}\left(1 - (\log_{\nabla_i} T^+_{\mathbb{k}_{p_{ij}}})^{L_1}\right)^{\chi_i}$$
$$\geq \sum_{i=1}^{n}\left(1 - (\log_{\nabla_i} T^-_{\mathbb{k}_{k_{ij}}})^{L_1}\right)^{\chi_i} + \sum_{i=1}^{n}\left(1 - (\log_{\nabla_i} T^+_{\mathbb{k}_{k_{ij}}})^{L_1}\right)^{\chi_i}$$

where χ_i denotes the weight of of $\mathbb{k}_i$, for $i = 1, 2, \ldots, n$, and

$$\sqrt[L_1]{1 - \sum_{i=1}^{n}\left(1 - (\log_{\nabla_i} T^-_{\mathbb{k}_{p_{ij}}})^{L_1}\right)^{\chi_i}} + \sqrt[L_1]{1 - \sum_{i=1}^{n}\left(1 - (\log_{\nabla_i} T^+_{\mathbb{k}_{p_{ij}}})^{L_1}\right)^{\chi_i}}$$
$$\leq \sqrt[L_1]{1 - \sum_{i=1}^{n}\left(1 - (\log_{\nabla_i} T^-_{\mathbb{k}_{k_{ij}}})^{L_1}\right)^{\chi_i}} + \sqrt[L_1]{1 - \sum_{i=1}^{n}\left(1 - (\log_{\nabla_i} T^+_{\mathbb{k}_{k_{ij}}})^{L_1}\right)^{\chi_i}}.$$

For any i, $(\log_{\nabla_i} I^-_{\mathbb{k}_{p_{ij}}})^{L_2} + (\log_{\nabla_i} I^+_{\mathbb{k}_{p_{ij}}})^{L_2} \geq (\log_{\nabla_i} I^-_{\mathbb{k}_{k_{ij}}})^{L_2} + (\log_{\nabla_i} I^+_{\mathbb{k}_{k_{ij}}})^{L_2}$. Therefore,

$$1 - (\log_{\nabla_i} I^-_{\mathbb{k}_{p_{ij}}})^{L_2} + (1 - (\log_{\nabla_i} I^+_{\mathbb{k}_{p_{ij}}})^{L_2} \leq 1 - (\log_{\nabla_i} I^-_{\mathbb{k}_{k_{ij}}})^{L_2} + 1 - (\log_{\nabla_i} I^+_{\mathbb{k}_{k_{ij}}})^{L_2}.$$

Hence,
$$\sum_{i=1}^{n}\left(1 - (\log_{\nabla_i} I^-_{\mathbb{k}_{p_{ij}}})^{L_2}\right)^{\chi_i} + \sum_{i=1}^{n}\left(1 - (\log_{\nabla_i} I^+_{\mathbb{k}_{p_{ij}}})^{L_2}\right)^{\chi_i}$$
$$\leq \sum_{i=1}^{n}\left(1 - (\log_{\nabla_i} I^-_{\mathbb{k}_{k_{ij}}})^{L_2}\right)^{\chi_i} + \sum_{i=1}^{n}\left(1 - (\log_{\nabla_i} I^+_{\mathbb{k}_{k_{ij}}})^{L_2}\right)^{\chi_i}$$

implies that

$$\sqrt[L_2]{1 - \sum_{i=1}^{n}\left(1 - (\log_{\nabla_i} I^-_{\mathbb{k}_{p_{ij}}})^{L_2}\right)^{\chi_i}} + \sqrt[L_2]{1 - \sum_{i=1}^{n}\left(1 - (\log_{\nabla_i} I^+_{\mathbb{k}_{p_{ij}}})^{L_2}\right)^{\chi_i}}$$

$$\geq \sqrt[L_2]{1-\sum_{i=1}^{n}\left(1-(\log_{\nabla_i} I^-_{\Bbbk_{k_{ij}}})^{L_2}\right)^{\Bbbk_i}} + \sqrt[L_2]{1-\sum_{i=1}^{n}\left(1-(\log_{\nabla_i} I^+_{\Bbbk_{k_{ij}}})^{L_2}\right)^{\chi_i}}.$$

Hence,

$$1-\sqrt[L_2]{1-\sum_{i=1}^{n}\left(1-(\log_{\nabla_i} I^-_{\Bbbk_{p_{ij}}})^{L_2}\right)^{\chi_i}} + 1-\sqrt[L_2]{1-\sum_{i=1}^{n}\left(1-(\log_{\nabla_i} I^+_{\Bbbk_{p_{ij}}})^{L_2}\right)^{\chi_i}}$$

$$\leq 1-\sqrt[L_2]{1-\sum_{i=1}^{n}\left(1-(\log_{\nabla_i} I^-_{\Bbbk_{k_{ij}}})^{L_2}\right)^{\chi_i}} + 1-\sqrt[L_2]{1-\sum_{i=1}^{n}\left(1-(\log_{\nabla_i} I^+_{\Bbbk_{k_{ij}}})^{L_2}\right)^{\chi_i}}.$$

For any i, $(\log_{\nabla_i} F^-_{\Bbbk_{p_{ij}}})^{L_3} + (\log_{\nabla_i} F^+_{\Bbbk_{p_{ij}}})^{L_3} \leq (\log_{\nabla_i} F^-_{\Bbbk_{k_{ij}}})^{L_3} + (\log_{\nabla_i} F^+_{\Bbbk_{k_{ij}}})^{L_3}$. Therefore,

$$1-\sum_{i=1}^{n}(\log_{\nabla_i} F^-_{\Bbbk_{p_{ij}}})^{L_3} + 1-\sum_{i=1}^{n}(\log_{\nabla_i} F^+_{\Bbbk_{p_{ij}}})^{L_3} \leq 1-\sum_{i=1}^{n}(\log_{\nabla_i} F^-_{\Bbbk_{k_{ij}}})^{L_3} + 1-\sum_{i=1}^{n}(\log_{\nabla_i} F^+_{\Bbbk_{k_{ij}}})^{L_3}.$$

$$\left[\begin{array}{l}\left(\sqrt[L_3]{1-\sum_{i=1}^{n}\left(1-(\log_{\nabla_i} T^-_{\Bbbk_{p_{ij}}})^{L_3}\right)^{\chi_i}} + \sqrt[L_3]{1-\sum_{i=1}^{n}\left(1-(\log_{\nabla_i} T^+_{\Bbbk_{p_{ij}}})^{L_3}\right)^{\chi_i}}\right)^3 \\ +1-\sqrt[L_3]{1-\sum_{i=1}^{n}\left(1-(\log_{\nabla_i} I^-_{\Bbbk_{p_{ij}}})^{L_3}\right)^{\chi_i}} + 1-\left(\sqrt[L_3]{1-\sum_{i=1}^{n}\left(1-(\log_{\nabla_i} I^+_{\Bbbk_{p_{ij}}})^{L_3}\right)^{\chi_i}}\right)^3 \\ +1-\left(\sum_{i=1}^{n}(\log_{\nabla_i} F^-_{\Bbbk_{p_{ij}}})\right)^3 + 1-\left(\sum_{i=1}^{n}(\log_{\nabla_i} F^+_{\Bbbk_{p_{ij}}})\right)^3\end{array}\right]$$

$$\leq \left[\begin{array}{l}\left(\sqrt[L_3]{1-\sum_{i=1}^{n}\left(1-(\log_{\nabla_i} T^-_{\Bbbk_{k_{ij}}})^{L_3}\right)^{\chi_i}} + \sqrt[L_3]{1-\sum_{i=1}^{n}\left(1-(\log_{\nabla_i} T^+_{\Bbbk_{k_{ij}}})^{L_3}\right)^{\chi_i}}\right)^3 \\ +1-\sqrt[L_3]{1-\sum_{i=1}^{n}\left(1-(\log_{\nabla_i} I^-_{\Bbbk_{k_{ij}}})^{L_3}\right)^{\chi_i}} + 1-\left(\sqrt[L_3]{1-\sum_{i=1}^{n}\left(1-(\log_{\nabla_i} I^+_{\Bbbk_{k_{ij}}})^{L_3}\right)^{\chi_i}}\right)^3 \\ +1-\left(\sum_{i=1}^{n}(\log_{\nabla_i} F^-_{\Bbbk_{k_{ij}}})\right)^3 + 1-\left(\sum_{i=1}^{n}(\log_{\nabla_i} F^+_{\Bbbk_{k_{ij}}})\right)^3\end{array}\right]$$

Hence, $(\Bbbk_1, \Bbbk_2, \ldots, \Bbbk_n) \leq (\mathcal{W}_1, \mathcal{W}_2, \ldots, \mathcal{W}_n)$. □

Definition 15. *Let $\Bbbk_1, \Bbbk_2, \ldots, \Bbbk_n$ be FNVSSs. Then, the new-type generalized Fermatean neutrosophic vague weighted average (GFNVWA) operator for $\Bbbk_1, \Bbbk_2, \ldots, \Bbbk_n$ is defined as*

$$(\Bbbk_1, \Bbbk_2, \ldots, \Bbbk_n) = \left(\sum_{i=1}^{n}\chi_i \Bbbk_i^{\alpha}\right)^{1/\alpha},$$

where χ_i denotes the weight of $\Bbbk_i$, for $i = 1, 2, \ldots, n$, and $\alpha > 0$ is a normalized parameter to ensure the aggregated result remains in a valid range between [0,1].

Theorem 2. *For $i = 1, 2, \ldots, n$, let $\Bbbk_i = \left\langle \log[T^-_{\Bbbk_{p_{ij}}}, T^+_{\Bbbk_{p_{ij}}}], \log[I^-_{\Bbbk_{p_{ij}}}, I^+_{\Bbbk_{p_{ij}}}], \log[F^-_{\Bbbk_{p_{ij}}}, F^+_{\Bbbk_{p_{ij}}}] \right\rangle$ be the family of FNVSSs. Then, the new-type GFNVWA operator for the FNVSS is given by:*

$$
(\mathbb{k}_1, \mathbb{k}_2, \ldots, \mathbb{k}_n) = \left[\begin{array}{c}
\left[\left(\sqrt[L_1]{1 - \left(\sum_{i=1}^{n} \left(1 - \left((\log_{\nabla_i} T^{-}_{\mathbb{k}_i})^{L_1} \right)^{L_1} \right)^{\chi_i} \right)} \right)^{1/\alpha}, \right. \\
\left. \left(\sqrt[L_1]{1 - \left(\sum_{i=1}^{n} \left(1 - \left((\log_{\nabla_i} T^{+}_{\mathbb{k}_i})^{L_1} \right)^{L_1} \right)^{\chi_i} \right)} \right)^{1/\alpha} \right], \\
\left[\left(\sqrt[L_2]{1 - \left(\sum_{i=1}^{n} \left(1 - \left((\log_{\nabla_i} I^{-}_{\mathbb{k}_i})^{L_2} \right)^{L_2} \right)^{\chi_i} \right)} \right)^{1/\alpha}, \right. \\
\left. \left(\sqrt[L_2]{1 - \left(\sum_{i=1}^{n} \left(1 - \left((\log_{\nabla_i} I^{+}_{\mathbb{k}_i})^{L_2} \right)^{L_2} \right)^{\chi_i} \right)} \right)^{1/\alpha} \right], \\
\left[\left(\sqrt[L_3]{1 - \left(1 - \left(\sum_{i=1}^{n} \left(\sqrt[L_3]{1 - (\log_{\nabla_i} F^{-}_{\mathbb{k}_i})^{L_3}} \right)^{\chi_i} \right)^{L_3} \right)^{1/\alpha}} \right), \right. \\
\left. \left(\sqrt[L_3]{1 - \left(1 - \left(\sum_{i=1}^{n} \left(\sqrt[L_3]{1 - (\log_{\nabla_i} F^{+}_{\mathbb{k}_i})^{L_3}} \right)^{\chi_i} \right)^{L_3} \right)^{1/\alpha}} \right) \right].
\end{array} \right].
$$

Proof. We have

$$
\sum_{i=1}^{n} \chi_i \mathbb{k}_i^{\alpha} = \left[\begin{array}{c}
\left[\sqrt[L_1]{1 - \sum_{i=1}^{n} \left(1 - \left((\log_{\nabla_i} T^{-}_{\mathbb{k}_i})^{L_1} \right) \right)^{\alpha_i}},\ \sqrt[L_1]{1 - \sum_{i=1}^{n} \left(1 - \left((\log_{\nabla_i} T^{+}_{\mathbb{k}_i})^{L_1} \right) \right)^{\alpha_i}} \right] \\
\left[\sqrt[L_2]{1 - \sum_{i=1}^{n} \left(1 - \left((\log_{\nabla_i} I^{-}_{\mathbb{k}_i})^{L_2} \right) \right)^{\alpha_i}},\ \sqrt[L_2]{1 - \sum_{i=1}^{n} \left(1 - \left((\log_{\nabla_i} I^{+}_{\mathbb{k}_i})^{L_2} \right) \right)^{\alpha_i}} \right] \\
\left[\sum_{i=1}^{n} \left(\sqrt[L_3]{1 - \left(1 - (\log_{\nabla_i} F^{-}_{\mathbb{k}_i})^{L_3} \right)} \right)^{\alpha_i}, \sum_{i=1}^{n} \left(\sqrt[L_3]{1 - \left(1 - (\log_{\nabla_i} F^{+}_{\mathbb{k}_i})^{L_3} \right)} \right)^{\alpha_i} \right]
\end{array} \right].
$$

If $n = 2$, then

$$
\chi_1 \mathbb{k}_1 \oplus \chi_2 \mathbb{k}_2 = \left[\begin{array}{c}
\left[\sqrt[L_1]{\left(\sqrt[L_1]{1 - \left(1 - \left((\log_{\nabla_i} T^{-}_{\mathbb{k}_1})^{L_1} \right) \right)^{\chi_1}} + \sqrt[L_1]{1 - \left(1 - \left((\log_{\nabla_i} T^{-}_{\mathbb{k}_2})^{L_1} \right) \right)^{\chi_1}} \right)^{L_1} \right. \\
\left. - \left(\sqrt[L_1]{1 - \left(1 - \left((\log_{\nabla_i} T^{-}_{\mathbb{k}_1})^{L_1} \right) \right)^{\chi_1}} \cdot \sqrt[L_1]{1 - \left(1 - \left((\log_{\nabla_i} T^{-}_{\mathbb{k}_2})^{L_1} \right) \right)^{\chi_1}} \right)^{L_1}}, \right. \\
\sqrt[L_1]{\left(\sqrt[L_1]{1 - \left(1 - \left((\log_{\nabla_i} T^{+}_{\mathbb{k}_1})^{L_1} \right) \right)^{\chi_1}} + \sqrt[L_1]{1 - \left(1 - \left((\log_{\nabla_i} T^{+}_{\mathbb{k}_2})^{L_1} \right) \right)^{\chi_1}} \right)^{L_1} \right. \\
\left. \left. - \left(\sqrt[L_1]{1 - \left(1 - \left((\log_{\nabla_i} T^{+}_{\mathbb{k}_1})^{L_1} \right) \right)^{\chi_1}} \cdot \sqrt[L_1]{1 - \left(1 - \left((\log_{\nabla_i} T^{+}_{\mathbb{k}_2})^{L_1} \right) \right)^{\chi_1}} \right)^{L_1}} \right] \\
\left[\sqrt[L_2]{\left(\sqrt[L_2]{1 - \left(1 - \left((\log_{\nabla_i} I^{-}_{\mathbb{k}_1})^{L_2} \right) \right)^{\chi_1}} + \sqrt[L_2]{1 - \left(1 - \left((\log_{\nabla_i} I^{-}_{\mathbb{k}_2})^{L_2} \right) \right)^{\chi_1}} \right)^{L_2} \right. \\
\left. - \left(\sqrt[L_2]{1 - \left(1 - \left((\log_{\nabla_i} I^{-}_{\mathbb{k}_1})^{L_2} \right) \right)^{\chi_1}} \cdot \sqrt[L_2]{1 - \left(1 - \left((\log_{\nabla_i} I^{-}_{\mathbb{k}_2})^{L_2} \right) \right)^{\chi_1}} \right)^{L_2}}, \right. \\
\sqrt[L_2]{\left(\sqrt[L_2]{1 - \left(1 - \left((\log_{\nabla_i} I^{+}_{\mathbb{k}_1})^{L_2} \right) \right)^{\chi_1}} + \sqrt[L_2]{1 - \left(1 - \left((\log_{\nabla_i} I^{+}_{\mathbb{k}_2})^{L_2} \right) \right)^{\chi_1}} \right)^{L_2} \right. \\
\left. \left. - \left(\sqrt[L_2]{1 - \left(1 - \left((\log_{\nabla_i} I^{+}_{\mathbb{k}_1})^{L_2} \right) \right)^{\chi_1}} \cdot \sqrt[L_2]{1 - \left(1 - \left((\log_{\nabla_i} I^{+}_{\mathbb{k}_2})^{L_2} \right) \right)^{\chi_1}} \right)^{L_2}} \right] \\
\left[\left(\sqrt[L_3]{1 - \left(1 - (\log_{\nabla_i} F^{-}_{\mathbb{k}_1})^{L_3} \right)^{L_3}} \cdot \sqrt[L_3]{1 - \left(1 - (\log_{\nabla_i} F^{+}_{\mathbb{k}_2})^{L_3} \right)^{L_3}} \right)^{\chi_1}, \right. \\
\left(\sqrt[L_3]{1 - \left(1 - (\log_{\nabla_i} F^{-}_{\mathbb{k}_1})^{L_3} \right)^{L_3}} \cdot \sqrt[L_3]{1 - \left(1 - (\log_{\nabla_i} F^{+}_{\mathbb{k}_2})^{L_3} \right)^{L_3}} \right)^{\chi_1}
\end{array} \right]
$$

$$= \left[\begin{array}{c} \left[\sqrt[L_1]{1-\sum_{i=1}^{l_1}\left(1-\left(\left(\log_{\nabla_i} T_{\mathbb{k}_1}^{-}\right)^{L_1}\right)\right)^{\chi_i}}, \sqrt[L_1]{1-\sum_{i=1}^{l_1}\left(1-\left(\left(\log_{\nabla_i} T_{\mathbb{k}_1}^{+}\right)^{L_1}\right)\right)^{\chi_i}}\right], \\ \left[\sqrt[L_2]{1-\sum_{i=1}^{l_2}\left(1-\left(\left(\log_{\nabla_i} I_{\mathbb{k}_1}^{-}\right)^{L_2}\right)\right)^{\chi_i}}, \sqrt[L_2]{1-\sum_{i=1}^{l_2}\left(1-\left(\left(\log_{\nabla_i} I_{\mathbb{k}_1}^{+}\right)^{L_2}\right)\right)^{\chi_i}}\right], \\ \left[\sum_{i=1}^{L_3}\left(\sqrt[L_3]{1-\left(1-\left(\log_{\nabla_i} F_{\mathbb{k}_i}^{-}\right)^{L_3}\right)^{L_3}}\right)^{\chi_i}, \sum_{i=1}^{L_3}\left(\sqrt[L_3]{1-\left(1-\left(\log_{\nabla_i} F_{\mathbb{k}_i}^{+}\right)^{L_3}\right)^{L_3}},\right)^{\chi_i}\right] \end{array}\right].$$

It is valid for $n = L$ and $L \geq 3$. Hence,

$$\sum_{i=1}^{L} \chi_i \mathbb{k}_i^{\alpha}$$

$$= \left[\begin{array}{c} \left[\sqrt[L_1]{1-\sum_{i=1}^{L}\left(1-\left(\left(\log_{\nabla_i} T_{\mathbb{k}_1}^{-}\right)^{L_1}\right)^{L_1}\right)^{\chi_i}}, \sqrt[L_1]{1-\sum_{i=1}^{L}\left(1-\left(\left(\log_{\nabla_i} T_{\mathbb{k}_1}^{+}\right)^{L_1}\right)^{L_1}\right)^{\chi_i}}\right] \\ \left[\sqrt[L_2]{1-\sum_{i=1}^{L}\left(1-\left(\left(\log_{\nabla_i} I_{\mathbb{k}_1}^{-}\right)^{L_2}\right)^{L_2}\right)^{\chi_i}}, \sqrt[L_2]{1-\sum_{i=1}^{L}\left(1-\left(\left(\log_{\nabla_i} I_{\mathbb{k}_1}^{+}\right)^{L_2}\right)^{L_2}\right)^{\chi_i}}\right] \\ \left[\sum_{i=1}^{L}\left(\sqrt[L_3]{1-\left(1-\left(\log_{\nabla_i} F_{\mathbb{k}_i}^{-}\right)^{L_3}\right)^{L_3}}\right)^{\chi_i}, \sum_{i=1}^{L}\left(\sqrt[L_3]{1-\left(1-\left(\log_{\nabla_i} F_{\mathbb{k}_i}^{+}\right)^{L_3}\right)^{L_3}}\right)^{\chi_i}\right] \end{array}\right].$$

If $n = L + 1$, then

$$\sum_{i=1}^{L} \chi_i \mathbb{k}_i^{\alpha} + \chi_{L+1} \mathbb{k}_{L+1}^{\alpha} = \sum_{i=1}^{L+1} \chi_i \mathbb{k}_i^{\alpha}.$$

Now,

$$\sum_{i=1}^{L} \chi_i \mathbb{k}_i^{\alpha} + \chi_{L+1} \mathbb{k}_{L+1}^{\alpha} = \chi_1 \mathbb{k}_1^{\alpha} \oplus \chi_2 \mathbb{k}_2^{\alpha} \oplus \cdots \oplus \chi_L \mathbb{k}_L^{\alpha} \oplus \chi_{L+1} \mathbb{k}_{L+1}^{\alpha}$$

$$= \begin{bmatrix} \left[\sqrt[L_1]{\left(\sqrt[L_1]{1 - \sum_{i=1}^{L} \left(1 - \left((\log_{\nabla_i} T^-_{\Bbbk_i})^{L_1}\right)^{L_1}\right)^{\chi_i}} + \sqrt[L_1]{1 - \left(1 - \left((\log_{\nabla_i} T^-_{\Bbbk_{L+1}})^{L_1}\right)^{L_1}\right)^{\chi_1}} \right)^{L_1} - \left(\sqrt[L_1]{1 - \sum_{i=1}^{l} \left(1 - \left((\log_{\nabla_i} T^-_{\Bbbk_i})^{L_1}\right)^{L_1}\right)^{\chi_i}} \cdot \sqrt[L_1]{1 - \left(1 - \left((\log_{\nabla_i} T^-_{\Bbbk_{L+1}})^{L_1}\right)^{L_1}\right)^{\chi_1}} \right)^{L_1}}, \right. \\ \left. \sqrt[L_1]{\left(\sqrt[L_1]{1 - \sum_{i=1}^{L} \left(1 - \left((\log_{\nabla_i} T^+_{\Bbbk_i})^{L_1}\right)^{L_1}\right)^{\chi_i}} + \sqrt[L_1]{1 - \left(1 - \left((\log_{\nabla_i} T^+_{\Bbbk_{L+1}})^{L_1}\right)^{L_1}\right)^{\chi_1}} \right)^{L_1} - \left(\sqrt[L_1]{1 - \sum_{i=1}^{l} \left(1 - \left((\log_{\nabla_i} T^+_{\Bbbk_i})^{L_1}\right)^{L_1}\right)^{\chi_i}} \cdot \sqrt[L_1]{1 - \left(1 - \left((\log_{\nabla_i} T^+_{\Bbbk_{L+1}})^{L_1}\right)^{L_1}\right)^{\chi_1}} \right)^{L_1}} \right], \\ \left[\sqrt[L_2]{\left(\sqrt[L_2]{1 - \sum_{i=1}^{L} \left(1 - \left(\log_{\nabla_i} I^-_{\Bbbk_i})^{L_2}\right)^{L_2}\right)^{\chi_i}} + \sqrt[L_2]{1 - \left(1 - \left((\log_{\nabla_i} I^-_{\Bbbk_{L+1}})^{L_2}\right)^{L_2}\right)^{\chi_1}} \right)^{L_2} - \left(\sqrt[L_2]{1 - \sum_{i=1}^{L} \left(1 - \left((\log_{\nabla_i} I^-_{\Bbbk_i})^{L_2}\right)^{L_2}\right)^{\chi_i}} \cdot \sqrt[L_2]{1 - \left(1 - \left((\log_{\nabla_i} I^-_{\Bbbk_{L+1}})^{L_2}\right)^{L_2}\right)^{\chi_1}} \right)^{L_2}}, \right. \\ \left. \sqrt[L_2]{\left(\sqrt[L_2]{1 - \sum_{i=1}^{L} \left(1 - \left(\log_{\nabla_i} I^+_{\Bbbk_i})^{L_2}\right)^{L_2}\right)^{\chi_i}} + \sqrt[L_2]{1 - \left(1 - \left((\log_{\nabla_i} I^+_{\Bbbk_{L+1}})^{L_2}\right)^{L_2}\right)^{\chi_1}} \right)^{L_2} - \left(\sqrt[L_2]{1 - \sum_{i=1}^{L} \left(1 - \left((\log_{\nabla_i} I^+_{\Bbbk_i})^{L_2}\right)^{L_2}\right)^{\chi_i}} \cdot \sqrt[L_2]{1 - \left(1 - \left((\log_{\nabla_i} I^+_{\Bbbk_{L+1}})^{L_2}\right)^{L_2}\right)^{\chi_1}} \right)^{L_2}} \right] \\ \left[\sum_{i=1}^{L} \left(\sqrt[L_3]{1 - \left(1 - (\log_{\nabla_i} F^-_{\Bbbk_i})^{L_3}\right)^{L_3}} \right)^{\chi_i} \cdot \left(\sqrt[L_3]{1 - \left(1 - (\log_{\nabla_i} F^-_{\Bbbk_{L+1}})^{L_3}\right)^{L_3}} \right)^{\chi_1}, \right. \\ \left. \sum_{i=1}^{L} \left(\sqrt[L_3]{1 - \left(1 - (\log_{\nabla_i} F^+_{\Bbbk_i})^{L_3}\right)^{L_3}} \right)^{\chi_i} \cdot \left(\sqrt[L_3]{1 - \left(1 - (\log_{\nabla_i} F^+_{\Bbbk_{L+1}})^{L_3}\right)^{L_3}} \right)^{\chi_1} \right] \end{bmatrix}.$$

Thus,

$$\sum_{i=1}^{L+1} \chi_i \Bbbk_i^{\alpha} = \begin{bmatrix} \left[\sqrt[L_1]{1 - \sum_{i=1}^{L+1} \left(1 - \left((\log_{\nabla_i} T^-_{\Bbbk_1})^{L_1}\right)^{L_1}\right)^{\chi_i}}, \sqrt[L_1]{1 - \sum_{i=1}^{L+1} \left(1 - \left((\log_{\nabla_i} T^+_{\Bbbk_1})^{L_1}\right)^{L_1}\right)^{\chi_i}} \right] \\ \left[\sqrt[L_2]{1 - \sum_{i=1}^{L+1} \left(1 - \left((\log_{\nabla_i} I^-_{\Bbbk_1})^{L_2}\right)^{l_2}\right)^{\chi_i}}, \sqrt[L_2]{1 - \sum_{i=1}^{L+1} \left(1 - \left((\log_{\nabla_i} I^+_{\Bbbk_1})^{L_2}\right)^{l_2}\right)^{\chi_i}} \right] \\ \left[\sum_{i=1}^{L+1} \left(\sqrt[L_3]{1 - \left(1 - (\log_{\nabla_i} F^-_{\Bbbk_i})^{L_3}\right)^{L_3}} \right)^{\chi_i}, \sum_{i=1}^{L+1} \left(\sqrt[L_3]{1 - \left(1 - (\log_{\nabla_i} F^+_{\Bbbk_i})^{L_3}\right)^{L_3}} \right)^{\chi_i} \right] \end{bmatrix}.$$

Hence,

$$\left(\sum_{i=1}^{L+1}\chi_i\Bbbk_i^{\alpha}\right)^{1/\alpha}=\begin{bmatrix}\left[\left(\sqrt[L_1]{\left(1-\sum_{i=1}^{l+1}\left(1-\left(\left(\log_{\nabla_i}T^{-}_{\Bbbk_1}\right)^{L_1}\right)^{L_1}\right)^{\chi_i}\right)}\right)^{\frac{1}{\alpha}},\right.\\ \left.\left(\sqrt[L_1]{\left(1-\sum_{i=1}^{l+1}\left(1-\left(\left(\log_{\nabla_i}T^{+}_{\Bbbk_1}\right)^{L_1}\right)^{L_1}\right)^{\chi_i}\right)}\right)^{\frac{1}{\alpha}}\right]\\ \left[\left(\sqrt[L_2]{\left(1-\sum_{i=1}^{L+1}\left(1-\left(\left(\log_{\nabla_i}I^{-}_{\Bbbk_1}\right)^{L_2}\right)^{L_2}\right)^{\chi_i}\right)}\right)^{\frac{1}{\alpha}},\right.\\ \left.\left(\sqrt[L_2]{\left(1-\sum_{i=1}^{L+1}\left(1-\left(\left(\log_{\nabla_i}I^{+}_{\Bbbk_1}\right)^{L_2}\right)^{L_2}\right)^{\chi_i}\right)}\right)^{\frac{1}{\alpha}}\right]\\ \left[\sqrt[L_3]{1-\left(1-\left(\sum_{i=1}^{L+1}\left(\sqrt[l_3]{1-\left(1-\left(\log_{\nabla_i}F^{-}_{\Bbbk_i}\right)^{L_3}\right)^{L_3}}\right)^{\chi_i}\right)^{L_3}\right)}^{\frac{1}{\alpha}},\right.\\ \left.\sqrt[L_3]{1-\left(1-\left(\sum_{i=1}^{L+1}\left(\sqrt[l_3]{1-\left(1-\left(\log_{\nabla_i}F^{+}_{\Bbbk_i}\right)^{L_3}\right)^{L_3}}\right)^{\chi_i}\right)^{L_3}\right)}^{\frac{1}{\alpha}}\right]\end{bmatrix}.$$

It is valid for $L \geq 1$. $\square$

5. Fermatean Neutrosophic Vague TOPSIS to Solve Decision-Making Problems

We use FNVSS theory in conjunction with multicriteria decision-making (MCDM) approaches to address this challenge by modeling uncertainty, imprecision, and reluctance in expert assessments (Figure 1). The TOPSIS method is applied to identify the most suitable solar panel.

5.1. Mathematical Formulation of FNVSS-TOPSIS Method

Step 1: Construct the decision matrix as follows:

$$DM=\begin{bmatrix}\langle T_{11},I_{11},1-F_{11}\rangle & \langle T_{12},I_{12},1-F_{12}\rangle & \cdots & \langle T_{1q},I_{1q},1-F_{1q}\rangle\\ \langle T_{21},I_{21},1-F_{21}\rangle & \langle T_{22},I_{22},1-F_{22}\rangle & \cdots & \langle T_{2q},I_{2q},1-F_{2q}\rangle\\ \vdots & \vdots & \ddots & \vdots\\ \langle T_{p1},I_{p1},1-F_{p1}\rangle & \langle T_{p2},I_{p2},1-F_{p2}\rangle & \cdots & \langle T_{pq},I_{pq},1-F_{pq}\rangle\end{bmatrix}$$

Step 2: Normalize the decision matrix by normalizing each Fermatean vague value $FNVS_{ij}=\langle T_{ij},I_{ij},1-F_{ij}\rangle$ as:

$$T'_{ij}=\frac{T^{+}_{ij}}{\max(T^{+}_{ij})},\quad I'_{ij}=\frac{I^{+}_{ij}}{\max(I^{+}_{ij})},\quad 1-F'_{ij}=\frac{1-F^{+}_{ij}}{\max(1-F^{+}_{ij})},$$

$$T''_{ij}=\frac{T^{-}_{ij}}{\max(T^{-}_{ij})},\quad I''_{ij}=\frac{I^{-}_{ij}}{\max(I^{-}_{ij})},\quad 1-F''_{ij}=\frac{1-F^{-}_{ij}}{\max(1-F^{-}_{ij})}.$$

where the following are true:

i. $\max(T^{+}_{j})$ is the highest upper truth value in the row j;

ii. $\max(I^{+}_{j})$ is the highest indeterminacy value in the row j;

iii. $\max(1-F^{+}_{j})$ is the highest falsity value in the row$j$.

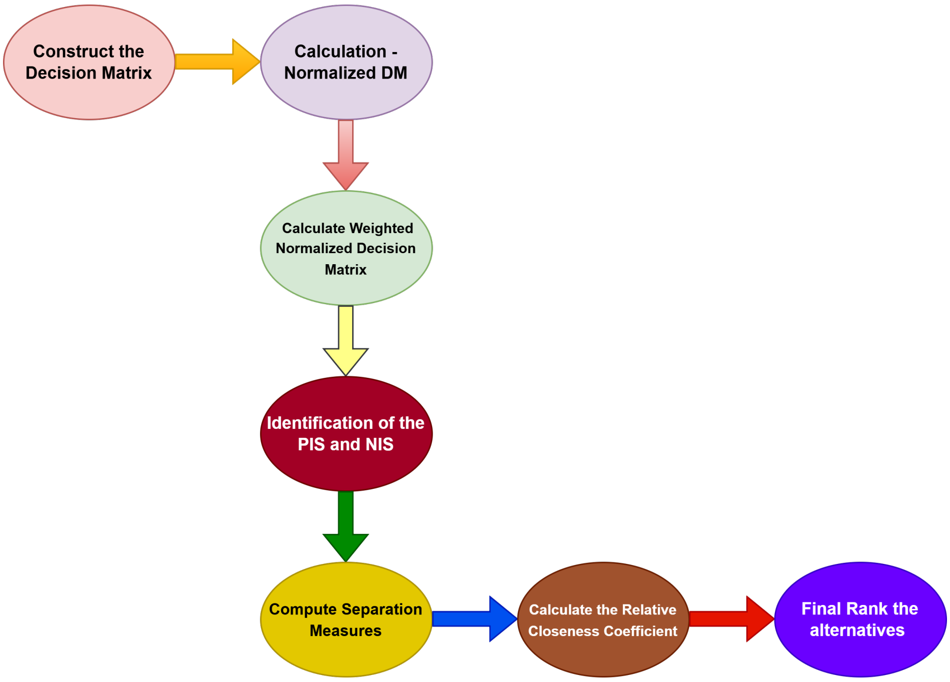

Figure 1. Graphical model of FNVSS-TOPSIS method.

Step 3: Calculate weighted normalized decision matrix by:

$$T_{ij}^{\oplus} = w_j \cdot (T'_{ij}), \quad I_{ij}^{\oplus} = w_j \cdot (I'_{ij}), \quad 1 - F_{ij}^{\oplus} = w_j \cdot (1 - F'_{ij})$$

$$T_{ij}^{\ominus} = w_j \cdot (T''_{ij}), \quad I_{ij}^{\ominus} = w_j \cdot (I''_{ij}), \quad 1 - F_{ij}^{\ominus} = w_j \cdot (1 - F''_{ij})$$

Step 4: Identification of the PIS and NIS by:

- Positive Ideal Solution (PIS):

$$P^{+} = \langle \max(T_{ij}^{\oplus}), \min(I_{ij}^{\oplus}), \min(1 - F_{ij}^{\oplus}) \rangle$$

- Negative Ideal Solution (NIS):

$$N^{-} = \langle \max(T_{ij}^{\ominus}), \min(I_{ij}^{\ominus}), \min(1 - F_{ij}^{\ominus}) \rangle$$

Step 5: Compute separation measures by:

$$D_i^{+} = \sqrt{\sum_{j=1}^{n} \left(T_{ij}^{\oplus} - T'_j)^2 + (I_{ij}^{\oplus} - I'_{ij})^2 + ([1 - F_{ij}^{\oplus}] - [1 - F'_j])^2 \right)}$$

$$D_i^{-} = \sqrt{\sum_{j=1}^{n} \left(T_{ij}^{\ominus} - T''_j)^2 + (I_{ij}^{\ominus} - I''_{ij})^2 + ([1 - F_{ij}^{\ominus}] - [1 - F''_j])^2 \right)}$$

Step 6: Calculate the relative closeness coefficient by:

$$C_i = \frac{D_i^{-}}{D_i^{+} + D_i^{-}}$$

Step 7: Rank the alternatives based on C_i (higher values are preferred).

5.2. Application of TOPSIS Method Based on FNVSS

A renewable energy firm intends to implement a large-scale solar power system to optimize energy efficiency while reducing prices and environmental effect. Four alternatives of solar panels—which are x_1: Monocrystalline, x_2: Polycrystalline, x_3: Thin-Film, and x_4: PERC—are assessed for these criteria: c_1: efficiency, c_2: installation expenses, c_3: durability, and c_4: ecological effect. Nevertheless, professional judgments entail uncertainty, rendering accurate assessments challenging. Monocrystalline panels provide excellent efficiency at a greater cost, whereas Thin-Film panels are more economical but exhibit less durability. This trade-off requires a decision-making strategy that considers both quantitative accuracy and expert opinion.

Solution By TOPSIS

Step 1: Construct the Decision Matrix (Table 1)

Table 1. Decision matrix D = $[x_{ij}]_{m\times n}$.

Solar Panels	(Ef)	(C)	(D)	(EI)
Mono-C	[0.3,0.4], [0.6,0.5] [0.7,0.9]	[0.7,0.4], [0.1,0.5] [0.8,0.9]	[0.2,0.6], [0.4,0.4] [0.8,0.8]	[0.5,0.2], [0.3,0.5] [0.6,0.3]
Poly-C	[0.2,0.6], [0.4,0.4] [0.6,0.3]	[0.5,0.2], [0.3,0.5] [0.8,0.8]	[0.6,0.5], [0.3,0.4] [0.6,0.3]	[0.2,0.2], [0.1,0.6] [0.6,0.8]
Thin-Film	[0.5,0.8], [0.3,0.3] [0.7,0.9]	[0.6,0.8], [0.2,0.4] [0.7,0.8]	[0.3,0.4], [0.6,0.5] [0.4,0.2]	[0.3,0.1], [0.4,0.5] [0.7,0.9]
PERC	[0.2,0.6], [0.4,0.4] [0.8,0.8]	[0.5,0.2], [0.3,0.5] [0.8,0.8]	[0.2,0.6], [0.4,0.4] [0.6,0.3]	[0.2,0.6], [0.4,0.4] [0.8,0.8]

Step 2: Normalize the Decision Matrix (Table 2)

Table 2. Calculating $[T'_{ij}]$, $[T''_{ij}]$, $[I'_{ij}]$, $[I''_{ij}]$, $[1-F'_{ij}]$, $[1-F''_{ij}]$.

Solar Panels	(Ef)	(C)	(D)	(EI)
Mono-C	[0.428,0.666], [1.0,1.0] [0.875,1.0]	[1.0,0.666], [0.166,1.0] [1.0,1.0]	[0.285,0.666], [0.666,0.8] [1.0,0.888]	[0.714,0.333], [0.5,1.0] [0.75,0.333]
Poly-C	[0.333,1.0], [1.0,0.666] [0.75,0.375]	[0.833,0.333], [0.75,0.833] [1.0,1.0]	[1.0,0.833], [0.75,0.666] [0.75,0.375]	[0.333,0.333], [0.25,1.0] [0.75,1.0]
Thin-Film	[0.833,1.0], [0.5,0.6] [1.0,1.0]	[1.0,1.0], [0.5,0.8] [1.0,0.8888]	[0.5,0.5], [1.0,1.0] [0.571,0.222]	[0.5,0.125], [0.666,1.0] [1.0,1.0]
PERC	[0.4,1.0],[1.0,0.8] [1.0,1.0]	[0.4,0.333],[0.75,1.0] [1.0,1.0]	[0.4,1.0],[1.0,0.8] [0.75,0.375]	[0.4,1.0],[1.0,0.8] [1.0,1.0]

To normalize the decision matrix, divide each entry, for example:

$$T'_{ij} = \frac{T^+_{ij}}{\max(T^+_{ij})}, \quad I'_{ij} = \frac{I^+_{ij}}{\max(I^+_{ij})}, \quad 1-F'_{ij} = \frac{1-F^+_{ij}}{\max(1-F^+_{ij})},$$

$$T'_{11} = \frac{0.3}{\max(0.3,0.7,0.2,0.5)} = \frac{0.3}{0.7} = 0.428, \quad I'_{11} = \frac{0.6}{\max(0.6,0.1,0.4,0.3)} = \frac{0.6}{0.6} = 1.0,$$

$$1-F'_{11} = \frac{0.7}{\max(0.7,0.8,0.8,0.6)} = \frac{0.7}{0.8} = 0.875,$$

$$T''_{ij} = \frac{T^-_{ij}}{\max(T^-_{ij})}, \quad I''_{ij} = \frac{I^-_{ij}}{\max(I^-_{ij})}, \quad 1-F''_{ij} = \frac{1-F^-_{ij}}{\max(1-F^-_{ij})}.$$

$$T''_{11} = \frac{0.4}{\max(0.4,0.4,0.6,0.2)} = \frac{0.4}{0.6} = 0.666, \quad I''_{11} = \frac{0.5}{\max(0.5,0.5,0.4,0.5)} = \frac{0.5}{0.5} = 1.0,$$

$$1-F''_{11} = \frac{0.9}{\max(0.9,0.9,0.8,0.3)} = \frac{0.9}{0.9} = 1.0.$$

Step 3: Computation of the Weight Matrix (Table 3)

The weights assigned by the experts (decision makers) to the criteria are given by the matrix:

$$W = \left[w_1\text{ (Ef)} = 0.3, \quad w_2\text{ (C)} = 0.25, \quad w_3\text{ (D)} = 0.2, \quad w_4\text{ (EI)} = 0.25\right]$$

$$T_{ij}^{\oplus} = w_j \cdot (T'_{ij}), \quad I_{ij}^{\oplus} = w_j \cdot (I'_{ij}), \quad 1 - F_{ij}^{\oplus} = w_j \cdot (1 - F'_{ij})$$

$$T_{ij}^{\ominus} = w_j \cdot (T''_{ij}), \quad I_{ij}^{\ominus} = w_j \cdot (I''_{ij}), \quad 1 - F_{ij}^{\ominus} = w_j \cdot (1 - F''_{ij})$$

Table 3. Weighted normalized decision matrix.

Weights w_j	**0.3**	**0.25**	**0.2**	**0.25**
	Ef	**C**	**D**	**EI**
Mono-C	[0.128,0.199], [0.3,0.3] [0.262,0.3]	[0.25,0.041], [0.041,0.25] [0.25,0.25]	[0.057,0.133], [0.133,0.16] [0.2,0.177]	[0.178,0.083], [0.15,0.3] [0.187,0.833]
Poly-C	[0.099,0.3], [0.3,0.199] [0.225,0.112]	[0.208,0.083], [0.187,0.208] [0.25,0.25]	[0.2,0.166], [0.15,0.133] [0.15,0.075]	[0.083,0.066], [0.05,0.2] [0.187,0.25]
Thin-Film	[0.249,0.3], [0.15,0.18] [0.3,0.3]	[0.25,0.25], [0.125,0.2] [0.25,0.222]	[0.1,0.1], [0.2,0.2] [0.114,0.044]	[0.125,0.031], [0.166,0.25] [0.25,0.25]
PERC	[0.12,0.3], [0.3,0.24] [0.3,0.3]	[0.1,0.083], [0.187,0.25] [0.25,0.25]	[0.08,0.2], [0.2,0.16] [0.15,0.075]	[0.1,0.25], [0.25,0.2] [0.25,0.25]

Step 4: Identification of PIS and NIS

To find the PIS P^+,

$$P^+ = \langle \max(T_{ij}^{\oplus}), \min(I_{ij}^{\oplus}), \min(1 - F_{ij}^{\oplus}) \rangle$$

$$P^+ = \begin{bmatrix} ([0.25, 0.199], [0.041, 0.16], [0.187, 0.075]), \\ ([0.208, 0.3], [0.05, 0.133], [0.187, 0.075]), \\ ([0.249, 0.3], [0.125, 0.18], [0.114, 0.044]), \\ ([0.12, 0.3], [0.187, 0.16], [0.15, 0.075]) \end{bmatrix}$$

To find the NIS N^-,

$$N^- = \langle \min(T_{ij}^{\ominus}), \max(I_{ij}^{\ominus}), \max(1 - F_{ij}^{\ominus}) \rangle$$

$$N^- = \begin{bmatrix} ([0.057, 0.041], [0.3, 0.3], [0.262, 0.833]), \\ ([0.083, 0.066], [0.3, 0.2], [0.25, 0.25]), \\ ([0.1, 0.031], [0.2, 0.25], [0.3, 0.3]), \\ ([0.08, 0.083], [0.3, 0.25], [0.3, 0.3]) \end{bmatrix}$$

Step 5: Compute Separation Measures (Table 4)

$$D_i^+ = \sqrt{\sum_{j=1}^{n} \left(T_{ij}^{\oplus} - T'_j)^2 + (I_{ij}^{\oplus} - I'_{ij})^2 + ([1 - F_{ij}^{\oplus}] - [1 - F'_j])^2 \right)}$$

$$D_i^- = \sqrt{\sum_{j=1}^{n} \left(T_{ij}^{\ominus} - T''_j)^2 + (I_{ij}^{\ominus} - I''_{ij})^2 + ([1 - F_{ij}^{\ominus}] - [1 - F''_j])^2 \right)}$$

Table 4. Calculation of D_i^+, D_i^-.

	D_i^+	D_i^-
Mono-C	(0.3885)	(0.5921)
Poly-C	(0.4102)	(0.6109)
Thin-Film	(0.4571)	(0.6774)
PERC	(0.4352)	(0.6422)

Step 6: Relative Closeness to Ideal Solution

The RCC to the ideal solution C_i is computed as follows:

$$C_{Mono} = \frac{D_1^-}{D_1^+ + D_1^-} = \frac{0.5921}{0.5921 + 0.3885} = 0.6038$$

Similarly, we can obtain

$$C_{Poly} = 0.5983,$$

$$C_{Thin} = 0.5971,$$

$$C_{PERC} = 0.5965.$$

Step 7: Ranking Closeness to Ideal Solution

The final ranking indicates that Mono-C emerged as the preferred choice due to its strong overall performance across multiple evaluation criteria, such as efficiency, durability, and cost-effectiveness. This suggests that decision-makers prioritized a balance between high energy output and long-term reliability, even if certain alternatives might have had advantages in specific isolated factors. The results reflect a trade-off where slightly higher costs or installation complexity were considered acceptable in exchange for superior long-term benefits and consistent performance.

5.3. Comparative Analysis

We conduct a comparison between the AHP and VIKOR methods. The results are shown in Table 5. From the above analysis, it is evident that the three methods present consistent results in identifying the best alternative, although some variations exist in the detailed ranking orders. Specifically, both the FNVSS-based TOPSIS method and the VIKOR method select C_{Mono} as the best alternative, while the AHP method identifies C_{Thin} as the top choice. The proposed FNVSS-based TOPSIS approach effectively addresses the MAGDM problem under fuzzy environments. Additionally, compared to traditional AHP and VIKOR methods, our method demonstrates higher robustness by consistently ranking C_{Mono} at the top, thereby providing more reliable decision support in the selection of alternatives.

Table 5. Comparison of ranking results based on different methods.

Methods	Ranking Orders	Best Alternative
Proposed Method (FNVSS-TOPSIS)	$C_{Mono} \succ C_{Poly} \succ C_{Thin} \succ C_{PERC}$	C_{Mono}
AHP	$C_{Thin} \succ C_{Poly} \succ C_{PERC} \succ C_{PERC}$	C_{Thin}
VIKOR	$C_{Mono} \succ C_{Poly} \succ C_{PERC} \succ C_{Thin}$	C_{Mono}

6. Analysis and Comparisons of FNVSS with Traditional Models

FNVSSs further refine the representation of uncertainty and vagueness by integrating the triple-valued logic of neutrosophic theory with the augmented constraints of Fermatean logic, governed by the inequality $T^p + I^p + F^p \leq 1$, where $p > 1$, typically $p = 3$. This generalized situation permits the concurrent existence of elevated membership and non-membership degrees, facilitating a more accommodating and articulate framework for expert assessment and divergent viewpoints. Furthermore, the vague characteristics of truth, indeterminacy, and varying degrees of untruth within FNVSSs provide interval-based evaluations, effectively encapsulating indeterminacy, ambiguity, and contradiction with greater precision than traditional models.

Consequently, FNVSS provides enhanced modeling capabilities for practical decision-making contexts, particularly in areas such as intelligent medical diagnosis, trust assessment in blockchain systems, environmental evaluation, and renewable energy prioritization, where information frequently lacks clarity, precision, or completeness. As shown in Table 6, this comparison emphasizes the advantages of FNVSS in relation to traditional and contemporary fuzzy frameworks, focusing on essential semantic, structural, and application-oriented criteria.

Table 6. Comparison of FS, IFS, PFS, NSS, NVSS, and Fermatean neutrosophic vague soft set models.

Parameter	**FS**	**IFS**	**PFS**	**NSS**	**NVSS**	**FNVSS**
Membership Model	μ	(μ, ν)	(μ, ν)	(T, I, F)	Vague (T, I, F)	Vague (T, I, F)
Membership Constraint	$\mu \in [0,1]$	$\mu + \nu \leq 1$	$\mu^2 + \nu^2 \leq 1$	None	None	$T^p + I^p + F^p \leq 1$
Hesitation Degree	Not Defined	$1 - \mu - \nu$	$1 - \mu^2 - \nu^2$	Explicit I	Vague I	Enhanced Vague I
Handles Indeterminacy	No	Indirect	Indirect	Yes	Yes	Yes
Handles Contradiction	No	No	Partial	Yes	Yes	Yes
Vagueness Support	No	Limited	Limited	Partial	Strong	Very strong
Degree of Freedom	Low	Moderate	High	High	High	Very High
Soft Set Structure	No	Moderate	Moderate	Full	Full	Full
Uncertainty Flexibility	Low	Moderate	High	High	Very High	Extreme
MCDM Suitability	Weak	Moderate	High	High	Very High	Excellent
Linguistic Input Support	No	Limited	Limited	Moderate	Strong	Very Strong
Model Complexity	Low	Medium	High	Medium	High	High
TOPSIS Compatibility	Weak	Moderate	Strong	Strong	Very Strong	Excellent
Application Scope	Limited	Moderate	Strong	Strong	Broad	Very Broad
Expert Hesitation Modeling	No	Partial	Partial	Yes	Yes	Full

7. Conclusions

This study applied the TOPSIS method integrated with FNVSSs to evaluate solar panel technologies based on efficiency, cost, durability, and environmental impact. The proposed model provided a structured and robust decision-making framework, effectively managing uncertainty and prioritizing the most suitable solar panels for sustainable energy applications.

The results highlight the significant advantage of using FNVSS in multicriteria analysis, ensuring more precise and reliable rankings by accounting for both vagueness and indeterminacy in expert evaluations. This approach enhanced the selection process, optimizing performance, affordability, and sustainability in solar energy deployment.

In general, this research marks an important step towards more efficient, intelligent, and environmentally responsible energy systems while opening new pathways to apply FNVSS models in diverse and impactful domains.

Future Research

Future research drawn from [40–48], studying the evolution of HIV transmission, can boost simulations of fluctuating uncertainty by implementing dynamic system theory into Fermatean vague set structures. Determining bifurcation problems and transitioning network characteristics can give new perspectives on resilience and decision processes in the context of indeterminacy. The mathematical description of decision premises may be further improved by integrating notes of symmetry implications from Z_2-equivariant mechanisms with convoluted center movements. Also, higher-order numerical methods complemented by transparency and recurrence studies offer a solid conceptual basis for managing challenging decision-making problems using nonlinear uncertainty and reluctance.

Author Contributions: Conceptualization, A.A.-s.; Formal analysis, N.A.; Investigation, S.S.A.; Resources, S.A. and S.S.A.; Writing—original draft, A.A.-s.; Writing—review and editing, K.L.W., K.B.W. and A.A.-s.; Supervision, K.L.W., K.B.W. and K.A.; Project administration, S.A.; Funding acquisition, N.A. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: The raw data supporting the conclusions of this article will be made available by the authors on request.

Acknowledgments: The authors acknowledge the support of Prince Sultan University, Riyadh, Saudi Arabia, for paying the article processing charges for this publication.

Conflicts of Interest: The authors declare no conflicts of interest.

Abbreviations

The following abbreviations are used in this manuscript:

DM	decision-making
FS	fuzzy set
VS	vague set
VSS	vague soft set
NS	neutrosophic set
NSS	neutrosophic soft set
NVSS	neutrosophic vague soft set
IFS	intuitionistic fuzzy set
FNS	Fermatean neutrosophic set
PFS	Pythagorean fuzzy set
MCDM	multicriteria decision-making
MAGDM	Multiattribute Group Decision-Making
PIS	Positive Ideal Solution
NIS	Negative Ideal Solution
FNVSS	Fermatean neutrosophic vague soft set
FNVNWA	Fermatean neutrosophic vague number weighted aggregation
GFNVNWA	generalized Fermatean neutrosophic vague number weighted aggregation
TOPSIS	Technique for Order Preference by Similarity to Ideal Solution
AHP	Analytic Hierarchy Process
VIKOR	VIsekriterijumsko KOmpromisno Rangiranje

References

1. Zadeh, L.A. Fuzzy sets. *Inf. Control* **1965**, *8*, 338–353. [CrossRef]
2. Atanassov, K. Intuitionistic fuzzy sets. *Fuzzy Sets Syst.* **1986**, *20*, 87–96. [CrossRef]
3. Atanassov, K.T. Operators over interval valued intuitionistic fuzzy sets. *Fuzzy Sets Syst.* **1994**, *64*, 159–174. [CrossRef]
4. Molodtsov, D. Soft set theory—First results. *Comput. Math. Appl.* **1999**, *37*, 19–31. [CrossRef]

5. Maji, P.K.; Roy, A.R.; Biswas, R. Fuzzy softs Sets. *J. Fuzzy Math.* **2001**, *9*, 589–602.
6. Gau, W.L.; Buehrer, D.J. Vague sets. *IEEE Trans. Syst. Man Cybern.* **1993**, *23*, 610–614. [CrossRef]
7. Bustince, H.; Burillo, P. Vague sets are intuitionistic fuzzy sets. *Fuzzy Sets Syst.* **1996**, *79*, 403–405. [CrossRef]
8. Chen, S.M. Similarity measures between vague sets and between elements. *IEEE Trans. Syst. Man, Cybern. Part B Cybern.* **1997**, *27*, 153–158. [CrossRef]
9. Chen, S.M. Analyzing fuzzy system reliability using vague set theory. *Int. J. Appl. Sci. Eng.* **2003**, *1*, 82–88.
10. Smarandache, F. Neutrosophic set-a generalization of the intuitionistic fuzzy set. *Int. J. Pure Appl. Math.* **2005**, *24*, 287.
11. Maji, P.K. Neutrosophic soft set. *Ann. Fuzzy Math. Inform.* **2013**, *5*, 157–168.
12. Alkhazaleh, S.; Zureigat, H.; Batiha, B.; Al-Khateeb, A.; Al-Shboul, A. Fermatean Neutrosophic Soft Set. *Int. J. Neutrosophic Sci. IJNS* **2025**, *25*, 469–488. [CrossRef]
13. Alkhazaleh, S.; Batiha, B.; Al-khateeb, A.; Zureigat, H.; Al-shboul, A.; Batiha, K. Possibility Fermatean Neutrosophic Soft Set. *Int. J. Neutrosophic Sci. IJNS* **2024**, *24*, 105–125.
14. Al-shboul, A.; Alhazaymeh, K.; Wang, K.L.; Wong, K.B. Fermatean Vague Soft Set and Its Application in Decision Making. *Mathematics* **2024**, *12*, 3699. [CrossRef]
15. Hong, D.H.; Choi, C.H. Multicriteria fuzzy decision-making problems based on vague set theory. *Fuzzy Sets Syst.* **2000**, *114*, 103–113. [CrossRef]
16. Gorzałczany, M.B. A method of inference in approximate reasoning based on interval-valued fuzzy sets. *Fuzzy Sets Syst.* **1987**, *21*, 1–17. [CrossRef]
17. Kumar, A.; Yadav, S.P.; Kumar, S. Fuzzy reliability of a marine power plant using interval valued vague sets. *Int. J. Appl. Sci. Eng.* **2006**, *4*, 71–82.
18. Kumar, A.; Prasad Yadav, S.; Kumar, S. Fuzzy system reliability analysis using T ω (the weakest t-norm) based arithmetic operations on L-R type interval valued vague sets. *Int. J. Qual. Reliab. Manag.* **2007**, *24*, 846–860. [CrossRef]
19. Alhazaymeh, K.; Hassan, N. Einstein-operations on vague soft set. In Proceedings of the 4th International Conference on Mathematical Sciences: Mathematical Sciences: Championing the Way in a Problem Based and Data Driven Society, Putrajaya, Malaysia, 15–17 November 2017; Volume 1830, p. 070001.
20. Al-Qudah, Y.; Alhazaymeh, K.; Hassan, N.; Qoqazeh, H.; Almousa, M.; Alaroud, M. Transitive Closure of Vague Soft Set Relations and its Operators. *Int. J. Fuzzy Log. Intell. Syst.* **2022**, *22*, 59–68.
21. Alhazaymeh, K.; Al-Qudah, Y.; Hassan, N.; Nasruddin, A.M. Cubic vague set and its application in decision making. *Entropy* **2020**, *22*, 963. [CrossRef]
22. Alhazaymeh, K.; Hassan, N. Possibility interval-valued vague soft set. *Appl. Math. Sci.* **2013**, *7*, 6989–6994. [CrossRef]
23. Alhazaymeh, K.; Hassan, N.; Alhazaymeh, K. Generalized interval-valued vague soft set. *Appl. Math. Sci.* **2013**, *7*, 6983–6988. [CrossRef]
24. Alhazaymeh, K.; Gulistan, M.; Khan, M.; Kadry, S. Neutrosophic cubic Einstein hybrid geometric aggregation operators with application in prioritization using multiple attribute decision-making method. *Mathematics* **2019**, *7*, 346. [CrossRef]
25. Wang, J.; Liu, S.Y.; Zhang, J.; Wang, S.Y. On the parameterized OWA operators for fuzzy MCDM based on vague set theory. *Fuzzy Optim. Decis. Mak.* **2006**, *5*, 5–20. [CrossRef]
26. Zhou, L.; Wu, W.Z. On generalized intuitionistic fuzzy rough approximation operators. *Inf. Sci.* **2008**, *178*, 2448–2465. [CrossRef]
27. Shahzad, A.; Shoaib, A.; Mlaiki, N.; Subhi Aiadi, S. Results for fuzzy mappings and stability of fuzzy sets with applications. *Fractal Fract.* **2022**, *6*, 556. [CrossRef]
28. Rahim, M.; Amin, F.; Shah, K.; Abdeljawad, T.; Ahmad, S. Some distance measures for pythagorean cubic fuzzy sets: Application selection in optimal treatment for depression and anxiety. *MethodsX* **2024**, *12*, 102678. [CrossRef]
29. Khan, M.K.; Kamran; Sajjad Ali Khan, M.; Aloqaily, A.; Mlaiki, N. Covering-Based Intuitionistic Hesitant Fuzzy Rough Set Models and Their Application to Decision-Making Problems. *Symmetry* **2024**, *16*, 693. [CrossRef]
30. Fahmi, A.; Khan, A.; Abdeljawad, T. Group decision making based on cubic fermatean Einstein fuzzy weighted geometric operator. *Ain Shams Eng. J.* **2024**, *15*, 102737. [CrossRef]
31. Fahmi, A.; Ahmed, R.; Aslam, M.; Abdeljawad, T.; Khan, A. Disaster decision-making with a mixing regret philosophy DDAS method in Fermatean fuzzy number. *Aims Math.* **2023**, *8*, 3860–3884. [CrossRef]
32. Liang, J.; Yang, K.; Tan, C.; Wang, J.; Yin, G. Enhancing high-speed cruising performance of autonomous vehicles through integrated deep reinforcement learning framework. *IEEE Trans. Intell. Transp. Syst.* **2024**, *26*, 835–848. [CrossRef]
33. Wu, X.; Li, L.; Tao, X.; Xing, F.; Yuan, J. Happiness Prediction With Domain Knowledge Integration and Explanation Consistency. *IEEE Trans. Comput. Soc. Syst.* **2025**. [CrossRef]
34. Pan, H.; Tong, S.; Song, H.; Chu, X. A miner mental state evaluation scheme with decision level fusion based on multidomain EEG information. *IEEE Trans. Hum. Mach. Syst.* **2025**, *55*, 289–299. [CrossRef]
35. Li, Z.; Hu, J.; Leng, B.; Xiong, L.; Fu, Z. An integrated of decision making and motion planning framework for enhanced oscillation-free capability. *IEEE Trans. Intell. Transp. Syst.* **2023**, *25*, 5718–5732. [CrossRef]

36. Hwang, C.L.; Yoon, K. Methods for multiple attribute decision making. In *Multiple Attribute Decision Making: Methods and Applications a State-of-the-Art Survey*; Springer: Berlin/Heidelberg, Germany, 1981; pp. 58–191.
37. Chen, S.J.; Hwang, C.L. Fuzzy multiple attribute decision making methods. In *Fuzzy Multiple Attribute Decision Making: Methods and Applications*; Springer: Berlin/Heidelberg, Germany, 1992; pp. 289–486.
38. Sweety, C.A.C.; Jansi, R. Fermatean neutrosophic sets. *Int. J. Adv. Res. Comput. Commun. Eng* **2021**, *10*, 24–27.
39. Xu, W.; Ma, J.; Wang, S.; Hao, G. Vague soft sets and their properties. *Comput. Math. Appl.* **2010**, *59*, 787–794. [CrossRef]
40. Chen, T.; Li, F.; Yu, P. Nilpotent center conditions in cubic switching polynomial Liénard systems by higher-order analysis. *J. Differ. Equ.* **2024**, *379*, 258–289. [CrossRef]
41. Li, F.; Chen, T.; Liu, Y.; Yu, P. A complete classification on the center-focus problem of a generalized cubic Kukles system with a nilpotent singular point. *Qual. Theory Dyn. Syst.* **2024**, *23*, 8. [CrossRef]
42. Li, F.; Liu, Y.; Yu, P.; Wang, J. Complex integrability and linearizability of cubic Z_2-equivariant systems with two 1:Q resonant singular points. *J. Differ. Equ.* **2021**, *300*, 786–813. [CrossRef]
43. Li, F.; Jin, Y.; Tian, Y.; Yu, P. Integrability and linearizability of cubic Z_2 systems with non-resonant singular points. *J. Differ. Equ.* **2020**, *269*, 9026–9049. [CrossRef]
44. Li, F.; Liu, Y.; Liu, Y.; Yu, P. Complex isochronous centers and linearization transformations for cubic Z_2-equivariant planar systems. *J. Differ. Equ.* **2020**, *268*, 3819–3847. [CrossRef]
45. Li, F.; Yu, P.; Liu, Y.; Liu, Y. Centers and isochronous centers of a class of quasi-analytic switching systems. *Sci. China Math.* **2018**, *61*, 1201–1218. [CrossRef]
46. Li, F.; Liu, Y.; Liu, Y.; Yu, P. Bi-center problem and bifurcation of limit cycles from nilpotent singular points in Z_2-equivariant cubic vector fields. *J. Differ. Equ.* **2018**, *265*, 4965–4992. [CrossRef]
47. Li, F.; Yu, P.; Tian, Y.; Liu, Y. Center and isochronous center conditions for switching systems associated with elementary singular points. *Commun. Nonlinear Sci. Numer. Simul.* **2015**, *28*, 81–97. [CrossRef]
48. Li, F.; Wang, J. Analysis of an HIV infection model with logistic target-cell growth and cell-to-cell transmission. *Chaos Solitons Fractals* **2015**, *81*, 136–145. [CrossRef]