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ORIGINAL PAPER

On q-Gamma Operators and Their Applications to Classes of Bessel Functions

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Abstract

In this paper, we introduce two innovative variations of a gamma integral operator in the context of q-calculus theory. These q-analogues are derived from well-known q-analogues of the exponential function, with a focus on functions that satisfy specific exponential growth conditions. We also explore the application of these q-analogues in different categories of q-Bessel functions, including types one, two, and three. Furthermore, we derive various formulas and corollaries that demonstrate the practical applications of our findings. Additionally, we consider the finite products of Bessel functions of the same type in our analysis.

Keywords Bessel function · Integral operator · q-Hypergeometric function · Quantum calculus · q-Analogue · Gamma integral

Mathematics Subject Classification 26A33 · 44A20 · 05A30

Introduction

The quantum calculus or q-calculus is an area of calculus initiated by Jackson [1, 2] and developed by Euler to exchange the traditional derivative by a difference operator. The quantum theory of calculus connects mathematics and physics and receives attention of many investigators due its mainly numerous applications in various mathematical aspects including number theory, orthogonal polynomials, the theory of geometric functions, the theory of relativity, combinatorics and mechanics [3–7]. The concerned theory has realized many developments in mathematical physics including q-hypergeometric functions, polynomials, the area of partitions and the theory of numbers [4, 5, 5–12]. As an example on this interest, the q-hypergeometric functions are used in the fields of vector spaces, combinatorial analysis, particle physics, lie theory, nonlinear electric circuit theory, theory of heat conduction, mechanical engineering, statistics and cosmology. However, investigation of various q-analogues of various classical integral transforms is a popular topic among mathematicians and physicists [13, 14]. Soon after the q-Jackson definition various authors including Purohit and Kalla [15], Vyas et al. [16], Salem et al. [14], Hahn [17], Atici [18], Albayrak

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et al. [19], Al-Omari [9], Ucar [12, 20], Won Sang et al. [21], Al-Omari [13], Al-Salam [22] have contributed largely in this q-integral theory, see also [26–34] for further details.

For an arbitrary function φ and a real number q such that $0 < q < 1$, the q-analogue of the difference operator is introduced by Jackson [1] as

$$\left(D_q\varphi\right)(\tau) = \frac{\varphi(\tau) - \varphi(q\tau)}{(1-q)\tau}, \tau \neq 0. \tag{1}$$

The complex number $\tau \in \mathbb{C}$, the natural number $j \in \mathbb{N}$ and the factorial of the natural number j have q-analogues given as [32]

$$[\tau]_q = \frac{1-q^\tau}{1-q}, \ [j]_q = \frac{1-q^j}{1-q} \text{ and } [j]_q! = [j]_q [j-1]_q \ldots [2]_q [1]_q, \ [0]_q! = 1, \tag{2}$$

respectively. On the other hand, the shifted factorials have q-analogues defined by [32]

$$(\tau; q)_j = \prod_{i=0}^{j-1}\left(1 - \tau q^i\right), \ (\tau; q)_0 = 1 \text{ and } (\tau; q)_\infty = \lim_{j\to\infty}(\tau; q)_j. \tag{3}$$

The q-analogues of the exponential function are defined in two forms as [19]

$$E_q(\tau) = \sum_{j=0}^{\infty} q^{\frac{j(j-1)}{2}} \frac{\tau^j}{[k]_q!} = (\tau; q)_\infty \text{ and } e_q(\tau) = \sum_{j=0}^{\infty} \frac{\tau^j}{[j]_q!} = \frac{1}{(\tau; q)_\infty}, \tau \in \mathbb{R}. \tag{4}$$

The definite and improper integrals have been, respectively, assigned q-analogues defined by [32]

$$\int_0^\gamma \psi(\gamma)\, d_q\gamma = (1-q)\sum_{i\geq 0}\psi\left(q^i\gamma\right)\gamma q^i \tag{5}$$

and

$$\int_0^{\frac{\infty}{\tau}} \psi(\gamma)\, d_q\gamma = (1-q)\sum_{i\in\mathbb{Z}}\frac{q^i}{\tau}\psi\left(\frac{q^i}{\tau}\right). \tag{6}$$

The gamma function has two q-analogues defined on the basis of the q -exponential functions (4) as

$$\Gamma_q(\delta) = \int_0^{\frac{1}{1-q}} \gamma^{\delta-1} E_q(q(1-q)\gamma)\, d_q\gamma \tag{7}$$

and

$$\Gamma_q(\delta) = k(w; \delta)\int_0^{\frac{\infty}{w(1-q)}} \gamma^{\delta-1} e_q(-(1-q)\gamma)\, d_q\gamma, \tag{8}$$

where

$$k(w; \delta) = w^{\delta-1}\frac{(-q/w; q)_\infty (-w; q)_\infty}{\left(-q^\delta/w; q\right)_\infty \left(-wq^{1-\delta}; q\right)_\infty}. \tag{9}$$

Likewise, for real numbers $\tau > 0$ and $\beta > 0$, the beta function has q-analogue defined by the integral formula [32]

$$B(\tau, \beta) = \int_0^1 \delta^{\tau-1}(1-q)_q^{\beta-1}\, d_q\delta.$$

For further analysis, the gamma operators are defined for functions φ of certain exponential growth conditions in the form [33]

$$\left(\mathcal{G}_n\varphi\right)(\delta) = \frac{n^n}{\delta^n \Gamma(n)} \int_0^\infty \varphi(\tau)\, \tau^{n-1} e^{\frac{-n\tau}{\delta}} d\tau, \delta \in [0,\infty) \text{ and } n \in \mathbb{N}. \tag{10}$$

And, in a sense of quantum calculus theory, the q-analogues of the gamma operators are defined by [34]

$$\mathcal{G}_{n,q}(\varphi;\delta) = \frac{n^n}{\delta^n \Gamma_q(n)} \int_0^\infty \varphi(\tau)\, \tau^{n-1} e_q\left(\frac{-qn\tau}{\delta}\right) d_q\tau, 0 < q < 1. \tag{11}$$

In what follows, we introduce the q-analogues of the gamma function as follows:

Definition 1.1 Let φ be a function of certain exponential growth conditions. Then, we introduce a q-analogue of first kind for the gamma function in the form

$$\mathcal{G}_{n,q}(\varphi;\delta) = \frac{n^n}{(1-q)\,\delta^n \Gamma_q(n)} \int_0^\delta \varphi(\tau)\, \tau^{n-1} E_q\left(\frac{qn\tau}{\delta}\right) d_q\tau, \delta \in [0,\infty), \tag{12}$$

whereas we introduce a q-analogue of second type for the gamma function as

$$\bar{\mathcal{G}}_{n,q}(\varphi;\delta) = \frac{n^n}{(1-q)\,\delta^n \Gamma_q(n)} \int_0^\infty \varphi(\tau)\, \tau^{n-1} e_q\left(\frac{-n\tau}{\delta}\right) d_q\tau, \delta \in [0,\infty). \tag{13}$$

Indeed, the operators $\mathcal{G}_{n,q}$ and $\bar{\mathcal{G}}_{n,q}$ are positive and linear satisfying the relation $\mathcal{G}_{n,q}$ and $\bar{\mathcal{G}}_{n,q} \to \mathcal{G}_n$ as $q \to 1^-$.

Here, it is also interesting to mention here that the q-analogue (3) can be deduced in terms of series representation as follows.

Lemma 1.2 *Let φ be a function of certain exponential growth conditions. Then, we have*

$$\mathcal{G}_{n,q}(\varphi;\delta) = A_n^q \sum_{k=0}^{\infty} \frac{\varphi\left(\delta q^k\right)}{(nq;q)_k}, \tag{14}$$

where $A_n^q = \dfrac{n^n\,(nq;q)_\infty\, q^n}{\Gamma_q(n)}$.

Proof By utilizing the definition of the Jackson q -integral, the q-analogue (14) is deduced in terms of the series representation as

$$\mathcal{G}_{n,q}(\varphi;\delta) = \frac{n^n q^n}{\Gamma_q(n)} \sum_{k=0}^{\infty} \varphi\left(\delta q^k\right) E_q\left(nq^{k+1}\right).$$

Therefore, the preceding equation can be written in terms of the fact $E_q(\tau) = (\tau;q)_\infty$ as

$$\left(\mathcal{G}_{n,q}\right)(\varphi;\delta) = \frac{n^n q^n}{\Gamma_q(n)} \sum_{k=0}^{\infty} \varphi\left(\delta q^k\right) \left(nq^{k+1};q\right)_\infty.$$

Therefore, by employing the fact [20]

$$(c;q)_\tau = \frac{(c;q)_\infty}{(cq^\tau;q)_\infty},$$

the equation as above can be expressed as

$$\mathcal{G}_{n,q}\left(\varphi;\delta\right)=\frac{n^n q^n}{\Gamma_q\left(n\right)}\sum_{k=0}^{\infty}\varphi\left(\delta q^k\right)\frac{\left(nq;q\right)_\infty}{\left(nq;q\right)_k}=\frac{n^n q^n\left(nq;q\right)_\infty}{\Gamma_q\left(n\right)}\sum_{k=0}^{\infty}\frac{\varphi\left(\delta q^k\right)}{\left(nq;q\right)_k}.$$

□

This finishes the proof of the lemma.
The remarkable Bessel function

$$J_\gamma\left(\tau\right)=\sum_{j=0}^{\infty}\frac{\left(-1\right)^j\left(\tau/2\right)^{\gamma+2j}}{j!\Gamma\left(\gamma+j+1\right)}$$

has q-analogues which were first proposed by [33] and discussed subsequently by [17] as

$$J_\gamma^1\left(\tau;q\right)=\left(\frac{\tau}{2}\right)^\gamma\sum_{j=0}^{\infty}\frac{\left(\frac{-1}{4}\tau^2\right)^j}{\left(q;q\right)_{\gamma+j}\left(q;q\right)_j},\ \ |\tau|<2, \tag{15}$$

and

$$J_\gamma^2\left(\tau;q\right)=\left(\frac{\tau}{2}\right)^\gamma\sum_{j=0}^{\infty}\frac{q^{j(j+\gamma)}\left(\frac{-1}{4}\tau^2\right)^j}{\left(q;q\right)_{\gamma+j}\left(q;q\right)_j},\ \ \tau\in\mathbb{C}. \tag{16}$$

The Hahn–Exton q-analogue of the Bessel function which was discussed by Hahn [17] and Exton is given by [39]

$$J_\gamma^3\left(\tau;q\right)=\tau^\gamma\sum_{j=0}^{\infty}\frac{\left(-1\right)^j q^{\frac{j(j-1)}{2}}\left(q\tau^2\right)^j}{\left(q;q\right)_{\gamma+j}\left(q;q\right)_j},\tau\in\mathbb{C}. \tag{17}$$

Following is a result which is needful in the sequel.

$$\Gamma_q\left(\alpha\right)=\left(1-q\right)_{\alpha-1}\frac{G\left(q^\alpha\right)}{G\left(q\right)}\left(1-q\right)^{\alpha-1}G\left(q^\alpha\right)=\frac{1}{\left(q^\alpha,q\right)_\infty}\left(1-q\right)^{1-\alpha}. \tag{18}$$

In this article, we extend our results into four sections. In Sect. 1, certain definitions and preliminary results are introduced. In Sect. 2, the q-gamma integral operators of the first type are defined and employed to certain sets of Bessel function type. In Sect. 3, the q-gamma integral operators of the second type are employed to certain three sets of Bessel functions. Section 4 discusses some applications to the obtained results.

$\mathcal{G}_{n,q}$ of Finite Product of q-Bessel Functions

This section aims to discuss the $\mathcal{G}_{n,q}$ integral operator and its application to a finite product of q-analogues of Bessel functions of type one, two and three. The assigned products are multiplied by a polynomial to obtain more general cases.

Theorem 2.1 *Let* $S=\{J_{2\gamma_i}^1\left(2\left(c_1\tau\right)^{\frac{1}{2}};q\right),i=1,\ldots,r\}$ *be a set of first kind q-analogues of Bessel functions and*

$$\varphi\left(t\right)=\tau^{\Delta-1}\Pi_{i=1}^r J_{2\gamma_i}^1\left(2\left(c_i\tau\right)^{\frac{1}{2}};q\right),$$

then we have

$$\mathcal{G}_{n,q}(\varphi;\delta)=B_n^q\Pi_{i=1}^r\sum_{j=0}^{\infty}(-1)^j\,\frac{(c_i\delta)^{\gamma_i+j}\left(q^{2\gamma_i+j+1};q\right)_\infty}{(q;q)_\infty}\sum_{k=0}^{\infty}\frac{q^{k(\Delta-1+\gamma_i+j)}}{(nq;q)_k},$$

where $B_n^q=\dfrac{A_n^q}{\delta^{1-\Delta}}$.

Proof By using (12) , we write

$$\mathcal{G}_{n,q}(\varphi;\delta)=A_n^q\sum_{k=0}^{\infty}\frac{\varphi\left(\delta q^k\right)}{(nq;q)_k}.\tag{19}$$

Therefore, by employing (15) , (19) can be expressed in the form

$$\mathcal{G}_{n,q}(\varphi;\delta)=A_n^q\sum_{k=0}^{\infty}\left(\delta q^k\right)^{\Delta-1}\Pi_{i=1}^r\left(c_i\delta q^k\right)^{\gamma_i}\sum_{j=0}^{\infty}\frac{\left(-c_i\delta q^k\right)^j}{(q;q)_{2\gamma_i+j}\,(q;q)_j}\frac{1}{(nq;q)_k}.\tag{20}$$

Hence, (20) reveals to have

$$\mathcal{G}_{n,q}(\varphi;\delta)=A_n^q\Pi_{i=1}^r(c_i\delta)^{\gamma_i}\sum_{j=0}^{\infty}\frac{(-c_i\delta)^j}{(q;q)_{2\gamma_i+j}\,(q;q)_j}\sum_{k=0}^{\infty}\left(\delta q^k\right)^{\Delta-1}\frac{q^{k\gamma_i+jk}}{(nq;q)_k}.$$

That is,

$$\mathcal{G}_{n,q}(\varphi;\delta)=\frac{A_n^q}{\delta^{1-\Delta}}\Pi_{i=1}^r(c_i\delta)^{\gamma_i}\sum_{j=0}^{\infty}(-1)^n\,\frac{c_i^j\delta^j}{(q;q)_{2\gamma_i+j}\,(q;q)_j}\sum_{k=0}^{\infty}\frac{q^{k(\Delta-1+\gamma_i+j)k}}{(nq;q)_k}.\tag{21}$$

But, invoking

$$(c;q)_x=\frac{(c;q)_\infty}{(cq^x;q)_\infty}$$

in (21) yields

$$\mathcal{G}_{n,q}(\varphi;\delta)=\frac{A_n^q}{\delta^{1-\Delta}}\Pi_{i=1}^r\sum_{j=0}^{\infty}(-1)^j\,\frac{(c_i\delta)^{\gamma_i+j}\left(q^{2\gamma_i+j+1};q\right)_\infty}{(q;q)_\infty}\sum_{k=0}^{\infty}\frac{q^{k(\Delta-1+\gamma_i+j)}}{(nq;q)_k}.$$

□

This finishes our proof.

Theorem 2.2 *Let* $S=\{J_{2\gamma_i}^2\left(2\left(c_1\tau\right)^{\frac{1}{2}};q\right),i=1,\ldots,r\}$ *be a set of second type* q*-analogues of Bessel functions and*

$$\varphi(\tau)=\tau^{\Delta-1}\Pi_{i=1}^rJ_{2\gamma_i}^2\left(2\left(c_i\tau\right)^{\frac{1}{2}};q\right),$$

then

$$\mathcal{G}_{n,q}(\varphi;\delta)=C_n^q\Pi_{i=1}^r\sum_{j=0}^{\infty}(-1)^j\,q^{j(j+2\gamma_i)+kj}\frac{(\delta c_i)^{j+\gamma_i}\left(q^{2\gamma_i+j+1};q\right)_\infty}{(q;q)_j}$$
$$\sum_{k=0}^{\infty}\frac{q^{k(\Delta+\gamma_i)}}{(nq;q)_k},$$

where $C_n^q = \dfrac{A_n^q \delta^{\Delta-1}}{(q;q)_\infty}$.

Proof With the benefit of (12) we derive

$$\begin{aligned}\mathcal{G}_{n,q}(\varphi;\delta) &= A_n^q \sum_{k=0}^{\infty} \frac{\varphi\left(\delta q^k\right)}{(nq;q)_k} \\ &= A_n^q \sum_{k=0}^{\infty} \left(\delta q^k\right)^{\Delta-1} \Pi_{i=1}^r J_{2\gamma_i}^2\left(2\left(c_i \delta q^k\right)^{\frac{1}{2}};q\right) \frac{1}{(nq;q)_k}.\end{aligned}$$

By utilizing (16) we establish that

$$\mathcal{G}_{n,q}(\varphi;\delta) = A_n^q \sum_{k=0}^{\infty} \left(\delta q^k\right)^{\Delta-1} \Pi_{i=1}^r \left(c_i\delta q^k\right)^{\gamma_i} \sum_{j=0}^{\infty} \frac{q^{j(j+2\gamma_i)}\left(-c_i\delta q^k\right)^j}{(q;q)_{2\gamma_i+j}(q;q)_j} \frac{1}{(nq;q)_k}.$$

Or, equivalently, we have

$$\mathcal{G}_{n,q}(\varphi;\delta) = \frac{A_n^q}{(q;q)_\infty} \Pi_{i=1}^r \left(c_i\delta q^k\right)^{\gamma_i} \sum_{j=0}^{\infty} \frac{q^{j(j+2\gamma_i)}\left(-c_i\delta q^k\right)^j}{(q;q)_{2\gamma_i+j}(q;q)_j} \sum_{k=0}^{\infty} \left(\delta q^k\right)^{\Delta-1} \frac{1}{(nq;q)_k}. \tag{22}$$

By employing the fact

$$(c;q)_x = \frac{(c;q)_\infty}{(cq^x;q)_\infty},$$

equation (22) gives rise to

$$\mathcal{G}_{n,q}(\varphi;\delta) = \frac{A_n^q}{(q;q)_\infty} \Pi_{i=1}^r (c_i\delta)^{\gamma_i} \sum_{j=0}^{\infty} q^{j(j+2\gamma_i)} \frac{\left(-c_i\delta q^k\right)^j}{(q;q)_j} \left(q^{2\gamma_i+j+1};q\right)_\infty \sum_{k=0}^{\infty} \frac{q^{k\gamma_i}}{(nq;q)_k}. \tag{23}$$

Modifying (23) suggests to have

$$\begin{aligned}\mathcal{G}_{n,q}(\varphi;\delta) = \frac{A_n^q \delta^{\Delta-1}}{(q;q)_\infty} \Pi_{i=1}^r \sum_{j=0}^{\infty} (-1)^j \frac{q^{j(j+2\gamma_i)+kj}(\delta c_i)^{j+\gamma_i}\left(q^{2\gamma_i+j+1};q\right)_\infty}{(q;q)_j} \\ \sum_{k=0}^{\infty} \frac{q^{k(\Delta+\gamma_i)}}{(nq;q)_k}.\end{aligned}$$

□

This finishes the proof.

Theorem 2.3 *Let* $S = \{J_{2\gamma_i}^3\left(\left(q^{-1}c_1\tau\right)^{\frac{1}{2}};q\right), i = 1,\ldots,r\}$ *be q-Bessel functions of the third type and*

$$\varphi(\tau) = \tau^{\Delta-1} \Pi_{i=1}^r J_{2\gamma_i}^3\left(\left(q^{-1}c_i\tau\right)^{\frac{1}{2}};q\right),$$

then

$$\mathcal{G}_{n,q}(\varphi;\delta) = \frac{A_n^q \delta^{\Delta-1}}{(q;q)_\infty} \Pi_{i=1}^r \sum_{j=0}^{\infty} (c_i\delta)^{j+\gamma_i} (-1)^j q^{j\left(\frac{j-1}{2}\right)} \frac{\left(q^{2\gamma_i+j+1};q\right)_\infty}{(q;q)_j}$$

$$\sum_{k=0}^{\infty} \frac{q^{k(\Delta-1+j+\gamma_i)}}{(nq;q)_k},$$

where $C_n^q = \dfrac{A_n^q \delta^{\Delta-1}}{(q;q)_\infty}$.

Proof By (12) and (9) , we write

$$\mathcal{G}_{n,q}(\varphi;\delta) = A_n^q \sum_{k=0}^{\infty} \frac{f\left(\delta q^k\right)}{(nq;q)_k} = A_n^q \sum_{k=0}^{\infty} \left(\delta q^k\right)^{\Delta-1} \Pi_{i=1}^r J_{2\gamma_i}^3 \left(\left(q^{-1} c_i \delta q^k\right)^{\frac{1}{2}}; q\right) \frac{1}{(nq;q)_k}.$$

This can be simplified to yield

$$\mathcal{G}_{n,q}(\varphi;\delta) = A_n^q \sum_{k=0}^{\infty} \left(\delta q^k\right)^{\Delta-1} \Pi_{i=1}^r \left(c_i \delta q^{k-1}\right)^{\gamma_i} \sum_{j=0}^{\infty} (-1)^j \frac{q^{\frac{j(j-1)}{2}} \left(c_i \delta q^k\right)^j}{(q;q)_{2\gamma_i+j} (q;q)_j (nq;q)_k}. \tag{24}$$

By using the fact $(c;q)_x = \frac{(c;q)_\infty}{(cq^x;q)_\infty}$, (24) can be set into the form

$$\begin{aligned}\mathcal{G}_{n,q}(\varphi;\delta) = {} & \frac{A_n^q}{(q;q)_\infty} \Pi_{i=1}^r (c_i\delta)^{\mu_i} \sum_{j=0}^{\infty} (-1)^j q^{\frac{j(j-1)}{2}} \frac{(c_i\delta)^j}{(q;q)_j} (q^{2\mu_i+j+1};q)_\infty \\ & \sum_{k=0}^{\infty} \left(\delta q^k\right)^{\Delta-1} \frac{q^{kj+(k-1)\mu_i}}{(nq;q)_k} \\ = {} & \frac{A_n^q \delta^{\Delta-1}}{(q;q)_\infty} \Pi_{i=1}^r \sum_{j=0}^{\infty} (c_i\delta)^{j+\mu_i} (-1)^j q^{j\left(\frac{j-1}{2}\right)} \frac{\left(q^{2\mu_i+j+1};q\right)_\infty}{(q;q)_j} \\ & \sum_{k=0}^{\infty} \frac{q^{k(\Delta-1+j+\mu_i)}}{(nq;q)_k}.\end{aligned}$$

□

This finishes the proof.

$\bar{\mathcal{G}}_{n,q}$ of Finite Product of q-Bessel Functions

Analogous to results obtained in the previous section, this section investigates the second q-gamma integral $\bar{\mathcal{G}}_{n,q}$ and functions involving a finite product of q-Bessel functions of type one, two and three multiplied by a polynomials of different orders.

Remark 3.1 Let φ be a function of certain exponential growth conditions. Then, we have

$$\bar{\mathcal{G}}_{n,q}(\varphi,\delta) = T_n^q \sum_{k\in\mathbb{Z}} \left(-n\delta^{-1};q\right)_k q^k f\left(q^k\right),$$

where $T_n^q = \frac{n^n \delta^{1-n}}{\left(-n\delta^{-1};q\right)_\infty \Gamma_q(n)}$.

Proof By the series q-representation, (13) can be nicely expressed as

$$\bar{\mathcal{G}}_{n,q}(\varphi,\delta) = \frac{n^n \delta(1-q)}{(1-q)\delta^n \Gamma_q(n)} \sum_{k\in\mathbb{Z}} q^k \varphi\left(q^k\right) q^{k-1} e_q\left(\frac{-nq^k}{\delta}\right).$$

By the fact that $e_q(\tau) = \dfrac{1}{(\tau;q)_\infty}$, we rewrite the above equation in the form

$$\bar{\mathcal{G}}_{n,q}(\varphi,\delta) = \frac{n^n\delta^{1-n}}{\Gamma_q(n)}\sum_{k\in\mathbb{Z}} q^k \frac{\varphi\left(q^k\right)}{\left(-nq^k\delta^{-1};q\right)_\infty}$$
$$= \frac{n^n\delta^{1-n}}{\left(-n\delta^{-1};q\right)_\infty \Gamma_q(n)}\sum_{k\in\mathbb{Z}}\left(-n\delta^{-1};q\right)_k q^k\varphi\left(q^k\right). \tag{25}$$

Finally, setting $T_n^q = \frac{n^n\delta^{1-n}}{(-n\delta^{-1};q)_\infty\Gamma_q(n)}$ finishes the proof of our remark. □

Theorem 3.2 *Let* $S = \{J^1_{2\gamma_1}\left(2(c_1\tau)^{\frac{1}{2}};q\right),\ldots,J^1_{2\gamma_r}\left(2(c_r\tau)^{\frac{1}{2}};q\right)\}$ *and* $\varphi(\tau) = \tau^{\Delta-1}\Pi_{i=1}^r J^1_{2\gamma_1}\left(2(c_i\tau)^{\frac{1}{2}};q\right)$. *Then*

$$\bar{\mathcal{G}}_{n,q}(\varphi,s) = H_n^q\Pi_{i=1}^r\sum_{j=0}^\infty(-1)^j\frac{\left(q^{2\gamma_i+j-1};q\right)_\infty c_i^{\gamma_i+j}}{(q;q)_j}\frac{(1-q)^{\gamma_i+j-1}\Gamma_q(\Delta+\gamma_i+j)}{K\left(\frac{n}{\delta};\Delta+\gamma_i+j\right)},$$

where $H_n^q = \dfrac{(1-q)^\Delta\left(-n\delta^{-1};q\right)_\infty T_n^q}{(q;q)_\infty}$.

Proof By following (13) and (25) we establish that

$$\bar{\mathcal{G}}_{n,q}(\varphi,\delta) = T_n^q\sum_{k\in\mathbb{Z}}\left(-n\delta^{-1};q\right)_k q^k\varphi\left(q^k\right)$$
$$= T_n^q\sum_{k\in\mathbb{Z}}\left(-n\delta^{-1};q\right)_k q^{k\Delta}\Pi_{i=1}^r J^1_{2\gamma_i}\left(2\left(c_iq^k\right)^{\frac{1}{2}};q\right). \tag{26}$$

Hence, making use of (15) yields

$$\bar{\mathcal{G}}_{n,q}(\varphi,\delta) = T_n^q\sum_{k\in\mathbb{Z}}q^{k\Delta}\left(-n\delta^{-1};q\right)_k\Pi_{i=1}^r\left(c_iq^k\right)^{\mu_i}\sum_{j=0}^\infty\frac{\left(-c_iq^k\right)^j}{(q;q)_{2\mu_i+j}(q;q)_j}.$$

Thus, we have consequently obtained that

$$\bar{\mathcal{G}}_{n,q}(\varphi,\delta) = T_n^q\Pi_{i=1}^r\sum_{j=0}^\infty\frac{(-1)^j c_i^{\gamma_i+j}}{(q;q)_{2\gamma_i+j}(q;q)_j}\sum_{k\in\mathbb{Z}}q^{k(\Delta+\gamma_i+j)}\left(-n\delta^{-1};q\right)_k. \tag{27}$$

Therefore, by following the fact [20],(26)

$$_q\Gamma(\alpha) = \frac{K(A,\alpha)_\infty}{(1-q)^{\alpha-1}\left(-\frac{1}{A},q\right)_\infty}\sum_{k\in\mathbb{Z}}\left(\frac{q^k}{A}\right)\left(-\frac{1}{A};q\right)_k,$$

and setting $A = \dfrac{n}{\delta}$ and $\alpha = \Delta+\gamma_i+j$, we get

$$\bar{\mathcal{G}}_{n,q}(\varphi,\delta) = \frac{nT_n^q}{\delta}\Pi_{i=1}^r\sum_{j=0}^\infty\frac{(-1)^j(c_i)^{\gamma_i+1}}{(q;q)_{2\gamma_i+j}(q;q)_j}\frac{\Gamma_q(\Delta+\gamma_i+j)(1-q)^{\Delta+\gamma_i+j-1}}{K\left(\frac{n}{\delta},\Delta+\gamma_i+j\right)}, \tag{28}$$

where $K(A,\alpha)$ has the significance of

$$K(A,\alpha)=A^{\alpha-1}\frac{(-q/\alpha;q)_\infty(-\alpha;q)_\infty}{(-q^t/\alpha;q)_\infty\left(1-\alpha q^{1-t};q\right)_\infty}.$$

Now, owing to the fact $(c;q)_k=\dfrac{(c;q)_\infty}{\left(cq^k;q\right)_\infty}$ we get

$$\bar{\mathcal{G}}_{n,q}(\varphi,\delta)=\frac{(1-q)^\Delta\left(-n\delta^{-1};q\right)_\infty T_n^q}{(q;q)_\infty}\Pi_{i=1}^r\sum_{j=0}^\infty(-1)^j\frac{(c_i)^{\gamma_i+j}}{(q;q)_j}\left(q^{2\gamma_i+j-1};q\right)_\infty \frac{(1-q)^{\gamma_i+j-1}\Gamma_q(\Delta+\gamma_i+j)}{K\left(\frac{n}{\delta};\Delta+\gamma_i+j\right)}.$$

□

This finishes the proof.

Theorem 3.3 *Let* $S=\{J_{2\gamma_1}^2\left(2(c_i\tau)^{\frac{1}{2}};q\right),\ldots,J_{2\gamma_r}^2\left(2(c_i\tau)^{\frac{1}{2}};q\right)\}$ *and* $\varphi(\tau)=\tau^{\Delta-1}\Pi_{i=1}^r J_{2\gamma_1}^2\left(2(c_i\tau)^{\frac{1}{2}};q\right)$. *Then, we have*

$$\bar{\mathcal{G}}_{n,q}(\varphi;\delta)=K_n^q\Pi_{i=1}^r\sum_{j=0}^\infty(-1)^n\frac{q^{j(j+2\gamma_i)}c_i^{\gamma_i+j}}{(q;q)_j}\left(q^{2\gamma_i+j+1};q\right)_\infty \frac{(1-q)^{\gamma_i+j}\Gamma_q(\Delta+\gamma_i+j)}{K\left(\frac{n}{\delta};\Delta+\gamma_i+j\right)},$$

where $K_n^q=\dfrac{n(1-q)^{\Delta-1}H_n^q}{\delta}$.

Proof By utilizing (35) and (16) , we obtain

$$\begin{aligned}\bar{\mathcal{G}}_{n,q}(\varphi,\delta)&=T_n^q\sum_{k\in\mathbb{Z}}\left(-n\delta^{-1};q\right)_k q^k\varphi\left(q^k\right)\\&=T_n^q\sum_{k\in\mathbb{Z}}\left(-n\delta^{-1};q\right)_k q^{k\Delta}\Pi_{i=1}^r J_{2\gamma_i}^2\left(2\left(c_iq^k\right)^{\frac{1}{2}};q\right)\\&=T_n^q\sum_{k\in\mathbb{Z}}\left(-n\delta^{-1};q\right)_k q^{k\Delta}\Pi_{i=1}^r\left(c_iq^k\right)^{\gamma_i}\sum_{j=0}^\infty\frac{q^{j(j+2\gamma_i)}\left(-c_i;q^k\right)^j}{(q;q)_{2\gamma_i+j}(q;q)_j}.\end{aligned}$$

Indeed, this yields

$$\bar{\mathcal{G}}_{n,q}(\varphi,\delta)=T_n^q\Pi_{i=1}^r\sum_{j=0}^\infty\frac{(-1)^j q^{j(j+2\gamma_i)}c_i^{\gamma_i+j}}{(q;q)_{2\gamma_i+j}(q;q)_j}\sum_{k\in\mathbb{Z}}q^{k(\Delta+\gamma_i+j)}\left(-n\delta^{-1};q\right)_k. \tag{29}$$

Hence, by [5, 20]and setting $w=\dfrac{n}{\delta}$ and $\alpha=\Delta+\gamma_i+j$ and, the fact $(c;q)_x=\frac{(c;q)_\infty}{(cq^x;q)_\infty}$, we obtain

$$\bar{G}_{n,q}(\varphi;\delta)=K_n^q\Pi_{i=1}^r\sum_{j=0}^\infty(-1)^n\frac{q^{j(j+2\gamma_i)}(a_i)^{\gamma_i+j}}{(q;q)_j}\left(q^{2\gamma_i+j+1};q\right)_\infty$$

$$\frac{(1-q)^{\mu_i+j}\,\Gamma_q\left(\Delta+\gamma_i+j\right)}{K\left(\frac{n}{\delta};\Delta+\gamma_i+j\right)},$$

where $K_n^q = \dfrac{n\,(1-q)^{\Delta-1}\,H_n^q}{\delta}$. □

This finishes the proof.

Theorem 3.4 *Let* $S = \{J_{2\gamma_1}^3\left(\left(q^{-1}c_i\tau\right)^{\frac{1}{2}};q\right),\ldots,J_{2\gamma_r}^3\left(\left(q^{-1}c_r\tau\right)^{\frac{1}{2}};q\right)\}$ *and* $\varphi(\tau) = \tau^{\Delta-1}\Pi_{i=1}^r J_{2\gamma_i}^3\left(\left(q^{-1}c_i\tau\right)^{\frac{1}{2}};q\right)$. *Then,*

$$\bar{\mathcal{G}}_{n,q}(\varphi;\delta) = K_n^q\,\Pi_{i=1}^r \sum_{j=0} \frac{c_i^{j+1}(-1)^j\,q^{\frac{(j-1)}{2}-1}}{(q;q)_j}\left(q^{2\gamma_i+j+1};q\right)_\infty \frac{\Gamma_q\left(\Delta+j+1\right)(1-q)^j}{K\left(\frac{n}{\delta};\Delta+j+1\right)},$$

where $K_n^q = \dfrac{n\,(1-q)^{\Delta}\,H_n^q}{\delta}$.

Proof Assume the hypothesis of the theorem holds. Then, by Remark 3.1 and the definition of $J_{2\mu_i}^3$ given by (17), we get

$$\begin{aligned}\bar{\mathcal{G}}_{n,q}(\varphi;\delta) &= T_n^q \sum_{k\in\mathbb{Z}}\left(-n\delta^{-1};q\right)_k q^k\varphi\left(q^k\right)\\ &= T_n^q \sum_{k\in\mathbb{Z}}\left(-n\delta^{-1};q\right)_k q^{k\Delta}\,\Pi_{i=1}^r J_{2\gamma_i}^3\left(\left(q^{k-1}c_i\right)^{\frac{1}{2}};q\right)\\ &= T_n^q \sum_{k\in\mathbb{Z}}\left(-n\delta^{-1};q\right)_k q^{k\Delta}\,\Pi_{i=1}^r q^{k-1}c_i\sum_{j=0}^\infty(-1)^j\,\frac{q^{\frac{j(j-1)}{2}}\left(q^kc_i\right)^j}{(q;q)_{2\gamma_i+j}\,(q;q)_j}.\end{aligned}$$

Therefore, we have

$$\bar{\mathcal{G}}_{n,q}(\varphi;\delta) = T_n^q\,\Pi_{i=1}^r\sum_{j=0}^\infty \frac{c_i^{j+1}(-1)^j\,q^{\frac{j(j-1)}{2}-1}}{(q;q)_{2\gamma_i+j}\,(q;q)_j}\sum_{k\in\mathbb{Z}} q^{(\Delta+j+1)k}\left(-n\delta^{-1};q\right)_k. \tag{30}$$

Thus, by utilizing [5, 20] and setting $A = \dfrac{n}{\delta}$, $\alpha = \Delta + j + 1$ we obtain

$$\begin{aligned}\bar{G}_{n,q}(\varphi;\delta) = \frac{n\,(1-q)^{\Delta}\,H_n^q}{\delta}\,\Pi_{i=1}^r\sum_{j=0}\frac{c_i^{j+1}(-1)^j\,q^{\frac{(j-1)}{2}-1}}{(q;q)_j}\left(q^{2\gamma_i+j+1};q\right)_\infty\\ \frac{(1-q)^j\,\Gamma_q\left(\Delta+j+1\right)}{K\left(\frac{n}{\delta};\Delta+j+1\right)}.\end{aligned}$$

□

This finishes the proof.

Applications

In this application part, we consider several special cases of q-gamma functions of various kinds of functions including q-Bessel functions.

Corollary 4.1 *Let* $\varphi_1^1(\tau) = \tau^{\Delta-1} J_1^1\left(2(c\tau)^{\frac{1}{2}}, q\right)$, *where* J_1^1 *is a* q*-analogue of a Bessel function of first type. Then*

$$\mathcal{G}_{n,q}(\varphi_1;\delta) = B_n^q \sum_{j=0}^{\infty} (-1)^j c\delta^{\frac{1}{2}+j} \left(q^{j+2};q\right)_\infty \sum_{k=0}^{\infty} \frac{q^{k\left(\Delta-\frac{1}{2}+j\right)}}{(nq;q)_k},$$

where $B_n^q = \dfrac{A_n^q}{\delta^{1-\Delta}}$.

Proof The result is a truthful end results from Theorem 2.1, follows with the aid of putting $\mu = \dfrac{1}{2}$ and $r = 1$. □

Corollary 4.2 *Let* $\varphi_1^2(\tau) = \tau^{\Delta-1} J_1^2\left(2(c\tau)^{\frac{1}{2}}, q\right)$, *where* J_1^2 *is a* q*-analogue of a Bessel function of second type. Then*

$$\mathcal{G}_{n,q}\left(\varphi_1^2;\delta\right) = C_n^q \sum_{j=0}^{\infty} (-1)^j \frac{q^{j(j+1)}(\delta c)^{j+\frac{1}{2}}\left(q^{j+2};q\right)_\infty}{(q;q)_j} \sum_{k=0}^{\infty} \frac{q^{k\left(\Delta+\frac{1}{2}\right)}}{(nq;q)_k},$$

where $C_n^q = \dfrac{A\delta^{\Delta-1}}{(q;q)_\infty}$.

Proof The result is a truthful end results from Theorem 2.2, follows with the aid of putting $\mu = \dfrac{1}{2}$ and $r = 1$. □

Corollary 4.3 *Let* $\varphi_1^3(\tau) = \tau^{\Delta-1} J_1^3\left(2(c\tau)^{\frac{1}{2}}, q\right)$, *where* J_1^3 *is a Bessel function of third type. Then*

$$\mathcal{G}_{n,q}\left(\varphi_1^3;\delta\right) = C_n^q \sum_{j=0}^{\infty} \frac{(-1)^j (c\delta)^j q^{\frac{j(j-1)+1}{2}}\left(q^{j+2};q\right)_\infty}{(q;q)_j} \sum_{k=0}^{\infty} \frac{q^{k\left(\Delta+j-\frac{1}{2}\right)}}{(nq;q)_k},$$

where $C_n^q = \dfrac{A_n^q (c\delta)^{\frac{1}{2}} \delta^{\Delta-1}}{(q;q)_\infty}$.

Proof The result is a truthful end results from Theorem 2.3, follows with the aid of putting $\mu = \dfrac{1}{2}$ and $r = 1$. □

Corollary 4.4 *Let* $\varphi_1^1(\tau) = \tau^{\Delta-1} J_1^1\left(2(c\tau)^{\frac{1}{2}}, q\right)$, *where* J_1^1 *is a Bessel function of the first type. Then*

$$\bar{\mathcal{G}}_{n,q}\left(\varphi_1^1;\delta\right) = H_n^q \sum_{j=0}^{\infty} (-1)^j a^{j+\frac{1}{2}} \left(q^{j+2};q\right)_\infty \frac{\Gamma_q\left(\Delta+j+\frac{1}{2}\right)(1-q)^j}{K\left(\frac{n}{\delta};\Delta+j+\frac{1}{2}\right)},$$

where $H_n^q = \dfrac{\left(-n\delta^{-1}; q\right)_\infty (1-q)^{\Delta-\frac{1}{2}} T_n^q}{(q;q)_\infty}$.

Proof The result is a truthful end results from Theorem 3.2, follows with the aid of putting $\mu = \dfrac{1}{2}$ and $r = 1$. □

Corollary 4.5 *Let* $\varphi_1^2(\tau) = \tau^{\Delta-1} J_1^2\left(2(c\tau)^{\frac{1}{2}}, q\right)$, *where* J_1^2 *is a Bessel function of the second type. Then*

$$\bar{\mathcal{G}}_{n,q}\left(\varphi_1^2;\delta\right) = K_n^q \sum_{j=0}^{\infty} (-1)^j q^{j(j+1)} c^{j+\frac{1}{2}} \left(q^{j+2};q\right)_\infty \frac{\Gamma_q\left(\Delta+j+\frac{1}{2}\right)(1-q)^j}{(q;q)_j K\left(\frac{n}{\delta};\Delta+j+\frac{1}{2}\right)},$$

where $K_n^q = \dfrac{n(1-q)^{\Delta-\frac{1}{2}} H_n^q}{\delta}$.

Proof The result is a truthful end results from Theorem 3.3, follows with the aid of putting $\mu = \dfrac{1}{2}$ and $r = 1$. □

Corollary 4.6 *Let* $\varphi_1^3(\tau) = \tau^{\Delta-1} J_1^3\left(2(c\tau)^{\frac{1}{2}}, q\right)$, *where* J_1^3 *is a Bessel function. Then*

$$\bar{\mathcal{G}}_{n,q}\left(\varphi_1^3;\delta\right) = K_n^q \sum_{j=0}^{\infty} (-1)^j c^{j+1} q^{j\frac{(j+2)}{2}-1} \left(q^{j+2};q\right)_\infty \frac{\Gamma_q(\Delta+j+1)(1-q)^j}{(q;q)_j K\left(\frac{n}{\delta},\Delta+j+1\right)},$$

where $K_n^q = \dfrac{n(1-q)^{\Delta} H_n^q}{\delta}$.

Proof The end result is a truthful end results from Theorem 3.2, follows with the aid of putting $\mu = \dfrac{1}{2}$ and $r = 1$ □

Conclusion

This article introduces and discusses two q-analogues of the Gamma operator, focusing on various finite products of different types of q-Bessel functions. The findings presented here are also applied to specific instances of the aforementioned results.

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