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On r-Fuzzy Soft δ-Open Sets With Applications in Fuzzy Soft Topological Spaces

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Abstract. In this paper, we introduce the notion of r-fuzzy soft δ-open sets on fuzzy soft topological spaces in the sense of Šostak. Furthermore, we define and characterize the notions of fuzzy soft δ-closure (δ-interior) operators using r-fuzzy soft δ-closed (δ-open) sets. After that, we explore the notions of fuzzy soft δ-continuous (semi-continuous and pre-continuous) functions, which are weaker forms of fuzzy soft continuity. Moreover, we study some properties of these functions along with their mutual relationships with the help of some problems. We also present a decomposition of fuzzy soft semi-continuity and a decomposition of fuzzy soft α-continuity. Additionally, as a weaker form of fuzzy soft continuity, we define and study the notions of fuzzy soft almost (weakly) continuous functions. Lastly, we explore the notion of continuity in a very general setting called fuzzy soft $(\mathcal{L}, \mathcal{M}, \mathcal{N}, \mathcal{O})$-continuity and introduce a historical justification.

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1. Introduction and preliminaries

In [25], the author proposed a novel notion of soft set theory, which is a completely new approach for modeling uncertainty and vagueness. He studied many applications of this theory in solving different problems in engineering, social science, medical science, etc. The notion of soft sets was used to define soft topological spaces in [31]. The method in [31] was particularly important in the development of the field of soft topology (see [6, 19, 39, 44]). Generalizations of soft open sets play an effective role in soft topology through their use to

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improve on some known results or to open the door to explore some of the soft topological notions such as soft separation axioms [21], soft connectedness [40, 42], soft continuity [26], etc. Akdag and Ozkan [3] introduced and studied the notion of soft α-open sets in soft topological spaces. Also, the notion of soft β-open sets was introduced and studied by the authors of [2, 18]. Al-shami et al. [4] defined the notion of weakly soft β-open sets and obtained weakly soft β-continuity. Kaur et al. [22] initiated a new approach to studying soft continuous functions. Moreover, many authors have contributed to the theory of soft sets in the different fields such as topology, algebra; see [7, 10, 11, 16, 17, 27, 30].

Maji et al. [23] defined the notion of fuzzy soft sets which combines soft sets [25] and fuzzy sets [43]. The notion of fuzzy soft topology was defined and some of its properties such as fuzzy soft continuity, interior fuzzy soft set, closure fuzzy soft set, and fuzzy soft subspace topology were obtained in [15, 20] based on fuzzy topologies in Šostaks sense [41]. A novel approach to studying separation axioms and regularity axioms via fuzzy soft sets was defined by the author of [32, 36]. The notion of r-fuzzy soft regularly open sets was defined by Çetkin and Aygün [14]. Also, the notions of r-fuzzy soft β-open (pre-open) sets were also introduced by Taha [33]. In addition, several researchers have contributed to the theory of fuzzy soft sets in many fields such as topology; see [5, 28, 29].

The organization of this paper is as follows:

• In Section 2, we define new types of fuzzy soft sets in fuzzy soft topological spaces based on the paper by Aygünoğlu et al. [20]. Also, the relations of these sets with each other are established with the help of some examples. Moreover, the concept of r-fuzzy soft δ-connected sets is introduced and characterized with the help of fuzzy soft δ-closure operators.

• In Section 3, we define the concepts of fuzzy soft δ-continuous (semi-continuous and pre-continuous) functions, which are weaker forms of fuzzy soft continuity [20]. Some properties of these functions along with their mutual relationships are discussed. Also, a decomposition of fuzzy soft semi-continuity is obtained.

• In Section 4, as a weaker form of a fuzzy soft continuity, the concepts of fuzzy soft almost (weakly) continuous functions are introduced and some properties are specified. Also, we show that fuzzy soft continuity $\Rightarrow$ fuzzy soft almost continuity $\Rightarrow$ fuzzy soft weakly continuity, but the converse may not be true. In addition, we explore the notion of continuity in a very general setting called fuzzy soft $(\mathcal{L}, \mathcal{M}, \mathcal{N}, \mathcal{O})$-continuous functions and a historical justification is introduced.

• Finally, we close this paper with some conclusions and make a plan to suggest some future works in Section 5.

Throughout this article, nonempty sets will be denoted by U, V, etc. E is the set of all parameters for U and $A \subseteq E$. The family of all fuzzy sets on U is denoted by I^U (where

$I_\circ = (0, 1]$, $I = [0, 1]$), and for $t \in I$, $\underline{t}(u) = t$, for all $u \in U$. The following definitions will be used in the next sections:

Definition 1. *[1, 13, 20]* A fuzzy soft set f_A on U is a function from E to I^U such that $f_A(e)$ is a fuzzy set on U, for each $e \in A$ and $f_A(e) = \underline{0}$, if $e \notin A$. The family of all fuzzy soft sets on U is denoted by $\widetilde{(U, E)}$.

Definition 2. *[24]* A fuzzy soft point e_{u_t} on U is a fuzzy soft set defined as follows:

$$e_{u_t}(k) = \begin{cases} u_t, & \text{if} \quad k = e, \\ \underline{0}, & \text{if} \quad k \in E - \{e\}, \end{cases}$$

where u_t is a fuzzy point on U. e_{u_t} is said to belong to a fuzzy soft set f_A, denoted by $e_{u_t} \tilde{\in} f_A$, if $t \leq f_A(e)(u)$. The family of all fuzzy soft points on U is denoted by $\widetilde{P_t(U)}$.

Definition 3. *[12]* A fuzzy soft point $e_{u_t} \in \widetilde{P_t(U)}$ is called a soft quasi-coincident with $f_A \in \widetilde{(U, E)}$ and denoted by $e_{u_t} \widetilde{q} f_A$, if $t + f_A(e)(u) > 1$. A fuzzy soft set $f_A \in \widetilde{(U, E)}$ is called a soft quasi-coincident with $g_B \in \widetilde{(U, E)}$ and denoted by $f_A \widetilde{q} g_B$, if there is $e \in E$ and $u \in U$, such that $f_A(e)(u) + g_B(e)(u) > 1$. If f_A is not soft quasi-coincident with g_B, $f_A \widetilde{\not{q}} g_B$.

Definition 4. *[20]* A function $\tau : E \longrightarrow [0, 1]^{\widetilde{(U,E)}}$ is called a fuzzy soft topology on U if it satisfies the following conditions for every $e \in E$,

(i) $\tau_e(\Phi) = \tau_e(\widetilde{E}) = 1$,

(ii) $\tau_e(f_A \sqcap g_B) \geq \tau_e(f_A) \wedge \tau_e(g_B)$, for every $f_A, g_B \in \widetilde{(U, E)}$,

(iii) $\tau_e(\bigsqcup_{\delta \in \Delta} (f_A)_\delta) \geq \bigwedge_{\delta \in \Delta} \tau_e((f_A)_\delta)$, for every $(f_A)_\delta \in \widetilde{(U, E)}, \delta \in \Delta$.

Thus, (U, τ_E) is called a fuzzy soft topological space (FSTS) in Šostaks sense [41].

Definition 5. *[20]* Let (U, τ_E) and (V, τ^*_F) be an FSTSs. A fuzzy soft function $\varphi_\psi : \widetilde{(U, E)} \longrightarrow \widetilde{(V, F)}$ is said to be fuzzy soft continuous if $\tau_e(\varphi_\psi^{-1}(g_B)) \geq \tau^*_k(g_B)$ for every $g_B \in \widetilde{(V, F)}$, $e \in E$, and $(k = \psi(e)) \in F$.

Definition 6. *[14, 15]* In an FSTS (U, τ_E), for each $f_A \in \widetilde{(U, E)}$, $e \in E$, and $r \in I_0$, we define the fuzzy soft operators C_τ and $I_\tau : E \times \widetilde{(U, E)} \times I_\circ \to \widetilde{(U, E)}$ as follows:

$C_\tau(e, f_A, r) = \sqcap \{g_B \in \widetilde{(U, E)} : f_A \sqsubseteq g_B,\ \tau_e(g^c_B) \geq r\}$.

$I_\tau(e, f_A, r) = \sqcup \{g_B \in \widetilde{(U, E)} : g_B \sqsubseteq f_A,\ \tau_e(g_B) \geq r\}$.

Definition 7. Let (U, τ_E) be an FSTS and $r \in I_0$. A fuzzy soft set $f_A \in \widetilde{(U,E)}$ is said to be r-fuzzy soft regularly open [14] (pre-open [33] and β-open [33]) if $f_A = I_\tau(e, C_\tau(e, f_A, r), r)$ ($f_A \sqsubseteq I_\tau(e, C_\tau(e, f_A, r), r)$ and $f_A \sqsubseteq C_\tau(e, I_\tau(e, C_\tau(e, f_A, r), r), r)$) for every $e \in E$.

Lemma 1. *[33]* Every r-fuzzy soft regularly open set is r-fuzzy soft pre-open.

In general, the converse of Lemma 1 is not true, as shown by Example 1.

Example 1. *[5]* Let $U = \{u_1, u_2\}$, $E = \{e, k\}$, and define $g_E, f_E \in \widetilde{(U,E)}$ as follows: $g_E = \{(e, \{\frac{u_1}{0.3}, \frac{u_2}{0.4}\}), (k, \{\frac{u_1}{0.3}, \frac{u_2}{0.4}\})\}$, $f_E = \{(e, \{\frac{u_1}{0.6}, \frac{u_2}{0.2}\}), (k, \{\frac{u_1}{0.6}, \frac{u_2}{0.2}\})\}$. Define fuzzy soft topology $\tau_E : E \longrightarrow [0,1]^{\widetilde{(U,E)}}$ as follows:

$$\tau_e(m_E) = \begin{cases} 1, & \text{if} \quad m_E \in \{\Phi, \widetilde{E}\}, \\ \frac{1}{4}, & \text{if} \quad m_E = g_E, \\ \frac{1}{3}, & \text{if} \quad m_E = f_E, \\ \frac{1}{3}, & \text{if} \quad m_E = g_E \sqcap f_E, \\ \frac{1}{4}, & \text{if} \quad m_E = g_E \sqcup f_E, \\ 0, & \text{otherwise}, \end{cases} \qquad \tau_k(m_E) = \begin{cases} 1, & \text{if} \quad m_E \in \{\Phi, \widetilde{E}\}, \\ \frac{1}{4}, & \text{if} \quad m_E = g_E, \\ \frac{1}{2}, & \text{if} \quad m_E = f_E, \\ \frac{1}{2}, & \text{if} \quad m_E = g_E \sqcap f_E, \\ \frac{1}{4}, & \text{if} \quad m_E = g_E \sqcup f_E, \\ 0, & \text{otherwise}. \end{cases}$$

Thus, f_E is $\frac{1}{4}$-fuzzy soft pre-open set, but it is not $\frac{1}{4}$-fuzzy soft regularly open set.

The basic definitions and results that we need in the next sections are found in [15, 20].

2. On r-fuzzy soft δ-open sets

Here, we are going to give the concepts of r-fuzzy soft δ-open (semi-open) sets in an FSTS. Some properties of these sets along with their mutual relationships are investigated with the help of some examples. Also, the concept of an r-fuzzy soft δ-connected set is defined and studied with the help of fuzzy soft δ-closure operators.

Definition 8. Let (U, τ_E) be an FSTS. A fuzzy soft set $f_A \in \widetilde{(U,E)}$ is said to be an r-fuzzy soft δ-open (resp., semi-open and α-open [8]) if $I_\tau(e, C_\tau(e, f_A, r), r) \sqsubseteq C_\tau(e, I_\tau(e, f_A, r), r)$ (resp., $f_A \sqsubseteq C_\tau(e, I_\tau(e, f_A, r), r)$ and $f_A \sqsubseteq I_\tau(e, C_\tau(e, I_\tau(e, f_A, r), r), r)$) for every $e \in E$ and $r \in I_0$.

Remark 1. The concepts of an r-fuzzy soft δ-open set and r-fuzzy soft β-open set [33] are independent concepts, as shown by Examples 2 and 3.

Example 2. Let $U = \{u_1, u_2\}$, $E = \{e, k\}$, and define $h_E, g_E, f_E \in \widetilde{(U,E)}$ as follows: $h_E = \{(e, \{\frac{u_1}{0.4}, \frac{u_2}{0.5}\}), (k, \{\frac{u_1}{0.4}, \frac{u_2}{0.5}\})\}$, $g_E = \{(e, \{\frac{u_1}{0.2}, \frac{u_2}{0.3}\}), (k, \{\frac{u_1}{0.2}, \frac{u_2}{0.3}\})\}$, $f_E = \{(e, \{\frac{u_1}{0.8}, \frac{u_2}{0.7}\}), (k, \{\frac{u_1}{0.8}, \frac{u_2}{0.7}\})\}$. Define fuzzy soft topology $\tau_E : E \longrightarrow [0,1]^{\widetilde{(U,E)}}$ as follows:

$$\tau_e(m_E)=\begin{cases}1, & \text{if} \quad m_E\in\{\Phi,\widetilde{E}\},\\ \frac{1}{3}, & \text{if} \quad m_E=g_E,\\ \frac{2}{3}, & \text{if} \quad m_E=f_E,\\ 0, & \text{otherwise,}\end{cases} \qquad \tau_k(m_E)=\begin{cases}1, & \text{if} \quad m_E\in\{\Phi,\widetilde{E}\},\\ \frac{1}{3}, & \text{if} \quad m_E=g_E,\\ \frac{1}{2}, & \text{if} \quad m_E=f_E,\\ 0, & \text{otherwise.}\end{cases}$$

Thus, h_E is $\frac{1}{3}$-fuzzy soft β-open set, but it is neither $\frac{1}{3}$-fuzzy soft δ-open nor $\frac{1}{3}$-fuzzy soft semi-open.

Example 3. Let $U=\{u_1,u_2,u_3\}$, $E=\{e,k\}$, and define $h_E,g_E,f_E\in\widetilde{(U,E)}$ as follows: $h_E=\{(e,\{\frac{u_1}{0},\frac{u_2}{1},\frac{u_3}{1}\}),(k,\{\frac{u_1}{0},\frac{u_2}{1},\frac{u_3}{1}\})\}$, $g_E=\{(e,\{\frac{u_1}{0},\frac{u_2}{0},\frac{u_3}{1}\}),(k,\{\frac{u_1}{0},\frac{u_2}{0},\frac{u_3}{1}\})\}$, $f_E=\{(e,\{\frac{u_1}{0},\frac{u_2}{1},\frac{u_3}{0}\}),(k,\{\frac{u_1}{0},\frac{u_2}{1},\frac{u_3}{0}\})\}$. Define fuzzy soft topology $\tau_E:E\longrightarrow[0,1]^{\widetilde{(U,E)}}$ as follows:

$$\tau_e(m_E)=\begin{cases}1, & \text{if} \quad m_E\in\{\Phi,\widetilde{E}\},\\ \frac{1}{4}, & \text{if} \quad m_E=g_E,\\ \frac{1}{2}, & \text{if} \quad m_E=f_E,\\ \frac{1}{3}, & \text{if} \quad m_E=h_E,\\ 0, & \text{otherwise,}\end{cases} \qquad \tau_k(m_E)=\begin{cases}1, & \text{if} \quad m_E\in\{\Phi,\widetilde{E}\},\\ \frac{1}{3}, & \text{if} \quad m_E=g_E,\\ \frac{1}{2}, & \text{if} \quad m_E=f_E,\\ \frac{1}{3}, & \text{if} \quad m_E=h_E,\\ 0, & \text{otherwise.}\end{cases}$$

Thus, h^c_E is $\frac{1}{4}$-fuzzy soft δ-open set, but it is neither $\frac{1}{4}$-fuzzy soft β-open nor $\frac{1}{4}$-fuzzy soft semi-open.

Remark 2. The complement of an r-fuzzy soft δ-open (resp., semi-open, α-open and β-open) set is said to be an r-fuzzy soft δ-closed (resp., semi-closed, α-closed and β-closed).

Proposition 1. Let (U,τ_E) be an FSTS, $f_A\in\widetilde{(U,E)}$, $e\in E$, and $r\in I_0$. The following statements are equivalent:

(i) f_A is an r-fuzzy soft semi-open.

(ii) f_A is an r-fuzzy soft δ-open and r-fuzzy soft β-open.

Proof.

(i) $\Rightarrow$ (ii) Let f_A be an r-fuzzy soft semi-open, then $f_A\sqsubseteq C_\tau(e,I_\tau(e,f_A,r),r)\sqsubseteq C_\tau(e,I_\tau(C_\tau(e,f_A,r),r),r)$. This shows that f_A is r-fuzzy soft β-open. Moreover,

$$I_\tau(e,C_\tau(e,f_A,r),r)\sqsubseteq C_\tau(e,f_A,r)\sqsubseteq C_\tau(e,C_\tau(e,I_\tau(e,f_A,r),r),r)=C_\tau(e,I_\tau(e,f_A,r),r).$$

Therefore, f_A is r-fuzzy soft δ-open.

(ii) $\Rightarrow$ (i) Let f_A be an r-fuzzy soft δ-open and r-fuzzy soft β-open, then $I_\tau(e, C_\tau(e, f_A, r), r) \sqsubseteq C_\tau(e, I_\tau(e, f_A, r), r)$ and $f_A \sqsubseteq C_\tau(e, I_\tau(e, C_\tau(e, f_A, r), r), r)$. Thus,

$$f_A \sqsubseteq C_\tau(e, I_\tau(e, C_\tau(e, f_A, r), r), r) \sqsubseteq C_\tau(e, C_\tau(e, I_\tau(e, f_A, r), r), r) = C_\tau(e, I_\tau(e, f_A, r), r).$$

This shows that f_A is r-fuzzy soft semi-open.

Proposition 2. Let (U, τ_E) be an FSTS, $f_A \in \widetilde{(U, E)}$, $e \in E$, and $r \in I_0$. The following statements are equivalent:

(i) f_A is an r-fuzzy soft α-open.

(ii) f_A is an r-fuzzy soft δ-open and r-fuzzy soft pre-open.

Proof. (i) $\Rightarrow$ (ii) From Proposition 1 the proof is straightforward.

(ii) $\Rightarrow$ (i) Let f_A be an r-fuzzy soft pre-open and r-fuzzy soft δ-open. Then, $f_A \sqsubseteq I_\tau(e, C_\tau(e, f_A, r), r) \sqsubseteq I_\tau(e, C_\tau(e, I_\tau(e, f_A, r), r), r)$. This shows that f_A is r-fuzzy soft α-open.

Remark 3. From the previous definitions and results, we can summarize the relationships among different types of fuzzy soft open sets as in the next diagram.

$$\begin{tikzcd}
\textbf{fuzzy soft } \alpha\textbf{-open set} \arrow[d] \arrow[dr] & & \\
\textbf{fuzzy soft pre-open set} \arrow[r, phantom, "\nleftrightarrow"] \arrow[d] & \textbf{fuzzy soft semi-open set} \arrow[r] \arrow[dl] & \textbf{fuzzy soft } \delta\textbf{-open set} \\
\textbf{fuzzy soft } \beta\textbf{-open set} & &
\end{tikzcd}$$

Remark 4. In general, the converses of the above relationships are not true, as shown by Examples 2, 3, 4, 5, and 6.

Example 4. Let $U = \{u_1, u_2\}$, $E = \{e, k\}$, and define $g_E, f_E, h_E \in \widetilde{(U, E)}$ as follows: $g_E = \{(e, \{\frac{u_1}{0.3}, \frac{u_2}{0.4}\}), (k, \{\frac{u_1}{0.3}, \frac{u_2}{0.4}\})\}$, $f_E = \{(e, \{\frac{u_1}{0.6}, \frac{u_2}{0.2}\}), (k, \{\frac{u_1}{0.6}, \frac{u_2}{0.2}\})\}$, $h_E = \{(e, \{\frac{u_1}{0.7}, \frac{u_2}{0.5}\}), (k, \{\frac{u_1}{0.7}, \frac{u_2}{0.5}\})\}$. Define fuzzy soft topology $\tau_E : E \longrightarrow [0, 1]^{\widetilde{(U, E)}}$ as follows:

$$\tau_e(m_E) = \begin{cases} 1, & \text{if} \quad m_E \in \{\Phi, \widetilde{E}\}, \\ \frac{1}{2}, & \text{if} \quad m_E = g_E, \\ \frac{2}{3}, & \text{if} \quad m_E = f_E, \\ \frac{2}{3}, & \text{if} \quad m_E = g_E \sqcap f_E, \\ \frac{1}{2}, & \text{if} \quad m_E = g_E \sqcup f_E, \\ 0, & \text{otherwise}, \end{cases} \qquad \tau_k(m_E) = \begin{cases} 1, & \text{if} \quad m_E \in \{\Phi, \widetilde{E}\}, \\ \frac{1}{3}, & \text{if} \quad m_E = g_E, \\ \frac{1}{2}, & \text{if} \quad m_E = f_E, \\ \frac{1}{2}, & \text{if} \quad m_E = g_E \sqcap f_E, \\ \frac{1}{3}, & \text{if} \quad m_E = g_E \sqcup f_E, \\ 0, & \text{otherwise}. \end{cases}$$

Thus, h_E is $\frac{1}{3}$-fuzzy soft semi-open set, but it is neither $\frac{1}{3}$-fuzzy soft α-open nor $\frac{1}{3}$-fuzzy soft pre-open.

Example 5. Let $U = \{u_1, u_2, u_3\}$, $E = \{e, k\}$, and define $g_E, f_E \in \widetilde{(U, E)}$ as follows: $g_E = \{(e, \{\frac{u_1}{0.2}, \frac{u_2}{0.3}, \frac{u_3}{0.2}\}), (k, \{\frac{u_1}{0.2}, \frac{u_2}{0.3}, \frac{u_3}{0.2}\})\}$, $f_E = \{(e, \{\frac{u_1}{0.3}, \frac{u_2}{0.4}, \frac{u_3}{0.8}\}), (k, \{\frac{u_1}{0.3}, \frac{u_2}{0.4}, \frac{u_3}{0.8}\})\}$. Define fuzzy soft topology $\tau_E : E \longrightarrow [0, 1]^{\widetilde{(U,E)}}$ as follows:

$$\tau_e(m_E) = \begin{cases} 1, & \text{if} \quad m_E \in \{\Phi, \widetilde{E}\}, \\ \frac{1}{2}, & \text{if} \quad m_E = g_E, \\ 0, & \text{otherwise}, \end{cases} \qquad \tau_k(m_E) = \begin{cases} 1, & \text{if} \quad m_E \in \{\Phi, \widetilde{E}\}, \\ \frac{1}{3}, & \text{if} \quad m_E = g_E, \\ 0, & \text{otherwise}. \end{cases}$$

Thus, f_E is $\frac{1}{3}$-fuzzy soft β-open set, but it is not $\frac{1}{3}$-fuzzy soft pre-open.

Example 6. Let $U = \{u_1, u_2\}$, $E = \{e, k\}$, and define $g_E, f_E \in \widetilde{(U, E)}$ as follows: $g_E = \{(e, \{\frac{u_1}{0.4}, \frac{u_2}{0.5}\}), (k, \{\frac{u_1}{0.4}, \frac{u_2}{0.5}\})\}$, $f_E = \{(e, \{\frac{u_1}{0.3}, \frac{u_2}{0.4}\}), (k, \{\frac{u_1}{0.3}, \frac{u_2}{0.4}\})\}$. Define fuzzy soft topology $\tau_E : E \longrightarrow [0, 1]^{\widetilde{(U,E)}}$ as follows:

$$\tau_e(m_E) = \begin{cases} 1, & \text{if} \quad m_E \in \{\Phi, \widetilde{E}\}, \\ \frac{1}{4}, & \text{if} \quad m_E = g_E, \\ 0, & \text{otherwise}, \end{cases} \qquad \tau_k(m_E) = \begin{cases} 1, & \text{if} \quad m_E \in \{\Phi, \widetilde{E}\}, \\ \frac{1}{2}, & \text{if} \quad m_E = g_E, \\ 0, & \text{otherwise}. \end{cases}$$

Thus, f_E is $\frac{1}{4}$-fuzzy soft pre-open set, but it is neither $\frac{1}{4}$-fuzzy soft α-open nor $\frac{1}{4}$-fuzzy soft semi-open.

Theorem 1. Let (U, τ_E) be an FSTS, $f_A, g_B \in \widetilde{(U, E)}$, $e \in E$, and $r \in I_0$. If f_A is an r-fuzzy soft δ-open set such that $f_A \sqsubseteq g_B \sqsubseteq C_\tau(e, f_A, r)$, then g_B is also r-fuzzy soft δ-open.

Proof. Suppose that an f_A is r-fuzzy soft δ-open and $f_A \sqsubseteq g_B \sqsubseteq C_\tau(e, f_A, r)$. Then, $I_\tau(e, C_\tau(e, f_A, r), r) \sqsubseteq C_\tau(e, I_\tau(e, f_A, r), r) \sqsubseteq C_\tau(e, I_\tau(e, g_B, r), r)$. Since $g_B \sqsubseteq C_\tau(e, f_A, r)$, $I_\tau(e, C_\tau(e, g_B, r), r) \sqsubseteq I_\tau(e, C_\tau(e, f_A, r), r) \sqsubseteq C_\tau(e, I_\tau(e, g_B, r), r)$. This shows that g_B is r-fuzzy soft δ-open.

Definition 9. In an FSTS (U, τ_E), for each $f_A \in \widetilde{(U,E)}$, $e \in E$, and $r \in I_0$, we define a fuzzy soft δ-closure operator $\delta C_\tau : E \times \widetilde{(U,E)} \times I_\circ \to \widetilde{(U,E)}$ as follows: $\delta C_\tau(e, f_A, r) = \sqcap \{g_B \in \widetilde{(U,E)} : f_A \sqsubseteq g_B,\ g_B \text{ is } r\text{-fuzzy soft } \delta\text{-closed}\}$.

Theorem 2. In an FSTS (U, τ_E), for each $f_A, g_B \in \widetilde{(U,E)}$, $e \in E$, and $r \in I_0$, the operator $\delta C_\tau : E \times \widetilde{(U,E)} \times I_\circ \to \widetilde{(U,E)}$ satisfies the following properties.

(1) $\delta C_\tau(e, \Phi, r) = \Phi$.

(2) $f_A \sqsubseteq \delta C_\tau(e, f_A, r) \sqsubseteq C_\tau(e, f_A, r)$.

(3) $\delta C_\tau(e, f_A, r) \sqsubseteq \delta C_\tau(e, g_B, r)$ if $f_A \sqsubseteq g_B$.

(4) $\delta C_\tau(e, \delta C_\tau(e, f_A, r), r) = \delta C_\tau(e, f_A, r)$.

(5) $\delta C_\tau(e, f_A \sqcup g_B, r) \sqsupseteq \delta C_\tau(e, f_A, r) \sqcup \delta C_\tau(e, g_B, r)$.

(6) $\delta C_\tau(e, f_A, r) = f_A$ iff f_A is r-fuzzy soft δ-closed.

(7) $\delta C_\tau(e, C_\tau(e, f_A, r), r) = C_\tau(e, f_A, r)$.

Proof. (1), (2), (3), and (6) are easily proved from Definition 9.

(4) From (2) and (3), $\delta C_\tau(e, f_A, r) \sqsubseteq \delta C_\tau(e, \delta C_\tau(e, f_A, r), r)$.
Now, we show that $\delta C_\tau(e, f_A, r) \sqsupseteq \delta C_\tau(e, \delta C_\tau(e, f_A, r), r)$. Suppose that $\delta C_\tau(e, f_A, r)$ does not contain $\delta C_\tau(e, \delta C_\tau(e, f_A, r), r)$, then there is $u \in U$ and $t \in (0, 1)$ such that

$$\delta C_\tau(e, f_A, r)(e)(u) < t < \delta C_\tau(e, \delta C_\tau(e, f_A, r), r)(e)(u). \qquad (A)$$

Since $\delta C_\tau(e, f_A, r)(e)(u) < t$, by the definition of δC_τ, there is g_B as a r-fuzzy soft δ-closed with $f_A \sqsubseteq g_B$ such that $\delta C_\tau(e, f_A, r)(e)(u) \leq g_B(e)(u) < t$. Since $f_A \sqsubseteq g_B$, then $\delta C_\tau(e, f_A, r) \sqsubseteq g_B$. Again, by the definition of δC_τ, we have $\delta C_\tau(e, \delta C_\tau(e, f_A, r), r) \sqsubseteq g_B$. Hence, $\delta C_\tau(e, \delta C_\tau(e, f_A, r), r)(e)(u) \leq g_B(e)(u) < t$, which is a contradiction for (A). Thus, $\delta C_\tau(e, f_A, r) \sqsupseteq \delta C_\tau(e, \delta C_\tau(e, f_A, r), r)$, then $\delta C_\tau(e, \delta C_\tau(e, f_A, r), r) = \delta C_\tau(e, f_A, r)$.

(5) Since f_A and $g_B \sqsubseteq f_A \sqcup g_B$, hence by (3), $\delta C_\tau(e, f_A, r) \sqsubseteq \delta C_\tau(e, f_A \sqcup g_B, r)$ and $\delta C_\tau(e, g_B, r) \sqsubseteq \delta C_\tau(e, f_A \sqcup g_B, r)$. Thus, $\delta C_\tau(e, f_A \sqcup g_B, r) \sqsupseteq \delta C_\tau(e, f_A, r) \sqcup \delta C_\tau(e, g_B, r)$.

(7) From (6) and $C_\tau(e, f_A, r)$ is r-fuzzy soft δ-closed set, hence $\delta C_\tau(e, C_\tau(e, f_A, r), r) = C_\tau(e, f_A, r)$.

Theorem 3. In an FSTS (U, τ_E), for each $f_A \in \widetilde{(U,E)}$, $e \in E$, and $r \in I_0$, we define a fuzzy soft δ-interior operator $\delta I_\tau : E \times \widetilde{(U,E)} \times I_\circ \to \widetilde{(U,E)}$ as follows: $\delta I_\tau(e, f_A, r) = \sqcup \{g_B \in \widetilde{(U,E)} : g_B \sqsubseteq f_A,\ g_B$ is r-fuzzy soft δ-open$\}$. Then, for each f_A and $g_B \in \widetilde{(U,E)}$, the operator δI_τ satisfies the following properties.

(1) $\delta I_\tau(e, \widetilde{E}, r) = \widetilde{E}$.

(2) $I_\tau(e, f_A, r) \sqsubseteq \delta I_\tau(e, f_A, r) \sqsubseteq f_A$.

(3) $\delta I_\tau(e, f_A, r) \sqsubseteq \delta I_\tau(e, g_B, r)$ if $f_A \sqsubseteq g_B$.

(4) $\delta I_\tau(e, \delta I_\tau(e, f_A, r), r) = \delta I_\tau(e, f_A, r)$.

(5) $\delta I_\tau(e, f_A, r) \sqcap \delta I_\tau(e, g_B, r) \sqsupseteq \delta I_\tau(e, f_A \sqcap g_B, r)$.

(6) $\delta I_\tau(e, f_A, r) = f_A$ iff f_A is r-fuzzy soft δ-open.

(7) $\delta I_\tau(e, f_A^c, r) = (\delta C_\tau(e, f_A, r))^c$.

Proof. (1), (2), (3), and (6) are easily proved from the definition of δI_τ.

(4) and (5) are easily proved by a similar way in Theorem 2.

(7) For each $f_A \in \widetilde{(U,E)}$, $e \in E$, and $r \in I_0$, we have $\delta I_\tau(e, f_A^c, r) = \sqcup\{g_B \in \widetilde{(U,E)} : g_B \sqsubseteq f_A^c,\ g_B$ is r-fuzzy soft δ-open$\} = [\sqcap\{g_B^c \in \widetilde{(U,E)} : f_A \sqsubseteq g_B^c,\ g_B^c$ is r-fuzzy soft δ-closed$\}]^c$ $= (\delta C_\tau(e, f_A, r))^c$.

Definition 10. Let (U, τ_E) be an FSTS, $r \in I_0$, and $f_A, g_B \in \widetilde{(U,E)}$, then we have:

(1) Two fuzzy soft sets f_A and g_B are called r-fuzzy soft δ-separated iff $g_B \,\widetilde{\not{q}}\, \delta C_\tau(e, f_A, r)$ and $f_A \,\widetilde{\not{q}}\, \delta C_\tau(e, g_B, r)$ for each $e \in E$.

(2) Any fuzzy soft set which cannot be expressed as the union of two r-fuzzy soft δ-separated sets is called an r-fuzzy soft δ-connected.

Theorem 4. In an FSTS (U, τ_E), we have:

(1) If f_A and $g_B \in \widetilde{(U,E)}$ are r-fuzzy soft δ-separated and h_C, $t_D \in \widetilde{(U,E)}$ such that $h_C \sqsubseteq f_A$ and $t_D \sqsubseteq g_B$, then h_C and t_D are r-fuzzy soft δ-separated.

(2) If $f_A \widetilde{\not q}\, g_B$ and either both are r-fuzzy soft δ-open or both r-fuzzy soft δ-closed, then f_A and g_B are r-fuzzy soft δ-separated.

(3) If f_A and g_B are either both r-fuzzy soft δ-open or both r-fuzzy soft δ-closed, then $f_A \sqcap g^c_B$ and $g_B \sqcap f^c_A$ are r-fuzzy soft δ-separated.

Proof. (1) and (2) are obvious.

(3) Let f_A and g_B be an r-fuzzy soft δ-open. Since $f_A \sqcap g^c_B \sqsubseteq g^c_B$, $\delta C_\tau(e, f_A \sqcap g^c_B, r) \sqsubseteq g^c_B$ and hence $\delta C_\tau(e, f_A \sqcap g^c_B, r) \widetilde{\not q}\, g_B$. Then, $\delta C_\tau(e, f_A \sqcap g^c_B\ , r) \widetilde{\not q}\, (g_B \sqcap f^c_A)$.

Again, since $g_B \sqcap f^c_A \sqsubseteq f^c_A$, $\delta C_\tau(e, g_B \sqcap f^c_A, r) \sqsubseteq f^c_A$ and hence $\delta C_\tau(e, g_B \sqcap f^c_A, r) \widetilde{\not q}\, f_A$. Then, $\delta C_\tau(e, g_B \sqcap f^c_A, r) \widetilde{\not q}\, (f_A \sqcap g^c_B)$. Thus, $f_A \sqcap g^c_B$ and $g_B \sqcap f^c_A$ are r-fuzzy soft δ-separated. The other case follows similar lines.

Theorem 5. In an FSTS (U, τ_E), then f_A, $g_B \in \widetilde{(U, E)}$ are r-fuzzy soft δ-separated iff there exist two r-fuzzy soft δ-open sets h_C and t_D such that $f_A \sqsubseteq h_C$, $g_B \sqsubseteq t_D$, $f_A \widetilde{\not q}\, t_D$ and $g_B \widetilde{\not q}\, h_C$.

Proof. ($\Rightarrow$) Let f_A and $g_B \in \widetilde{(U, E)}$ be an r-fuzzy soft δ-separated, $f_A \sqsubseteq (\delta C_\tau(e, g_B, r))^c = h_C$ and $g_B \sqsubseteq (\delta C_\tau(e, f_A, r))^c = t_D$, where t_D and h_C are r-fuzzy soft δ-open, then $t_D \widetilde{\not q}\ \delta C_\tau(e, f_A, r)$ and $h_C \widetilde{\not q}\ \delta C_\tau(e, g_B, r)$. Thus, $g_B \widetilde{\not q}\ h_C$ and $f_A \widetilde{\not q}\ t_D$. Hence, we obtain the required result.

($\Leftarrow$) Let h_C and t_D be an r-fuzzy soft δ-open such that $g_B \sqsubseteq t_D$, $f_A \sqsubseteq h_C$, $g_B \widetilde{\not q}\, h_C$ and $f_A \widetilde{\not q}\, t_D$. Then, $g_B \sqsubseteq h^c_C$ and $f_A \sqsubseteq t^c_D$. Hence, $\delta C_\tau(e, g_B, r) \sqsubseteq h^c_C$ and $\delta C_\tau(e, f_A, r) \sqsubseteq t^c_D$. Then, $\delta C_\tau(e, g_B, r) \widetilde{\not q}\ f_A$ and $\delta C_\tau(e, f_A, r) \widetilde{\not q}\ g_B$. Thus, f_A and g_B are r- fuzzy soft δ-separated. Hence, we obtain the required result.

Theorem 6. In an FSTS (U, τ_E), if $g_B \in \widetilde{(U, E)}$ is r-fuzzy soft δ-connected such that $g_B \sqsubseteq f_A \sqsubseteq \delta C_\tau(e, g_B, r)$, then f_A is r-fuzzy soft δ-connected.

Proof. Suppose that f_A is not r-fuzzy soft δ-connected, then there is r-fuzzy soft δ-separated sets h^*_C and $t^*_D \in \widetilde{(U, E)}$ such that $f_A = h^*_C \sqcup t^*_D$. Let $h_C = g_B \sqcap h^*_C$ and $t_D = g_B \sqcap t^*_D$, then $g_B = t_D \sqcup h_C$. Since $h_C \sqsubseteq h^*_C$ and $t_D \sqsubseteq t^*_D$, hence by Theorem 4(1), h_C and t_D are r-fuzzy soft δ-separated, it is a contradiction. Thus, f_A is r-fuzzy soft δ-connected, as required.

3. A decomposition of fuzzy soft semi-continuity

Here, we introduce the concepts of fuzzy soft δ-continuous (semi-continuous and pre-continuous) functions, which are weaker forms of fuzzy soft continuity in an FSTSs in Šostaks sense. Also, we study several relationships related to fuzzy soft δ-continuity with the help of some problems. A decomposition of fuzzy soft semi-continuity is obtained.

Definition 11. Let (U, τ_E) and (V, τ^*_F) be an FSTSs. A fuzzy soft function $\varphi_\psi : \widetilde{(U,E)} \longrightarrow \widetilde{(V,F)}$ is said to be a fuzzy soft δ-continuous (resp., β-continuous [9], semi-continuous, pre-continuous, and α-continuous [8]) if $\varphi_\psi^{-1}(g_B)$ is r-fuzzy soft δ-open (resp., β-open, semi-open, pre-open, and α-open) set for every $g_B \in \widetilde{(V,F)}$ with $\tau^*_k(g_B) \geq r$, $e \in E$, $(k = \psi(e)) \in F$, and $r \in I_o$.

Remark 5. Fuzzy soft δ-continuity and fuzzy soft β-continuity are independent concepts, as shown by Examples 7 and 8.

Example 7. Let $U = \{u_1, u_2\}$, $E = \{e_1, e_2\}$, and define $h_E, g_E, f_E \in \widetilde{(U,E)}$ as follows: $h_E = \{(e_1, \{\frac{u_1}{0.4}, \frac{u_2}{0.5}\}), (e_2, \{\frac{u_1}{0.4}, \frac{u_2}{0.5}\})\}$, $g_E = \{(e_1, \{\frac{u_1}{0.2}, \frac{u_2}{0.3}\}), (e_2, \{\frac{u_1}{0.2}, \frac{u_2}{0.3}\})\}$, $f_E = \{(e_1, \{\frac{u_1}{0.8}, \frac{u_2}{0.7}\}), (e_2, \{\frac{u_1}{0.8}, \frac{u_2}{0.7}\})\}$. Define fuzzy soft topologies $\tau_E, \tau^*_E : E \longrightarrow [0,1]^{\widetilde{(U,E)}}$ as follows: $\forall e \in E$,

$$\tau_e(m_E) = \begin{cases} 1, & \text{if} \quad m_E \in \{\Phi, \widetilde{E}\}, \\ \frac{1}{3}, & \text{if} \quad m_E = g_E, \\ \frac{2}{3}, & \text{if} \quad m_E = f_E, \\ 0, & \text{otherwise,} \end{cases} \qquad \tau^*_e(m_E) = \begin{cases} 1, & \text{if} \quad m_E \in \{\Phi, \widetilde{E}\}, \\ \frac{1}{3}, & \text{if} \quad m_E = h_E, \\ 0, & \text{otherwise.} \end{cases}$$

Thus, the identity fuzzy soft function $\varphi_\psi : (U, \tau_E) \longrightarrow (U, \tau^*_E)$ is fuzzy soft β-continuous, but it is neither fuzzy soft δ-continuous nor fuzzy soft semi-continuous.

Example 8. Let $U = \{u_1, u_2, u_3\}$, $E = \{e_1, e_2\}$, and define $h_E, g_E, f_E \in \widetilde{(U,E)}$ as follows: $h_E = \{(e_1, \{\frac{u_1}{0}, \frac{u_2}{1}, \frac{u_3}{1}\}), (e_2, \{\frac{u_1}{0}, \frac{u_2}{1}, \frac{u_3}{1}\})\}$, $g_E = \{(e_1, \{\frac{u_1}{0}, \frac{u_2}{0}, \frac{u_3}{1}\}), (e_2, \{\frac{u_1}{0}, \frac{u_2}{0}, \frac{u_3}{1}\})\}$, $f_E = \{(e_1, \{\frac{u_1}{0}, \frac{u_2}{1}, \frac{u_3}{0}\}), (e_2, \{\frac{u_1}{0}, \frac{u_2}{1}, \frac{u_3}{0}\})\}$. Define fuzzy soft topologies $\tau_E, \tau^*_E : E \longrightarrow [0,1]^{\widetilde{(U,E)}}$ as follows: $\forall e \in E$,

$$\tau_e(m_E) = \begin{cases} 1, & \text{if} \quad m_E \in \{\Phi, \widetilde{E}\}, \\ \frac{1}{4}, & \text{if} \quad m_E = g_E, \\ \frac{1}{2}, & \text{if} \quad m_E = f_E, \\ \frac{1}{3}, & \text{if} \quad m_E = h_E, \\ 0, & \text{otherwise,} \end{cases} \qquad \tau^*_e(m_E) = \begin{cases} 1, & \text{if} \quad m_E \in \{\Phi, \widetilde{E}\}, \\ \frac{1}{4}, & \text{if} \quad m_E = h^c_E, \\ 0, & \text{otherwise.} \end{cases}$$

Thus, the identity fuzzy soft function $\varphi_\psi : (U, \tau_E) \longrightarrow (U, \tau^*_E)$ is fuzzy soft δ-continuous, but it is neither fuzzy soft β-continuous nor fuzzy soft semi-continuous.

Now, we have the following decomposition of fuzzy soft semi-continuity and decomposition of fuzzy soft α-continuity, according to Propositions 1 and 2.

Proposition 3. Let (U, τ_E) and (V, τ^*_F) be an FSTSs. $\varphi_\psi : \widetilde{(U, E)} \longrightarrow \widetilde{(V, F)}$ is fuzzy soft semi-continuous function iff it is both fuzzy soft δ-continuous and fuzzy soft β-continuous.

Proof. The proof is obvious by Proposition 1.

Proposition 4. Let (U, τ_E) and (V, τ^*_F) be an FSTSs. $\varphi_\psi : \widetilde{(U, E)} \longrightarrow \widetilde{(V, F)}$ is fuzzy soft α-continuous function iff it is both fuzzy soft δ-continuous and fuzzy soft pre-continuous.

Proof. The proof is obvious by Proposition 2.

Remark 6. From the previous definitions and results, we can summarize the relationships among different types of fuzzy soft continuity as in the next diagram.

$$\begin{array}{ccccc}
 & & \textbf{fuzzy soft continuity} & & \\
 & & \downarrow & & \\
 & & \textbf{fuzzy soft } \alpha\textbf{-continuity} & & \\
 & \swarrow & & \searrow & \\
\textbf{fuzzy soft pre-continuity} & \not\leftrightarrow & \textbf{fuzzy soft semi-continuity} & \rightarrow & \textbf{fuzzy soft } \delta\textbf{-continuity} \\
 & \searrow & & \swarrow & \\
 & & \textbf{fuzzy soft } \beta\textbf{-continuity} & &
\end{array}$$

Remark 7. In general, the converses of the above relationships are not true, as shown by Examples 7, 8, 9, 10, and 11.

Example 9. Let $U = \{u_1, u_2\}$, $E = \{e_1, e_2\}$, and define $g_E, f_E, h_E \in \widetilde{(U, E)}$ as follows: $g_E = \{(e_1, \{\frac{u_1}{0.3}, \frac{u_2}{0.4}\}), (e_2, \{\frac{u_1}{0.3}, \frac{u_2}{0.4}\})\}$, $f_E = \{(e_1, \{\frac{u_1}{0.6}, \frac{u_2}{0.2}\}), (e_2, \{\frac{u_1}{0.6}, \frac{u_2}{0.2}\})\}$, $h_E = \{(e_1, \{\frac{u_1}{0.7}, \frac{u_2}{0.5}\}), (e_2, \{\frac{u_1}{0.7}, \frac{u_2}{0.5}\})\}$. Define fuzzy soft topologies $\tau_E, \tau^*_E : E \longrightarrow [0, 1]^{\widetilde{(U,E)}}$ as follows: $\forall e \in E$,

$$\tau_e(m_E) = \begin{cases} 1, & \text{if} \quad m_E \in \{\Phi, \widetilde{E}\}, \\ \frac{1}{2}, & \text{if} \quad m_E = g_E, \\ \frac{2}{3}, & \text{if} \quad m_E = f_E, \\ \frac{2}{3}, & \text{if} \quad m_E = g_E \sqcap f_E, \\ \frac{1}{2}, & \text{if} \quad m_E = g_E \sqcup f_E, \\ 0, & \text{otherwise}, \end{cases} \qquad \tau^*_e(m_E) = \begin{cases} 1, & \text{if} \quad m_E \in \{\Phi, \widetilde{E}\}, \\ \frac{1}{3}, & \text{if} \quad m_E = h_E, \\ 0, & \text{otherwise}. \end{cases}$$

Thus, the identity fuzzy soft function $\varphi_\psi : (U, \tau_E) \longrightarrow (U, \tau^*_E)$ is fuzzy soft semi-continuous, but it is neither fuzzy soft α-continuous nor fuzzy soft pre-continuous.

Example 10. Let $U = \{u_1, u_2, u_3\}$, $E = \{e_1, e_2\}$, and define $g_E, f_E \in \widetilde{(U, E)}$ as follows: $g_E = \{(e_1, \{\frac{u_1}{0.2}, \frac{u_2}{0.3}, \frac{u_3}{0.2}\}), (e_2, \{\frac{u_1}{0.2}, \frac{u_2}{0.3}, \frac{u_3}{0.2}\})\}$, $f_E = \{(e_1, \{\frac{u_1}{0.3}, \frac{u_2}{0.4}, \frac{u_3}{0.8}\}), (e_2, \{\frac{u_1}{0.3}, \frac{u_2}{0.4}, \frac{u_3}{0.8}\})\}$. Define fuzzy soft topologies $\tau_E, \tau^*_E : E \longrightarrow [0,1]^{\widetilde{(U,E)}}$ as follows: $\forall e \in E$,

$$\tau_e(m_E) = \begin{cases} 1, & \text{if} \quad m_E \in \{\Phi, \widetilde{E}\}, \\ \frac{1}{2}, & \text{if} \quad m_E = g_E, \\ 0, & \text{otherwise,} \end{cases} \qquad \tau^*_e(m_E) = \begin{cases} 1, & \text{if} \quad m_E \in \{\Phi, \widetilde{E}\}, \\ \frac{1}{3}, & \text{if} \quad m_E = f_E, \\ 0, & \text{otherwise.} \end{cases}$$

Thus, the identity fuzzy soft function $\varphi_\psi : (U, \tau_E) \longrightarrow (U, \tau^*_E)$ is fuzzy soft β-continuous, but it is not fuzzy soft pre-continuous.

Example 11. Let $U = \{u_1, u_2\}$, $E = \{e_1, e_2\}$, and define $g_E, f_E \in \widetilde{(U, E)}$ as follows: $g_E = \{(e_1, \{\frac{u_1}{0.4}, \frac{u_2}{0.5}\}), (e_2, \{\frac{u_1}{0.4}, \frac{u_2}{0.5}\})\}$, $f_E = \{(e_1, \{\frac{u_1}{0.3}, \frac{u_2}{0.4}\}), (e_2, \{\frac{u_1}{0.3}, \frac{u_2}{0.4}\})\}$. Define fuzzy soft topologies $\tau_E, \tau^*_E : E \longrightarrow [0,1]^{\widetilde{(U,E)}}$ as follows: $\forall e \in E$,

$$\tau_e(m_E) = \begin{cases} 1, & \text{if} \quad m_E \in \{\Phi, \widetilde{E}\}, \\ \frac{1}{2}, & \text{if} \quad m_E = g_E, \\ 0, & \text{otherwise,} \end{cases} \qquad \tau^*_e(m_E) = \begin{cases} 1, & \text{if} \quad m_E \in \{\Phi, \widetilde{E}\}, \\ \frac{1}{4}, & \text{if} \quad m_E = f_E, \\ 0, & \text{otherwise.} \end{cases}$$

Thus, the identity fuzzy soft function $\varphi_\psi : (U, \tau_E) \longrightarrow (U, \tau^*_E)$ is fuzzy soft pre-continuous, but it is neither fuzzy soft α-continuous nor fuzzy soft semi-continuous.

Theorem 7. Let (U, τ_E) and (V, τ^*_F) be an FSTSs and $\varphi_\psi : \widetilde{(U, E)} \longrightarrow \widetilde{(V, F)}$ be a fuzzy soft function. The following statements are equivalent for every $g_B \in \widetilde{(V, F)}$, $e \in E$, $(k = \psi(e)) \in F$, and $r \in I_\circ$.

(i) φ_ψ is fuzzy soft β-continuous.

(ii) $I_\tau(e, C_\tau(e, I_\tau(e, \varphi^{-1}_\psi(g_B), r), r), r) \sqsubseteq \varphi^{-1}_\psi(g_B)$, if $\tau^*_k(g^c_B) \geq r$.

(iii) $I_\tau(e, C_\tau(e, I_\tau(e, \varphi^{-1}_\psi(g_B), r), r), r) \sqsubseteq \varphi^{-1}_\psi(C_{\tau^*}(k, g_B, r))$.

(iv) $\varphi^{-1}_\psi(I_{\tau^*}(k, g_B, r)) \sqsubseteq C_\tau(e, I_\tau(e, C_\tau(e, \varphi^{-1}_\psi(g_B), r), r), r)$.

Proof. (i) $\Rightarrow$ (ii) Let $g_B \in \widetilde{(V, F)}$ with $\tau^*_k(g^c_B) \geq r$. Then by Definition 11,

$$(\varphi^{-1}_\psi(g_B))^c = \varphi^{-1}_\psi(g^c_B) \sqsubseteq C_\tau(e, I_\tau(e, C_\tau(e, \varphi^{-1}_\psi(g^c_B), r), r), r) = (I_\tau(e, C_\tau(e, I_\tau(e, \varphi^{-1}_\psi(g_B), r), r), r))^c.$$

Thus, $I_\tau(e, C_\tau(e, I_\tau(e, \varphi_\psi^{-1}(g_B), r), r), r) \sqsubseteq \varphi_\psi^{-1}(g_B)$.

(ii) $\Rightarrow$ (iii) Obvious.

(iii) $\Rightarrow$ (iv) Since $(I_\tau(e, C_\tau(e, I_\tau(e, \varphi_\psi^{-1}(g_B), r), r), r))^c = C_\tau(e, I_\tau(e, C_\tau(e, \varphi_\psi^{-1}(g_B^c), r), r), r)$ and $(\varphi_\psi^{-1}(C_{\tau^*}(k, g_B, r)))^c = \varphi_\psi^{-1}(I_{\tau^*}(k, g_B^c, r))$. Then,

$$\varphi_\psi^{-1}(I_{\tau^*}(k, g_B, r)) \sqsubseteq C_\tau(e, I_\tau(e, C_\tau(e, \varphi_\psi^{-1}(g_B), r), r), r),$$

for each $g_B \in \widetilde{(V, F)}$.

(iv) $\Rightarrow$ (i) Let $g_B \in \widetilde{(V, F)}$ with $\tau_k^*(g_B) \geq r$. Then by (iv) and $g_B = I_{\tau^*}(k, g_B, r)$, $\varphi_\psi^{-1}(g_B) \sqsubseteq C_\tau(e, I_\tau(e, C_\tau(e, \varphi_\psi^{-1}(g_B), r), r), r)$. Thus, φ_ψ is fuzzy soft β-continuous.

The following theorem is similarly proved as in Theorem 7.

Theorem 8. Let (U, τ_E) and (V, τ_F^*) be an FSTSs and $\varphi_\psi : \widetilde{(U, E)} \longrightarrow \widetilde{(V, F)}$ be a fuzzy soft function. The following statements are equivalent for every $g_B \in \widetilde{(V, F)}$, $e \in E$, $(k = \psi(e)) \in F$, and $r \in I_\circ$.

(i) φ_ψ is fuzzy soft δ-continuous.

(ii) $I_\tau(e, C_\tau(e, \varphi_\psi^{-1}(g_B), r), r) \sqsubseteq C_\tau(e, I_\tau(e, \varphi_\psi^{-1}(g_B), r), r)$, if $\tau_k^*(g_B^c) \geq r$.

(iii) $I_\tau(e, C_\tau(e, \varphi_\psi^{-1}(g_B), r), r) \sqsubseteq C_\tau(e, I_\tau(e, \varphi_\psi^{-1}(C_{\tau^*}(k, g_B, r)), r), r)$

(iv) $I_\tau(e, C_\tau(e, \varphi_\psi^{-1}(I_{\tau^*}(k, g_B, r)), r), r) \sqsubseteq C_\tau(e, I_\tau(e, \varphi_\psi^{-1}(g_B), r), r)$.

Proposition 5. Let (U, τ_E), (V, τ_F^*), and (W, γ_H) be an FSTSs and $\varphi_\psi : \widetilde{(U, E)} \longrightarrow \widetilde{(V, F)}$, $\varphi^*_{\psi^*} : \widetilde{(V, F)} \longrightarrow \widetilde{(W, H)}$ be two fuzzy soft functions. Then, the composition $\varphi^*_{\psi^*} \circ \varphi_\psi$ is fuzzy soft δ-continuous (resp., β-continuous) if φ_ψ is fuzzy soft δ-continuous (resp., β-continuous) and $\varphi^*_{\psi^*}$ is fuzzy soft continuous.

Proof. Obvious.

4. Some weaker forms of fuzzy soft continuity

Here, as a weaker form of fuzzy soft continuity [20], the concepts of fuzzy soft almost (weakly) continuous functions are introduced and some properties are obtained. Furthermore, we show that fuzzy soft continuity $\Rightarrow$ fuzzy soft almost continuity $\Rightarrow$ fuzzy soft weakly continuity, but the converse may not be true. Finally, we introduce the notion of continuity in a very general setting called fuzzy soft $(\mathcal{L}, \mathcal{M}, \mathcal{N}, \mathcal{O})$-continuous functions.

Definition 12. Let (U, τ_E) and (V, τ_F^*) be an FSTSs. A fuzzy soft function $\varphi_\psi : \widetilde{(U, E)} \longrightarrow \widetilde{(V, F)}$ is said to be fuzzy soft almost (resp., weakly) continuous if for each $e_{u_t} \in \widetilde{P_t(U)}$ and each $g_B \in \widetilde{(V, F)}$ with $\tau_k^*(g_B) \geq r$ containing $\varphi_\psi(e_{u_t})$, there is $f_A \in \widetilde{(U, E)}$ with $\tau_e(f_A) \geq r$ containing e_{u_t}, such that $\varphi_\psi(f_A) \sqsubseteq I_{\tau^*}(k, C_{\tau^*}(k, g_B, r), r)$ (resp., $\varphi_\psi(f_A) \sqsubseteq C_{\tau^*}(k, g_B, r)$).

Theorem 9. Let (U, τ_E) and (V, τ_F^*) be an FSTSs and $\varphi_\psi : \widetilde{(U, E)} \longrightarrow \widetilde{(V, F)}$ be a fuzzy soft function. Suppose that one of the following holds for every $g_B \in \widetilde{(V, F)}$, $e \in E$, $(k = \psi(e)) \in F$, and $r \in I_\circ$:

(i) If $\tau_k^*(g_B) \geq r$, $\varphi_\psi^{-1}(g_B) \sqsubseteq I_\tau(e, \varphi_\psi^{-1}(I_{\tau^*}(k, C_{\tau^*}(k, g_B, r), r)), r)$.

(ii) $C_\tau(e, \varphi_\psi^{-1}(C_{\tau^*}(k, I_{\tau^*}(k, g_B, r), r)), r) \sqsubseteq \varphi_\psi^{-1}(g_B)$ if $\tau_k^*(g_B^c) \geq r$.

Then, φ_ψ is fuzzy soft almost continuous.

Proof. (i) $\Rightarrow$ (ii) Let $g_B \in \widetilde{(V, F)}$ with $\tau_k^*(g_B^c) \geq r$. From (i), it follows
$\varphi_\psi^{-1}(g_B^c) \sqsubseteq I_\tau(e, \varphi_\psi^{-1}(I_{\tau^*}(k, C_{\tau^*}(k, g_B^c, r), r)), r) = I_\tau(e, \varphi_\psi^{-1}((C_{\tau^*}(k, I_{\tau^*}(k, g_B, r), r))^c), r) = I_\tau(e, (\varphi_\psi^{-1}(C_{\tau^*}(k, I_{\tau^*}(k, g_B, r), r)))^c, r) = (C_\tau(e, \varphi_\psi^{-1}(C_{\tau^*}(k, I_{\tau^*}(k, g_B, r), r)), r))^c$.
Hence, $C_\tau(e, \varphi_\psi^{-1}(C_{\tau^*}(k, I_{\tau^*}(k, g_B, r), r)), r) \sqsubseteq \varphi_\psi^{-1}(g_B)$. Similarly, we get (ii) $\Rightarrow$ (i).

Suppose that (i) holds. Let $e_{u_t} \in \widetilde{P_t(U)}$ and $g_B \in \widetilde{(V, F)}$ with $\tau_k^*(g_B) \geq r$ containing $\varphi_\psi(e_{u_t})$. Then, by (i), $e_{u_t} \tilde{\in} I_\tau(e, \varphi_\psi^{-1}(I_{\tau^*}(k, C_{\tau^*}(k, g_B, r), r)), r)$, and so there is $f_A \in \widetilde{(U, E)}$ with $\tau_e(f_A) \geq r$ containing e_{u_t} such that $f_A \sqsubseteq \varphi_\psi^{-1}(I_{\tau^*}(k, C_{\tau^*}(k, g_B, r), r))$. Hence, $\varphi_\psi(f_A) \sqsubseteq I_{\tau^*}(k, C_{\tau^*}(k, g_B, r), r)$. Then, φ_ψ is fuzzy soft almost continuous.

Lemma 2. Every fuzzy soft continuous function [20] is fuzzy soft almost continuous.

Proof. It follows from Definitions 5 and 12.

Remark 8. In general, the converse of Lemma 2 is not true, as shown by Example 12.

Example 12. Let $U = \{u_1, u_2\}$, $E = \{e_1, e_2\}$, and define $g_E, f_E \in \widetilde{(U, E)}$ as follows: $g_E = \{(e_1, \{\frac{u_1}{0.4}, \frac{u_2}{0.5}\}), (e_2, \{\frac{u_1}{0.4}, \frac{u_2}{0.5}\})\}$, $f_E = \{(e_1, \{\frac{u_1}{0.3}, \frac{u_2}{0.4}\}), (e_2, \{\frac{u_1}{0.3}, \frac{u_2}{0.4}\})\}$. Define fuzzy soft topologies $\tau_E, \tau_E^* : E \longrightarrow [0, 1]^{\widetilde{(U,E)}}$ as follows: $\forall e \in E$,

$$\tau_e(m_E) = \begin{cases} 1, & \text{if} \quad m_E \in \{\Phi, \widetilde{E}\}, \\ \frac{1}{2}, & \text{if} \quad m_E = g_E, \\ 0, & \text{otherwise}, \end{cases} \qquad \tau_e^*(m_E) = \begin{cases} 1, & \text{if} \quad m_E \in \{\Phi, \widetilde{E}\}, \\ \frac{1}{4}, & \text{if} \quad m_E \in \{f_E, g_E\}, \\ 0, & \text{otherwise}. \end{cases}$$

Thus, the identity fuzzy soft function $\varphi_\psi : (U, \tau_E) \longrightarrow (U, \tau_E^*)$ is fuzzy soft almost continuous, but it is not fuzzy soft continuous.

Theorem 10. Let (U,τ_E) and (V,τ^*_F) be an FSTSs and $\varphi_\psi : \widetilde{(U,E)} \longrightarrow \widetilde{(V,F)}$ be a fuzzy soft function. Suppose that one of the following holds for every $g_B \in \widetilde{(V,F)}$, $e \in E$, $(k=\psi(e)) \in F$, and $r \in I_\circ$:

(i) $\varphi^{-1}_\psi(g_B) \sqsubseteq I_\tau(e, \varphi^{-1}_\psi(C_{\tau^*}(k,g_B,r)),r)$ if $\tau^*_k(g_B) \geq r$.

(ii) $C_\tau(e,\varphi^{-1}_\psi(I_{\tau^*}(k,g_B,r)),r) \sqsubseteq \varphi^{-1}_\psi(g_B)$ if $\tau^*_k(g^c_B) \geq r$.

Then, φ_ψ is fuzzy soft weakly continuous.

Proof. (i) ⇒ (ii) Let $g_B \in \widetilde{(V,F)}$ with $\tau^*_k(g^c_B) \geq r$. From (i), it follows

$$\begin{aligned}\varphi^{-1}_\psi(g^c_B) &\sqsubseteq I_\tau(e,\varphi^{-1}_\psi(C_{\tau^*}(k,g^c_B,r)),r)\\ &= I_\tau(e,\varphi^{-1}_\psi((I_{\tau^*}(k,g_B,r))^c),r)\\ &= I_\tau(e,(\varphi^{-1}_\psi(I_{\tau^*}(k,g_B,r)))^c,r)\\ &= (C_\tau(e,\varphi^{-1}_\psi(I_{\tau^*}(k,g_B,r)),r))^c.\end{aligned}$$

Hence, $C_\tau(e,\varphi^{-1}_\psi(I_{\tau^*}(k,g_B,r)),r) \sqsubseteq \varphi^{-1}_\psi(g_B)$. Similarly, we get (ii) ⇒ (i).

Suppose that (i) holds. Let $e_{u_t} \in \widetilde{P_t(U)}$ and $g_B \in \widetilde{(V,F)}$ with $\tau^*_k(g_B) \geq r$ containing $\varphi_\psi(e_{u_t})$. Then, by (i), $e_{u_t} \tilde{\in} I_\tau(e,\varphi^{-1}_\psi(C_{\tau^*}(k,g_B,r)),r)$, and so there is $f_A \in \widetilde{(U,E)}$ with $\tau_e(f_A) \geq r$ containing e_{u_t} such that $f_A \sqsubseteq \varphi^{-1}_\psi(C_{\tau^*}(k,g_B,r))$. Thus, $\varphi_\psi(f_A) \sqsubseteq C_{\tau^*}(k,g_B,r)$. Hence, φ_ψ is fuzzy soft weakly continuous.

Lemma 3. Every fuzzy soft almost continuous function is fuzzy soft weakly continuous.

Proof. It follows from Definition 12.

Remark 9. In general, the converse of Lemma 3 is not true, as shown by Example 13.

Example 13. Let $U=\{u_1,u_2,u_3\}$, $E=\{e_1,e_2\}$, and define $g_E, f_E \in \widetilde{(U,E)}$ as follows: $g_E=\{(e_1,\{\frac{u_1}{0.6},\frac{u_2}{0.6},\frac{u_3}{0.5}\}),(e_2,\{\frac{u_1}{0.6},\frac{u_2}{0.6},\frac{u_3}{0.5}\})\}$, $f_E=\{(e_1,\{\frac{u_1}{0.3},\frac{u_2}{0},\frac{u_3}{0.5}\}),(e_2,\{\frac{u_1}{0.3},\frac{u_2}{0},\frac{u_3}{0.5}\})\}$. Define fuzzy soft topologies $\tau_E, \tau^*_E : E \longrightarrow [0,1]^{\widetilde{(U,E)}}$ as follows: $\forall e \in E$,

$$\tau_e(m_E)=\begin{cases}1, & \text{if } m_E \in \{\Phi,\widetilde{E}\},\\ \frac{1}{2}, & \text{if } m_E = g_E,\\ 0, & \text{otherwise,}\end{cases} \qquad \tau^*_e(m_E)=\begin{cases}1, & \text{if } m_E \in \{\Phi,\widetilde{E}\},\\ \frac{1}{2}, & \text{if } m_E = f_E,\\ 0, & \text{otherwise.}\end{cases}$$

Thus, the identity fuzzy soft function $\varphi_\psi : (U,\tau_E) \longrightarrow (U,\tau^*_E)$ is fuzzy soft weakly continuous, but it is not fuzzy soft almost continuous.

Remark 10. From the previous results, we have: fuzzy soft continuity $\Rightarrow$ fuzzy soft almost continuity $\Rightarrow$ fuzzy soft weakly continuity.

In [38], the difference between f_A and g_B is a fuzzy soft set defined as follows:
$(f_A \,\overline{\sqcap}\, g_B)(e) = \left\{ \begin{array}{cc} \underline{0}, & \text{if } f_A(e) \leq g_B(e), \\ f_A(e) \wedge (g_B(e))^c, & \text{otherwise,} \end{array} \right. \quad \forall e \in E.$

Let $\mathcal{L}$ and $\mathcal{M}$: $E \times \widetilde{(U,E)} \times I_\circ \to \widetilde{(U,E)}$ be operators on $\widetilde{(U,E)}$, and $\mathcal{N}$ and $\mathcal{O}$: $F \times \widetilde{(V,F)} \times I_\circ \to \widetilde{(V,F)}$ be operators on $\widetilde{(V,F)}$.

Definition 13. Let (U, τ_E) and (V, τ^*_F) be an FSTSs. $\varphi_\psi : \widetilde{(U,E)} \longrightarrow \widetilde{(V,F)}$ is said to be a fuzzy soft $(\mathcal{L}, \mathcal{M}, \mathcal{N}, \mathcal{O})$-continuous function if

$$\mathcal{L}[e, \varphi_\psi^{-1}(\mathcal{O}(k, g_B, r)), r] \,\overline{\sqcap}\, \mathcal{M}[e, \varphi_\psi^{-1}(\mathcal{N}(k, g_B, r)), r] = \Phi$$

for each $g_B \in \widetilde{(V,F)}$ with $\tau^*_k(g_B) \geq r$, $e \in E$, and $(k = \psi(e)) \in F$.

In (2014), Aygünoğlu et al. [20] introduced the concept of fuzzy soft continuous functions: $\tau_e(\varphi_\psi^{-1}(g_B)) \geq \tau^*_k(g_B)$, for each $g_B \in \widetilde{(V,F)}$, $e \in E$, and $(k = \psi(e)) \in F$. We can see that Definition 4.2 generalizes the concept of fuzzy soft continuous functions when we choose $\mathcal{L}$ = identity operator, $\mathcal{M}$ = interior operator, $\mathcal{N}$ = identity operator, and $\mathcal{O}$ = identity operator.

A historical justification of Definition 13:

(1) In Section 3, we introduced the concept of fuzzy soft δ-continuous functions: $I_\tau(e, C_\tau(e, \varphi_\psi^{-1}(g_B), r), r) \sqsubseteq C_\tau(e, I_\tau(e, \varphi_\psi^{-1}(g_B), r), r)$, for each $g_B \in \widetilde{(V,F)}$ with $\tau^*_k(g_B) \geq r$. Here, $\mathcal{L}$ = interior closure operator, $\mathcal{M}$ = closure interior operator, $\mathcal{N}$ = identity operator, and $\mathcal{O}$ = identity operator.

(2) In Section 3, we introduced the concept of fuzzy soft β-continuous functions: $\varphi_\psi^{-1}(g_B) \sqsubseteq C_\tau(e, I_\tau(e, C_\tau(e, \varphi_\psi^{-1}(g_B), r), r), r)$, for each $g_B \in \widetilde{(V,F)}$ with $\tau^*_k(g_B) \geq r$. Here, $\mathcal{L}$ = identity operator, $\mathcal{M}$ = closure interior closure operator, $\mathcal{N}$ = identity operator, and $\mathcal{O}$ = identity operator.

(3) In Section 3, we introduced the concept of fuzzy soft semi-continuous functions: $\varphi_\psi^{-1}(g_B) \sqsubseteq C_\tau(e, I_\tau(e, \varphi_\psi^{-1}(g_B), r), r)$, for each $g_B \in \widetilde{(V,F)}$ with $\tau^*_k(g_B) \geq r$. Here, $\mathcal{L}$ =

identity operator, $\mathcal{M}$ = closure interior operator, $\mathcal{N}$ = identity operator, and $\mathcal{O}$ = identity operator.

(4) In Section 3, we introduced the concept of fuzzy soft pre-continuous functions: $\varphi_\psi^{-1}(g_B) \sqsubseteq I_\tau(e, C_\tau(e, \varphi_\psi^{-1}(g_B), r), r)$, for each $g_B \in \widetilde{(V,F)}$ with $\tau_k^*(g_B) \geq r$. Here, $\mathcal{L}$ = identity operator, $\mathcal{M}$ = interior closure operator, $\mathcal{N}$ = identity operator, and $\mathcal{O}$ = identity operator.

(5) In Section 3, we introduced the concept of fuzzy soft α-continuous functions: $\varphi_\psi^{-1}(g_B) \sqsubseteq I_\tau(e, C_\tau(e, I_\tau(e, \varphi_\psi^{-1}(g_B), r), r), r)$, for each $g_B \in \widetilde{(V,F)}$ with $\tau_k^*(g_B) \geq r$. Here, $\mathcal{L}$ = identity operator, $\mathcal{M}$ = interior closure interior operator, $\mathcal{N}$ = identity operator, and $\mathcal{O}$ = identity operator.

(6) In Section 4, we introduced the concept of fuzzy soft almost continuous functions: $\varphi_\psi^{-1}(g_B) \sqsubseteq I_\tau(e, \varphi_\psi^{-1}(I_{\tau^*}(k, C_{\tau^*}(k, g_B, r), r)), r)$, for each $g_B \in \widetilde{(V,F)}$ with $\tau_k^*(g_B) \geq r$. Here, $\mathcal{L}$ = identity operator, $\mathcal{M}$ = interior operator, $\mathcal{N}$ = interior closure operator, and $\mathcal{O}$ = identity operator.

(7) In Section 4, we introduced the concept of fuzzy soft weakly continuous functions: $\varphi_\psi^{-1}(g_B) \sqsubseteq I_\tau(e, \varphi_\psi^{-1}(C_{\tau^*}(k, g_B, r)), r)$, for each $g_B \in \widetilde{(V,F)}$ with $\tau_k^*(g_B) \geq r$. Here, $\mathcal{L}$ = identity operator, $\mathcal{M}$ = interior operator, $\mathcal{N}$ = closure operator, and $\mathcal{O}$ = identity operator.

5. Conclusion and future work

In this paper, some new types of a fuzzy soft open set called an r-fuzzy soft δ-open (semi-open) set have been introduced in an FSTSs based on the paper by Aygünoğlu et al. [20]. In addition, the concepts of fuzzy soft δ-closure (δ-interior) operators have been introduced and some properties of them have been investigated. Furthermore, the concept of r-fuzzy soft δ-connected sets has been defined and studied with the help of fuzzy soft δ-closure operators. Thereafter, the concepts of fuzzy soft δ-continuous (semi-continuous and pre-continuous) functions have been introduced and the relations of these functions with each other have been specified with the help of some illustrative examples. Moreover, a decomposition of fuzzy soft semi-continuity and a decomposition of fuzzy soft α-continuity have been obtained.

In the end, as a weaker form of fuzzy soft continuity [20], the concepts of fuzzy soft almost (weakly) continuous functions have been introduced and some properties have been obtained. Also, we have shown that fuzzy soft continuity $\Rightarrow$ fuzzy soft almost continuity $\Rightarrow$ fuzzy soft weakly continuity. Moreover, we have explored the notion of continuity in a very general setting namely fuzzy soft $(\mathcal{L}, \mathcal{M}, \mathcal{N}, \mathcal{O})$-continuous functions. It is also we have the following results:

• Fuzzy soft $(id_U, I_\tau, id_V, id_V)$-continuous function is a fuzzy soft continuous function [20].

• Fuzzy soft $(I_\tau(C_\tau), C_\tau(I_\tau), id_V, id_V)$-continuous function is a fuzzy soft δ-continuous function.

• Fuzzy soft $(id_U, C_\tau(I_\tau(C_\tau)), id_V, id_V)$-continuous function is a fuzzy soft β-continuous function.

• Fuzzy soft $(id_U, C_\tau(I_\tau), id_V, id_V)$-continuous function is a fuzzy soft semi-continuous function.

• Fuzzy soft $(id_U, I_\tau(C_\tau), id_V, id_V)$-continuous function is a fuzzy soft pre-continuous function.

• Fuzzy soft $(id_U, I_\tau(C_\tau(I_\tau)), id_V, id_V))$-continuous function is a fuzzy soft α-continuous function.

• Fuzzy soft $(id_U, I_\tau, I_{\tau^*}(C_{\tau^*}), id_V)$-continuous function is a fuzzy soft almost continuous function.

• Fuzzy soft $(id_U, I_\tau, C_{\tau^*}, id_V)$-continuous function is a fuzzy soft weakly continuous function.

In upcoming papers, we will use r-fuzzy soft δ-open sets to define and study some new higher separation axioms and to introduce the concept of δ-compact spaces in an FSTSs based on the paper by Aygünoğlu et al. [20] Also, introducing these novel notions given here in the frame of fuzzy soft ideals as defined in [34, 35, 37].

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