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On the Irreversible k-Threshold Conversion Number for Some Graph Products and Neutrosophic Graphs

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Abstract

An irreversible k-threshold conversion process on a graph $G = (V, E)$ is an iterative process that studies the spread of a one way change (from state 0 to 1) on $V(G)$. The process begins by choosing a set $S_0 \subseteq V$. For each step $t(t = 1,2, \ldots,)$, S_t is obtained from S_{t-1} by adjoining all vertices that have at least k neighbors in S_{t-1}. We call S_0 the seed set of the k-threshold conversion process and if $S_t = V(G)$ for some $t \geq 0$, then S_0 is called an irreversible k-threshold conversion set (IkCS) of G. The k-threshold conversion number of G (denoted by $(c_k(G))$ is the minimum cardinality of all the IkCSs of G. In this paper, we study IkCSs of toroidal grids and the tensor product of two paths. We determine $c_2(C_3 \times C_n)$ and we present upper and lower bounds for $c_2(C_m \times C_n)$ for $m, n \geq 3$. We also determine $c_2(P_2 \times P_n), c_2(P_3 \times P_n)$ and present an upper bound for $c_2(P_m \times P_n)$ when $m, n > 3$. Then we determine $c_3(P_m \times P_n)$ for $m = 2,3,4$ and arbitrary n. Finally, we determine $c_4(P_m \times P_n)$ for arbitrary $m, n.$. Also, we study the same concepts over some neutrosophic graphs with suggestions for future neutrosophic and fuzzy generalizations.

Keywords: Toroidal grid; Tensor product; Graph conversion process; k-threshold conversion set; Neutrosophic graph; Neutrosophic graph product

1. Introduction

Let $G(V, E)$ be a graph with $|V| = n$ vertices and $|E| = m$ edges. The open neighborhood of a vertex $v \in V$ is $N(v) = \{u \in V: uv \in E\}$ and the closed neighborhood of v is $N[v] = N(v) \cup \{v\}$. The degree of a vertex v (denoted by $\deg(v)$) is the number of all vertices that are adjacent to v. Therefore, $\deg(v) = |N(v)|$. For any undefined term in the paper, we refer to Harary [5]. Let $Y \subseteq V$ and let F be a subset of E such that F consists of all edges of G which have endpoints in Y, then $H = (Y, F)$ is called an induced subgraph of G by Y and is denoted by G_Y. An independent vertex set of a graph $G(V, E)$ is a subset of V such that no two vertices in the subset represent and edge of G. The independence number, denoted by $\alpha(G)$, is the cardinality of the largest independent vertex set of G. The cartesian product of two cycles (also called toroidal grid and denoted by $C_m \square C_n$) has the vertex set $V(C_m \square C_n) = \{(i, j): 1 \leq i \leq m,\ 1 \leq j \leq n\}$ and two vertices $(i, j), (i', j')$ are adjacent if and only if they satisfy one of the two following conditions:

- i is adjacent to i' and $j = j'$.
- $i = i'$ and j is adjacent to j'.

For more information on Toroidal grids see [12-24] The tensor product of two paths (denoted by $P_m \times P_n$) has the same vertex set as the cartesian product therefore $V(P_m \times P_n) = \{(i, j): 1 \leq i \leq m,\ 1 \leq j \leq n\}$ and two vertices

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$(i,j),(i',j')$ are adjacent if and only if i is adjacent to i' and j is adjacent to j'. For more information on the tensor product, see [6-11]. Irreversible Conversion processes study the spread of a one way change of state (from state 0 to state 1) through a specified society (the spread of disease through populations, the spread of opinion through social networks,..) where the conversion rule is determined at the beginning of the study. These processes can be modeled into graph theoretical models where the vertex set $V(G)$ represents the set of individuals on which the conversion is spreading. The irreversible k-threshold conversion process on a graph $G=(V,E)$ is an iterative process which begins by choosing a set $S_0 \subseteq V$, and for each step $t(t=1,2,\dots,)$, S_t is obtained from S_{t-1} by adjoining all vertices that have at least k neighbors in S_{t-1}. S_0 is called the seed set of the k-threshold conversion process and if $S_t = V(G)$ for some $t \geq 0$, then S_0 is an irreversible k-threshold conversion set (IkCS) of G. The k-threshold conversion number of G (denoted by $(c_k(G))$ is the minimum cardinality of all the IkCSs of G. The first graph model of the Irreversible k-threshold conversion problem was presented by Dreyer and Roberts in [3] where they determined the value of $c_2(G)$ for paths and cycles. They also determined $c_3(G)$ for toroidal grids $C_3 \square C_n$ and presented lower and upper bounds for $c_3(C_m \square C_n)$ when m,n are arbitraries, they also determined $c_4(C_m \square C_n)$ for arbitrary m,n. Adams *et al*., [1] presented upper bounds for $c_k(G)$ of the cartesian product and tensor product of two arbitrary graphs G and H. For further information on the irreversible k-threshold conversion problem on graphs see Centeno *et al*. [2], Kynčl *et al*. [7], Frances *et al*. [4], Takaoka and Ueno [11]. In [8] Mynhardt and Wodlinger presented a lower bound for $c_k(G)$ of graphs of maximum degree $k+1$. In [9], Mynhardt and Wodlinger gave an upper bound for $c_k(G)$ of regular graphs. In [10] Shaheen *et al*. studied irreversible k-threshold conversion processes on circulant graphs. In this paper, we determine $c_2(C_3 \square C_n)$ and we present upper and lower bounds for $c_2(C_m \square C_n)$ for $m,n>3$. We also determine $c_2(P_2 \times P_n), c_2(P_3 \times P_n)$ and present an upper bound for $c_2(P_m \times P_n)$ when $m,n>3$. We determine $c_3(P_m \times P_n)$ for $m=2,3,4$ and arbitrary n. We also determine $c_4(P_m \times P_n)$ for arbitrary m,n.

Proposition 1.1. [5] For $m \geq 2$ and $n \geq 3$:

$\alpha(P_m) = \left\lceil \frac{m}{2} \right\rceil$; $\alpha(C_n) = \left\lfloor \frac{n}{2} \right\rfloor$.

Proposition 1.2. [6] If G is a path or a cycle and H is a path or a cycle, then:

$$\alpha(G \times H) = \max\{\alpha(G)|V(H)|, \alpha(H)|V(G)|\}$$

Proposition 1.3. [3] For the toroidal grid graphs $C_m \square C_n$:

- $c_3(C_3 \square C_n) = n+1$.
- For $,n \geq 4$; $\frac{mn+2}{3} \leq c_3(C_m \square C_n) \leq \frac{mn}{3} + \frac{23m+13n-5}{12}$.
- For $m,n \geq 3$; $c_4(C_m \square C_n) = \begin{cases} max\left\{n\left\lceil \frac{m}{2}\right\rceil, m\left\lceil \frac{n}{2}\right\rceil\right\} if\ m\ or\ n\ is\ odd; \\ \frac{mn}{2}\ if\ m\ and\ n\ are\ even. \end{cases}$

Proposition 1.4. [1] Let G and H be two graphs, then $c_k(G \square H) \leq c_k(G)c_k(H)$.

Proposition 1.5.[1] Let G and H be two graphs without any isolated vertices. Then $c_k(G \times H) \leq \min\{\min\{c_k(G)|V(H)|\}, \min\{c_k(H)|V(G)|\}\}$.

Remark 1.1. Throughout this paper, we divide $V(P_m \times P_n)$ into three subsets (Q_1, Q_2, Q_3) defined as:

$Q_1 = \{v \in V; \deg(v) = 1\} = \{(1,1),(1,n),(m,1),(m,n)\}$;

$Q_2 = \{v \in V; \deg(v) = 2\} = \{(i,1),(i,n),(1,j),(m,j): 2 \leq i \leq m-1;\ 2 \leq j \leq n-1\}$;

$Q_3 = \{v \in V; \deg(v) = 4\} = \{(i,j): 2 \leq i \leq m-1;\ 2 \leq j \leq n-1\}$.

We notice that $|Q_1| = 4$; $|Q_2| = 2m+2n-8$; $|Q_3| = (m-2)(n-2) = mn-2m-2n+4$.

Remark 1.2. We denote the rows by $R_i: 1 \leq i \leq m$ and $R_l = \{(l,j): 1 \leq j \leq n\}$. In a similar way, we denote the grid columns by $CO_j: 1 \leq j \leq n$ and $CO_l = \{(i,l): 1 \leq i \leq m\}$ and we use these notations for $C_m \square C_n$ and for $P_m \times P_n$.

Remark 1.3. Throughout this paper, we call a column CO_j a seeded column if $CO_j \cap S_0 \neq \emptyset$, otherwise we call it an empty column. A column CO_j is even (odd) if j is even (odd) respectively. We use the same terminology for rows.

Remark 1.4. As an immediate consequence of the definition, $c_k(G) \geq k$ for any graph G.

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Remark 1.5. As a consequence of the definition, for any graph G; $1 \leq k \leq \Delta(G)$ where $\Delta(G) = max\{deg(v): v \in V(G)\}$.

Remark 1.6. As an immediate consequence of the definition, when studying an Irreversible k-threshold conversion process on a graph $G = (V, E)$ all vertices $\{v \in V\ ; \deg(v) < k\}$ must be included in the seed set S_0, otherwise the process will fail because none of these vertices can satisfy the conversion rule. These vertices are called k-immune vertices, see [9].

Remark 1.7. In every figure of this article, we represent the vertices as white circles, and we assign every converted vertex the number of the conversion step in which it gets converted by placing the number inside the circle.

2. Main Results

Definition 2.1. Let $U \subseteq V(G)$, then U is k-unconvertable if it satisfies two conditions:

- $U \cap S_0 = \emptyset$.
- For every $u \in U; deg(u) - |N(u) \cap U| < k$.

Therefore, no vertex of U can satisfy the conversion rule at any step of the conversion process. It is straightforward to see that $S \subseteq V$ is a k-conversion set of G if and only if $V - S$ does not contain a k-unconvertable set.

Remark 2.1. As an immediate consequence to Definition 2.1, when choosing the seed set S_0, we try to avoid leaving any versions of U on the studied graph.

2.1. $c_k(C_m \times C_n)$.

In this sub-section we determine $c_2(C_3 \times C_n)$. We also present a lower and an upper bound of $c_2(C_m \times C_n)$ when m, n are arbitraries.

Proposition 2.1. Let there be a 2-threshold conversion process initiated by a seed set S_0 on $C_m \times C_n$. The process fails if there are two adjacent empty columns or rows.

Proof: Let $CO_l, CO_{l+1}: l \in \{1,2, \dots, n-1\}$ be two adjacent empty columns. This means the following set $U = \{(i, l), (i, l+1): 1 \leq i \leq m\}$ satisfies the two conditions introduced in Definition 2.1 because $U \cap S_0 = \emptyset$ and every vertex $u \in U$ is of degree 4 and is adjacent to 3 other vertices of $u \in U$ therefore $deg(u) - |N(u) \cap U| = 1 < 2 = k$. This means no vertex of U can satisfy the conversion rule at any step of the conversion process, so it fails. The same argument applies if we have two adjacent empty rows.□

Proposition 2.2. $c_2(C_m \times C_n) \geq \begin{cases} \left\lceil \frac{m}{2} \right\rceil & if\ m \geq n; \\ \left\lceil \frac{n}{2} \right\rceil & if\ m < n. \end{cases}$

Proof. This result can be immediately concluded from Proposition 2.1.□

Theorem 2.1. For $n \geq 3$, $c_2(C_3 \times C_n) = \left\lceil \frac{n+1}{2} \right\rceil$.

Proof. Due to Proposition 2.2 we have $c_2(C_3 \times C_n) \geq \left\lceil \frac{n}{2} \right\rceil$. We consider the following cases:

Case 1. n is even.

Let D_0 be a seed set of cardinality $\left\lceil \frac{n}{2} \right\rceil = \frac{n}{2}$ on $C_3 \times C_n$. There are two ways to place the $\frac{n}{2}$ converted vertices on the columns without leaving two adjacent empty columns. We either include one vertex from each odd column in D_0 or we include one vertex from each even column in it. In both cases the process stops at the end of step $t = 1$ because of the following situation: Let $n = 6$. Let CO_1, CO_3, CO_5 be the seeded columns. No matter how we choose $(i_1, 1) \in CO_1 \cap D_0$; $(i_2, 3) \in CO_3 \cap D_0$; $(i_3, 5) \in CO_5 \cap D_0$, at $t = 1$ the conversion only spreads to:

- $(i_1, 2)$ if $i_1 = i_2 \neq i_3$.
- $(i_2, 4)$ if $i_2 = i_3 \neq i_1$.
- $(i_3, 6)$ if $i_3 = i_1 \neq i_2$.
- $(i_1, 2), (i_1, 4), (i_1, 6)$ if $i_2 = i_3 = i_1$.

However, in order to spread the conversion vertically (to different rows) at $t = 2$ we need another converted vertex that belongs to a different row from any column that is containing a converted vertex from D_1. Since there is no such vertex, it is impossible to convert any new vertices at $t = 2$ and the process fails. The same argument applies

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if the seeded columns are CO_2, CO_4, CO_6. Without loss of generality, we obtain the same result for any even value of n. We conclude that if n is even:

$$c_2(C_3 \times C_n) > \frac{n}{2} \tag{1}$$

Now let $S_0 = \{(1,2l+1): 0 \le l \le \frac{n}{2} - 1\} \cup \{(3,1)\}$ which is of cardinality $\frac{n}{2} + 1$ be the seed set. The process goes as follows:

$t = 0: S_0 = \{(1,2l+1): 0 \le l \le \frac{n}{2} - 1\} \cup \{(3,1)\}$.

$t = 1: S_1 = S_0 \cup \{(1,2l): 1 \le l \le \frac{n}{2}\} \cup \{(2,1)\}$.

For $2 \le t \le \frac{n}{2}: S_t = S_{t-1} \cup \{(2,t), (2, n-t+2), (3,t), (3, n-t+2)\}$.

The process ends at $t = \frac{n}{2} + 1$ for which $S_{\frac{n}{2}+1} = S_{\frac{n}{2}} \cup \left\{(2, \frac{n}{2} + 1), (3, \frac{n}{2} + 1)\right\} = V(C_3 \times C_n)$ therefore S_0 is an I2CS on $C_3 \times C_n$ which means:

$$c_2(C_3 \times C_n) \le \frac{n}{2} + 1 \tag{2}$$

Figure 1 illustrates a 2-conversion set on $C_3 \times C_{16}$ starting with S_0 of cardinality 9.

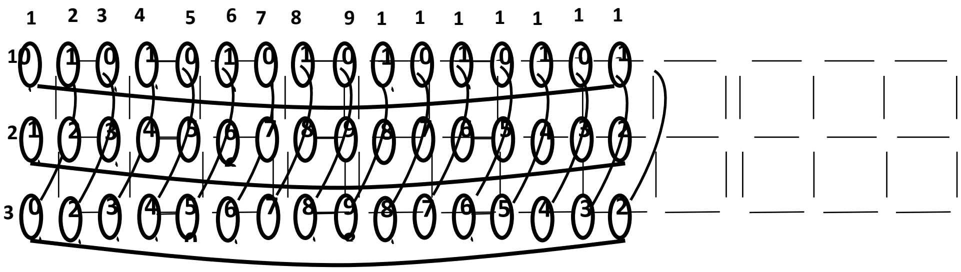

Figure 1. a 2-conversion set on $C_3 \times C_{16}$ starting with S_0 of cardinality 9.

From (1) and (2) we conclude that $c_2(C_3 \times C_n) = \frac{n}{2} + 1 = \left\lceil \frac{n+1}{2} \right\rceil$ if n is even.

Case 2. n is odd.

Due to Proposition 2.2 we know that $c_2(C_3 \times C_n) \ge \left\lceil \frac{n}{2} \right\rceil = \frac{n+1}{2}$. Let us now prove that $c_2(C_3 \times C_n) \le \frac{n+1}{2}$ when n is odd. Let $S_0 = \{(1,2l): 1 \le l \le \frac{n-1}{2}\} \cup \{(2,1)\}$ which is of cardinality $\frac{n+1}{2}$ be the seed set. The process goes as follows:

$t = 0: S_0 = \{(1,2l): 1 \le l \le \frac{n-1}{2}\} \cup \{(2,1)\}$.

$t = 1: S_1 = S_0 \cup \{(1,2l+1): 0 \le l \le \frac{n-3}{2}\} \cup \{(2,2)\}$.

$t = 2: S_2 = S_1 \cup \{(1,n), (2,3), (3,1), (3,2)\}$.

For $3 \le t \le \frac{n+1}{2}: S_t = S_{t-1} \cup \{(2,t+1), (2,n-t+3), (3,t), (3,n-t+3)\}$.

The process ends at $t = \frac{n+3}{2}$ for which $S_{\frac{n+3}{2}} = S_{\frac{n+1}{2}} \cup \{(3, \frac{n+3}{2})\} = V(C_3 \,\square\, C_n)$ which means $c_2(C_3 \,\square\, C_n) \le \frac{n+1}{2}$. We conclude that $c_2(C_3 \,\square\, C_n) = \frac{n+1}{2} = \left\lceil \frac{n+1}{2} \right\rceil$ if n is odd.

From Case 1 and Case 2 we conclude the requested. □

Theorem 2.2. For $m, n \ge 4$, $c_2(C_m \,\square\, C_n) \le \left\lceil \frac{m+n}{2} \right\rceil - 1$.

Proof. We consider the following cases for m, n:

Case 1. m, n are even.

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Let the seed set be $S_0 = \{(2l+1,1),(1,2d+1): 0 \le l \le \frac{m}{2}-1; 1 \le d \le \frac{n}{2}-1\}$ which is of cardinality $\frac{m+n}{2} - 1 = \left\lceil\frac{m+n}{2}\right\rceil - 1$.

We consider the following subcases:

Case 1.a. $m = n$.

This means $\left\lceil\frac{m+n}{2}\right\rceil - 1 = n-1$. The process goes as follows:

$t = 0: S_0 = \{(2l+1,1),(1,2d+1): 0 \le l \le \frac{n}{2}-1; 1 \le d \le \frac{n}{2}-1\}$.

$t = 1: S_1 = S_0 \cup \{(2l,1),(1,2l): 1 \le l \le \frac{n}{2}\}$.

$t = 2: S_2 = S_1 \cup \{(2,2),(2,n),(n,2),(n,n)\}$.

For $3 \le t \le \frac{n}{2}+1$:

$S_t = S_{t-1} \cup \{(2+l,t-l),(t-l,n-l),(n-l,t-l),(n-l,n-t+l+2) : 0 \le l \le t-2\}$.

For $\frac{n}{2}+1 < t < n$ which means for $t = \frac{n}{2}+1+h$; $1 \le h \le \frac{n}{2}-2$ and $h \in \mathbb{Z}$:

$S_t = S_{t-1} \cup \{(2+l,t-l),(t-l,n-l),(n-l,t-l),(n-l,n-t+l+2) : h \le l \le t-2-h\}$.

The process ends at step $t = n$ for which $h = \frac{n}{2}-1$ and:

$$S_n = S_{n-1} \cup \{(2+l,t-l),(t-l,n-l),(n-l,t-l),(n-l,n-t+l+2) : \frac{n}{2}-1 \le l \le \frac{n}{2}-1\}$$

$= S_{n-1} \cup \{(\frac{n}{2}+1,\frac{n}{2}+1)\} = V(C_m \,\square\, C_n)$. This means S_0 is an I2CS on $C_m \,\square\, C_n$, therefore $c_2(C_n \,\square\, C_n) \le n-1$ if n is odd. Figure 2 shows that $c_2(C_8 \,\square\, C_8) \le 7$.

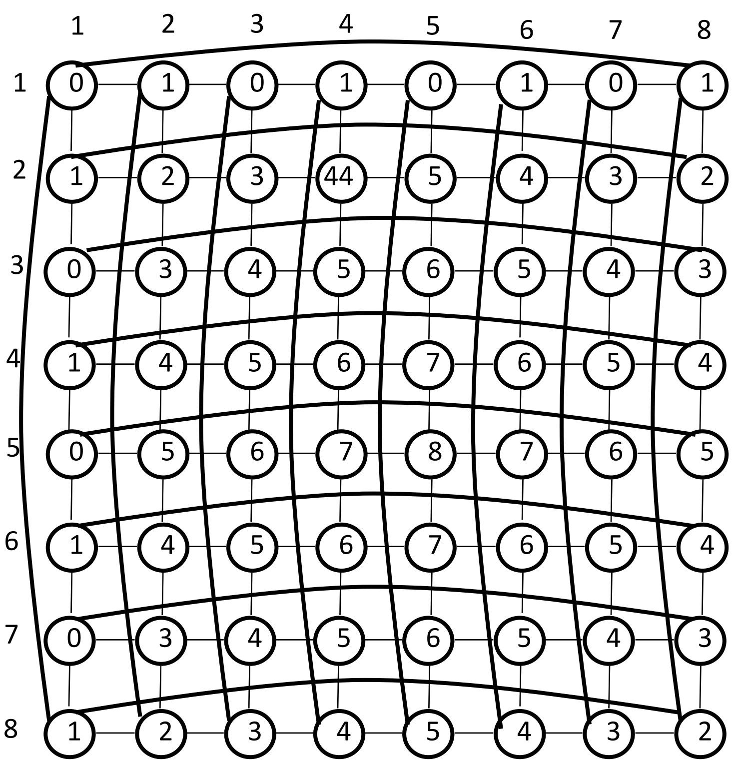

Figure 2. $c_2(C_8 \times C_8) \le 7$.

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Case 1.b. $m < n$.

This means $\left\lceil\frac{m+n}{2}\right\rceil - 1 = \frac{m+n}{2} - 1$. The process goes as follows:

$t = 0: S_0 = \{(2l+1,1),(1,2d+1): 0 \le l \le \frac{m}{2} - 1; 1 \le d \le \frac{n}{2} - 1\}$.

$t = 1: S_1 = S_0 \cup \{(2l,1),(1,2d): 1 \le l \le \frac{m}{2}; 1 \le l \le \frac{d}{2}\}$.

$t = 2: S_2 = S_1 \cup \{(2,2),(2,n),(m,2),(m,n)\}$.

For $3 \le t \le \frac{m}{2} + 1$:

$S_t = S_{t-1} \cup \{(2+l,t-l),(t-l,n-l),(n-l,t-l),(n-l,n-t+l+2) : 0 \le l \le t-2\}$.

For $\frac{m}{2} + 1 < t \le \frac{n}{2} + 1$ which means for $t = \frac{m}{2} + 1 + h; 1 \le h \le \frac{n-m}{2}$ and $h \in \mathbb{Z}$:

$S_t = S_{t-1} \cup \{(2+l,t-l),(t-p,n-p),(m-l,t-l),(m-l,n-t+l+2)$:

$0 \le l \le t-2-h; h \le p \le t-2\}$.

For $\frac{n}{2} + 1 < t \le \frac{n+m}{2} - 1$ which means for $t = \frac{n}{2} + 1 + \frac{n-m}{2} + x; 1 \le x \le \frac{m}{2} - 2$ and $x \in \mathbb{Z}$:

$S_t = S_{t-1} \cup \{(2+l,t-l),(t-p,n-p),(m-l,t-l),(m-l,n-t+l+2)$:

$x \le l \le t-2-\frac{n-m}{2} - x; \frac{n-m}{2} + x \le p \le t-2-x\}$.

The process ends at step $t = \frac{m+n}{2}$ for which $x = \frac{m}{2} - 1$ and:

$S_{\frac{m+n}{2}} = S_{\frac{m+n}{2}-1} \cup \{(2+l,t-l),(t-p,n-p),(n-l,t-l),(n-l,n-t+l+2)$:

$\frac{m}{2} - 1 \le l \le \frac{m}{2} - 1; \frac{n}{2} - 1 \le p \le \frac{n}{2} - 1\} = S_{\frac{m+n}{2}-1} \cup \{(\frac{m}{2}+1,\frac{n}{2}+1)\} = V(C_m \times C_n)$.

Therefore, $c_2(C_n \,\square\, C_n) \le \frac{m+n}{2} - 1$ if m, n are even and $m < n$.

Case 1.c. $m > n$.

This also means $\left\lceil\frac{m+n}{2}\right\rceil - 1 = \frac{m+n}{2} - 1$. The process goes as follows:

$t = 0: S_0 = \{(2l+1,1),(1,2d+1): 0 \le l \le \frac{m}{2} - 1; 1 \le d \le \frac{n}{2} - 1\}$.

$t = 1: S_1 = S_0 \cup \{(2l,1),(1,2d): 1 \le l \le \frac{m}{2}; 1 \le d \le \frac{n}{2}\}$.

$t = 2: S_2 = S_1 \cup \{(2,2),(2,n),(m,2),(m,n)\}$.

For $3 \le t \le \frac{n}{2} + 1$:

$S_t = S_{t-1} \cup \{(2+l,t-l),(t-l,n-l),(n-l,t-l),(n-l,n-t+l+2) : 0 \le l \le t-2\}$.

For $\frac{n}{2} + 1 < t \le \frac{m}{2} + 1$ which means for $t = \frac{n}{2} + 1 + h; 1 \le h \le \frac{m-n}{2}$ and $h \in \mathbb{Z}$:

$S_t = S_{t-1} \cup \{(2+l,t-l),(t-p,n-p),(m-l,t-l),(m-l,n-t+l+2)$:

$h \le l \le t-2; 0 \le p \le t-2-h\}$.

For $\frac{m}{2} + 1 < t \le \frac{n+m}{2} - 1$ which means for $t = \frac{n}{2} + 1 + \frac{m-n}{2} + x; 1 \le x \le \frac{n}{2} - 2$ and $x \in \mathbb{Z}$:

$S_t = S_{t-1} \cup \{(2+l,t-l),(t-p,n-p),(m-l,t-l),(m-l,n-t+l+2)$:

$\frac{m-n}{2} + x \le l \le t-2-x; x \le p \le t-2-\frac{m-n}{2} - x\}$.

The process ends at step $t = \frac{m+n}{2}$ for which $x = \frac{n}{2} - 1$ and:

$S_{\frac{m+n}{2}} = S_{\frac{m+n}{2}-1} \cup \{(2+l,t-l),(t-l,n-l),(n-l,t-l),(n-l,n-t+l+2)$:

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$\frac{m}{2}-1 \leq l \leq \frac{m}{2}-1; \frac{n}{2}-1 \leq p \leq \frac{n}{2}-1\} = S_{\frac{m+n}{2}-1} \cup \{(\frac{m}{2}+1, \frac{n}{2}+1)\} = V(C_m \times C_n)$ which means $c_2(C_n \square C_n) \leq \frac{m+n}{2}-1$ if m,n are even and $m > n$.

From subcases 1.a, 1.b and 1.c we conclude that $c_2(C_n \times C_n) \leq \frac{m+n}{2}-1$ if m,n are even.

Case 2. m,n are odd.

Let the seed set be $S_0 = \{(2l,1),(1,2d): 0 \leq l \leq \frac{m-1}{2}; 1 \leq d \leq \frac{n-1}{2}\}$ which is of cardinality

$\frac{m+n}{2}-1 = \left\lceil\frac{m+n}{2}\right\rceil - 1$.

Case 3. m is odd and n is even.

Let the seed set be $S_0 = \{(2l+1,1),(1,2d+1): 0 \leq l \leq \frac{m-1}{2}; 1 \leq d \leq \frac{n}{2}-1\}$ which is of cardinality $\frac{m+n-1}{2} = \left\lceil\frac{m+n}{2}\right\rceil - 1$.

Case 4. m is even and n is odd.

Let the seed set be $S_0 = \{(2l+1,1),(1,2d+1): 0 \leq l \leq \frac{m}{2}-1; 1 \leq d \leq \frac{n-1}{2}\}$ which is of cardinality $\frac{m+n-1}{2} = \left\lceil\frac{m+n}{2}\right\rceil - 1$.

In Case 2, Case 3 and Case 4 we can track the conversion process in a similar way to Case 1. In Case 2, the process ends successfully at step $t = \frac{m+n}{2}$ for which $S_{\frac{m+n}{2}} = V(C_m \times C_n)$.

In Case 3 and Case 4, the process ends successfully at $t = \frac{m+n-1}{2}$ and $S_{\frac{m+n-1}{2}} = V(C_m \times C_n)$ in both cases.

From all the previous cases and subcases we conclude the requested. □

2.2. $c_k(P_m \times P_n)$.

In this sub-section, we determine $c_2(P_m \times P_n)$ for $m = 2,3$ and we present an upper bound for $c_2(P_m \times P_n)$ when m,n are arbitraries. Then we determine $c_3(P_m \times P_n)$ for $m = 3,4,5$. Finally, we determine $c_4(P_m \times P_n)$ when m,n arbitraries are.

Theorem 2.3. For $n \geq 2$, $c_2(P_2 \times P_n) = 2\left\lceil\frac{n+1}{2}\right\rceil$.

Proof. It is obvious that $V(P_2 \times P_n) = Q_1 \cup Q_2$ only, where $Q_1 = \{(1,1),(1,n),(2,1),(2,n)\}$ while $Q_2 = \{(1,j),(2,j): 2 \leq j \leq n-1\}$. Due to Proposition 1.1 and Proposition 1.5, we conclude that $c_2(P_2 \times P_n) \leq 2\left\lceil\frac{n+1}{2}\right\rceil$. Since $k = 2$, all vertices of Q_1 are 2-immune therefore they must be contained in the seed set S_0 or else the process automatically fails. This means $S_0 = Q_1 \cup M$ where $M \subset Q_2$. We notice that any two adjacent unconverted vertices $u_1, u_2 \in Q_2$ form a 2-unconvertable set since $\deg(u_1) = \deg(u_2) = 2$ and $\deg(u_1) - |\mathrm{N}(u_1) \cap \{u_1,u_2\}| = \deg(u_2) - |\mathrm{N}(u_2) \cap \{u_1,u_2\}| = 1 < k$. This means that neither of them can satisfy the conversion rule at any step of the process, therefore $Q_2 - M$ must be independent, which means $|Q_2 - M| \leq \alpha(G_{Q_2})$. Since G_{Q_2} represents a $P_2 \times P_{n-2}$ graph and due to Proposition 1.2, we conclude that $|Q_2 - M| \leq 2\left\lceil\frac{n-2}{2}\right\rceil$. However, to make S_0 as small as possible we make $Q_2 - M$ as large as possible, therefore $|M| = |Q_2| - 2\left\lceil\frac{n-2}{2}\right\rceil$. We consider the following cases for n:

Case 1. n is odd.

Then $|M| = 2n - 4 - 2(\frac{n-1}{2}) = n - 3$ and $c_2(P_2 \times P_n) = |S_0| = |M| + |Q_1| = n - 3 + 4 = n + 1 = 2\left\lceil\frac{n+1}{2}\right\rceil$. We notice that $S_0 = \{(1,2l+1),(2,2l+1): 0 \leq l \leq \frac{n-1}{2}\}$ is the only I2CS of cardinality $2\left\lceil\frac{n+1}{2}\right\rceil$ on $P_2 \times P_n$ when n is odd.

Case 2. n is even.

Then $|M| = 2n - 4 - 2(\frac{n-2}{2}) = n - 2$ and $c_2(P_2 \times P_n) = |S_0| = |M| + |Q_1| = n - 2 + 4 = n + 2 = 2\left\lceil\frac{n+1}{2}\right\rceil$. We notice that:

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$S_0 = \{(1,2l+1),(2,2l+1): 0 \le l \le \frac{n}{2}-1\} \cup \{(1,n),(2,n)\}$,

$$B_0 = \{(1,2l),(2,2l): 1 \le l \le \frac{n}{2}\} \cup \{(1,1),(2,1)\}$$

are the only I2CSs of cardinality $2\left\lceil\frac{n+1}{2}\right\rceil$ on $P_2 \times P_n$ when n is even. From both cases we prove the requested□

Theorem 2.4. $c_2(P_3 \times P_n) = \begin{cases} 6 \; if \; n \in \{3,4\}; \\ n+2 \; if \; n \ge 5; \end{cases}$

Proof. As we implied in Remark 1,1, $V(P_3 \times P_n)$ can be divided into:

$$Q_1 = \{(1,1),(1,n),(3,1),(3,n)\};$$

$$Q_2 = \{(1,j),(3,j): 2 \le j \le n-1\} \cup \{(2,1),(2,n)\};$$

$$Q_3 = \{(2,j): 2 \le j \le n-1\}.$$

Let $n \ge 10$. Since $k = 2$, then all vertices of Q_1 are 2-immune which means $Q_1 \subset S_0$ otherwise, the process automatically fails. It also fails if one of the sets $U_i: 1 \le i \le 4$ defined as:
$U_1 = \{(1,2),(2,1)\}, U_2 = \{(2,1),(3,2)\}, U_3 = \{(1,n-1),(2,n)\}, U_4 = \{(2,n),(3,n-1)\}$ satisfies that $U_i \cap S_0 = \emptyset$ because each vertex of U_i is of degree 2 and is adjacent to one vertex of U_i which makes U_i 2-unconvertable. Therefore, we must include at least one vertex of U_i in S_0. For $3 \le j \le n-2$ We define some 2-uncorvetable sets on $P_3 \times P_n$ as:

$$W_j = \{(1,j-1),(1,j+1),(2,j),(3,j-1)\};$$
$$X_j = \{(1,j-1),(1,j+1),(2,j),(3,j+1)\};$$
$$Y_j = \{(1,j-1),(2,j),(3,j-1),(3,j+1)\};$$
$$Z_j = \{(1,j+1),(2,j),(3,j-1),(3,j+1)\}.$$

Each version of these sets is 2-unconvertable because it consists of:

- Three vertices of degree 1 that are adjacent to one vertex of the same set.
- One vertex of degree 4 that is adjacent to three vertices of the same set.

Figure 3 shows that W_3, X_3, Y_3 and Z_3 are 2-unconvertable on $P_3 \times P_5$ even if all the remaining vertices of V were included in S_0. (In Figure 3 we denote the vertices of W_3 by $\{w_1, w_2, w_3, w_4\}$ and we use the same notation for X_3, Y_3 and Z_3)

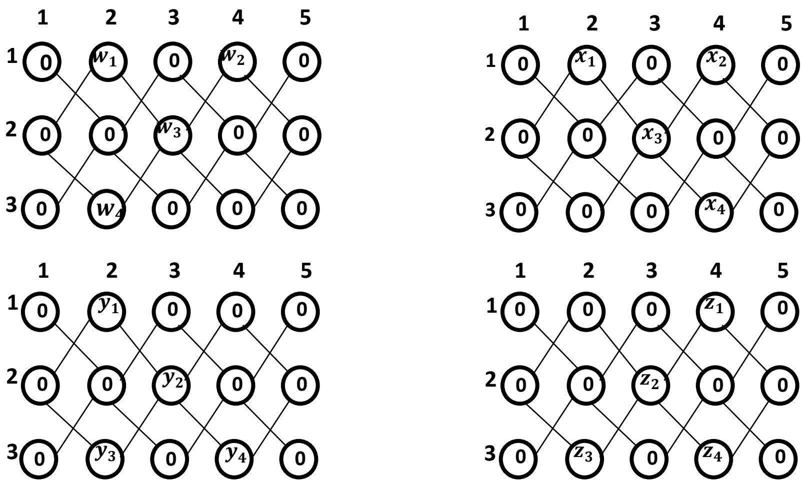

Figure 3. 2-unconvertable W_3, X_3, Y_3 and Z_3 on $P_3 \times P_5$.

Let us now try to distribute only three vertices from S_0 on the four columns CO_4, CO_5, CO_6, CO_7 without leaving any unconverted version of $W_j, X_j, Y_j, Z_j: j \in \{5,6\}$. We consider the following cases:

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Case 1. $(2,5),(2,6) \notin S_0$. Let $(1,4),(3,4),(1,7) \in S_0$. This would leave Y_6 unconverted therefore 2-unconvertable and the process fails. Without loss of generality, a version of a 2-unconvertable set will be left if we do not include any of $(2,4),(2,5)$ in S_0.

Case 2. Only one of $(2,5),(2,6)$ belongs to S_0. We assume that $(2,5) \in S_0$ taking into consideration that without loss of generality, the same argument applies if $(2,6) \in S_0$. Since $(2,5) \in S_0$, this leaves two converted vertices to be distributed in a way that does not leave any of W_6, X_6, Y_6, Z_6 unconverted. This is achievable if the two converted vertices were two vertices from $\{(1,5),(1,7),(3,5),(3,7)\}$.

Let us discuss the possibilities of the two chosen converted vertices in regards to the following sets:
$B_1 = \{(1,5),(2,4)\}, B_2 = \{(2,4),(3,5)\}, B_3 = \{(1,6),(2,7)\}, B_4 = \{(2,7),(3,6)\}$.
Case 2.a. The two converted vertices are $(1,5),(3,5)$. This would prevent leaving any of W_6, X_6, Y_6, Z_6 unconverted. However, it would also leave B_3, B_4 fully unconverted, we notice that we need to include both $(1,8),(3,8)$ in S_0 to avoid having unconverted W_7, X_7, Y_7, Z_7.This means we need to include five vertices from the five columns $CO_j: 4 \leq j \leq 8$ in S_0 or else the process automatically fails.

Case 2.b. Only one of the two chosen converted vertices belongs to $\{(1,5),(3,5)\}$, if it is $(1,5)$, then B_2, B_3, B_4 are all left unconverted therefore in addition to $(1,8),(3,8)$ we need to include one vertex from $\{(1,3),(3,3)\}$ in S_0 to avoid leaving Y_4 unconverted. This means we need to include six vertices from the six columns $CO_j: 3 \leq j \leq 8$ in S_0 or else the process automatically fails. Without loss of generality, the same result is obtained if $(3,5) \in S_0$.

Case 2.c. In order to prevent leaving any of B_1, B_2, B_3, B_4 entirely unconverted, we need the two chosen converted vertices to be $(2,4),(2,7)$. However, that would leave W_6, X_6, Y_6, Z_6 unconverted and the process automatically fails.
From all the cases and subcases and without loss of generality we conclude that the $n-4$ columns $CO_j: 3 \leq j \leq n-2$ must include $n-4$ vertices of S_0, and since we need to include Q_1 and one vertex from each of U_1, U_2, U_3, U_4 in S_0 (which means we need at least two vertices from U_1, U_2, U_3, U_4), then $|S_0| \geq |Q_1| + 2 + n - 4$. We conclude that for $n \geq 10$:

$$c_2(P_3 \times P_n) \geq n + 2 \qquad (3)$$

Let the seed set be $S_0 = Q_1 \cup \{(2,1).(2,n)\} \cup \{(2,j): 3 \leq j \leq n-2\}$ which is of cardinality $n+2$. The process goes as follows:

$t = 0: S_0 = Q_1 \cup \{(2,1),(2,n)\} \cup \{(2,j): 3 \leq j \leq n-2\}$.

$t = 1: S_1 = S_0 \cup \{(1,j),(3,j): 4 \leq j \leq n-3\} \cup \{(1,2),(1,n-1),(3,2).(3,n-1)\}$.

$t = 2: S_2 = S_1 \cup \{(1,3),(1,n-2),(3,3),(3,n-2)\} = V(P_3 \times P_n)$. This means S_0 is an I2CS on $P_3 \times P_n$ and $c_2(P_3 \times P_n) \leq n+2$. From (3) we conclude that $c_2(P_3 \times P_n) = n+2$ for $n \geq 10$. Let us now discuss the lower values for n.
It is easy to notice that the same argument used above applies to $P_3 \times P_n$ when $5 \leq n < 10$ and for each $5 \leq i < 10$; ${S_0}^i$ consists of:
- Q_1.
- two vertices from $U_1 \cup U_2 \cup U_3 \cup U_4$.
- $n-4$ vertices from $\bigcup_{j=3}^{j=n-2} CO_j$.

It is also obvious that $c_2(P_3 \times P_3) = c_2(P_3 \times P_4) = 6$ and $S_0 = Q_1 \cup \{(2,1).(2,n)\}$ could be the seed set for both of them. From all the above we conclude the requested.□

Theorem 2.5. For $m, n \geq 4$; $c_2(P_m \times P_n) \leq \begin{cases} 4\left\lfloor\frac{m-2}{4}\right\rfloor + n + 6 \;\; if\ m \equiv 0,1 (mod\ 4); \\ 4\left\lfloor\frac{m-2}{4}\right\rfloor + n + 2 \;\; if\ m \equiv 2 (mod\ 4); \\ 4\left\lfloor\frac{m-2}{4}\right\rfloor + n + 4 \;\; if\ m \equiv 3 (mod\ 4). \end{cases}$

Proof. We consider the following cases for m, n:
Case 1. $m \equiv 0 (mod\ 4)$ and n is arbitrary.

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Let the seed set be $S_0 = \{(4l+1,1),(4l+2,1),(4l+1,n),(4l+2,n): 0 \le l \le \left\lfloor \frac{m-2}{4} \right\rfloor - 1\} \cup \{(p,1),(p,n): m-3 \le p \le m\}$ which is of cardinality $4\left\lfloor \frac{m-2}{4} \right\rfloor + n + 6$. The process goes as follows:

$t = 0: S_0 = \{(4l+1,1),(4l+2,1),(4l+1,n),(4l+2,n): 0 \le l \le \left\lfloor \frac{m-2}{4} \right\rfloor - 1\} \cup \{(p,1),(p,n): m-3 \le p \le m\}$.

$t = 1: S_1 = S_0 \cup \{(2,l): 2 \le l \le n-1\} \cup \{(m-2,2),(m-2,n-1),(m-1,2),$
$(m-1,n-1)\}$.

For $2 \le t \le m-5$:

If $t \equiv 0,1 (mod\ 4)$; $S_t = S_{t-1} \cup \{(t+1,l): 2 \le l \le n-1\} \cup \{(t-1,1),(t-1,n)\}$.

If $t \equiv 2,3 (mod\ 4)$; $S_t = S_{t-1} \cup \{(t+1,l): 2 \le l \le n-1\} \cup \{(t-1,1),(t-1,n)\}$.

The final steps go as follows:

$t = m-4: S_{m-4} = S_{m-5} \cup \{(m-3,l): 2 \le l \le n-1\} \cup \{(m-5,1),(m-5,n)\}$.

$t = m-3: S_{m-3} = S_{m-4} \cup \{(m-2,l): 3 \le l \le n-2\} \cup \{(m-4,1),(m-4,n)\}$.

$t = m-2: S_{m-2} = S_{m-3} \cup \{(m-1,l): 3 \le l \le n-2\}$.

The process ends successfully at step $t = m-1$ for which:

$S_{m-1} = S_{m-2} \cup \{(m,l): 2 \le l \le n-1\} = V(P_m \times P_n)$. We conclude that S_0 is an I2CS on $P_m \times P_n$. Therefore $c_2(P_m \times P_n) \le 4\left\lfloor \frac{m-2}{4} \right\rfloor + n + 6$ if $m \equiv 0 (mod\ 4)$ and n is arbitrary. Figure 4 illustrates that $c_2(P_8 \times P_{10}) \le 20$.

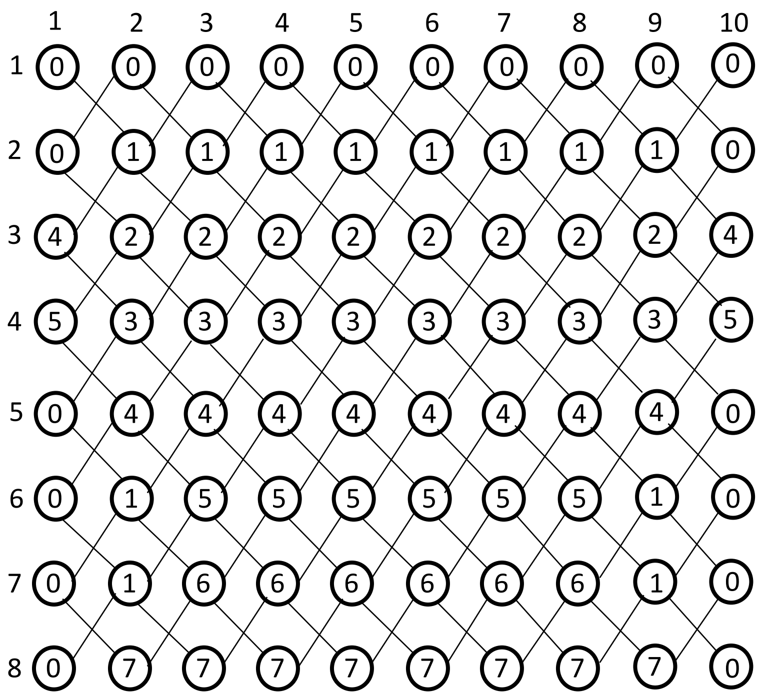

Figure 4. $c_2(P_8 \times P_{10}) \le 20$.

Case 2. $m \equiv 1 (mod\ 4)$ and n is arbitrary.

We choose $S_0 = \{(4l+1,1),(4l+2,1),(4l+1,n),(4l+2,n): 0 \le l \le \left\lfloor \frac{m-2}{4} \right\rfloor - 1\} \cup \{(m-4,1),(m-4,n),(m-3,1),(m-3,n),(m-1,1),(m-1,n),(m,1),(m,n)\}$ which is of cardinality $4\left\lfloor \frac{m-2}{4} \right\rfloor + n + 6$.

Case 3. $m \equiv 2 (mod\ 4)$ and n is arbitrary.

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We choose $S_0 = \{(4l+1,1), (4l+2,1), (4l+1,n), (4l+2,n) : 0 \le l \le \left\lfloor \frac{m-2}{4} \right\rfloor - 1\} \cup \{(m-1,1), (m-1,n), (m,1), (m,n)\}$ which is of cardinality $4\left\lfloor \frac{m-2}{4} \right\rfloor + n + 2$.

Case 4. $m \equiv 3 (mod\ 4)$ and n is arbitrary.

We choose $S_0 = \{(4l+1,1), (4l+2,1), (4l+1,n), (4l+2,n) : 0 \le l \le \left\lfloor \frac{m-2}{4} \right\rfloor - 1\} \cup \{(p,1), (p,n) : m-3 \le p \le m\}$ which is of cardinality $4\left\lfloor \frac{m-2}{4} \right\rfloor + n + 4$.

In Case 2, Case 3 and Case 4 we can track the conversion process in a similar way to Case 1 and in all these cases the process ends successfully at $t = m-1$ with $S_{m-1} = S_{m-2} \cup \{(m,l) : 2 \le l \le n-1\} = V(P_m \times P_n)$.

From all the cases we conclude the requested.□

Theorem 2.6. For $n \ge 3$:

i. $c_3(P_3 \times P_n) = 2n + 2$.
ii. $c_3(P_4 \times P_n) = 2n + 4$.

Proof. We consider the following cases for m:

Case 1. $m = 3$.

Since $k = 3$, then all vertices of $Q_1 \cup Q_2$ are 3-immuned and they must be contained in the seed set or else the process fails. This means $c_3(P_3 \times P_n) \ge |Q_1 \cup Q_2| = 2n + 2$, Now Let $S_0 = Q_1 \cup Q_2$ be the seed set. The process goes as follows:

$t = 0: S_0 = Q_1 \cup Q_2$.

$t = 1: S_1 = S_0 \cup Q_3 = V(P_3 \times P_n)$. Therefore S_0 is an I3CS on $P_3 \times P_n$ and $c_3(P_3 \times P_n) \le 2n + 2$ which means $c_3(P_3 \times P_n) \le 2n + 2$.

Case 2. $m = 4$. In this case:

$$Q_1 = \{(1,1), (1,n), (4,1), (4,n)\};$$

$$Q_2 = \{(1,j), (4,j) : 2 \le j \le n-1\} \cup \{(2,1), (2,n), (3,1), (3,n)\};$$

$$Q_3 = \{(2,j), (3,j) : 2 \le j \le n-1\}$$

In a similar way to Case 1, $S_0 = Q_1 \cup Q_2$ which is of cardinality $2n + 4$ and the process goes as follows:

$t = 0: S_0 = Q_1 \cup Q_2$.

$t = 1: S_1 = S_0 \cup Q_3 = V(P_4 \times P_n)$. Then S_0 is an I3CS on $P_4 \times P_n$ and we conclude that $c_3(P_4 \times P_n) = 2n + 4$.

From Case 1 and Case 2 we conclude the requested.□

Theorem 2.7. For $n \ge 5$; $c_3(P_5 \times P_n) = \begin{cases} 2n + 6 + \left\lfloor \frac{n-2}{4} \right\rfloor & if\ n \equiv 0,2,3 (mod\ 4); \\ 2n + 7 + \left\lfloor \frac{n-2}{4} \right\rfloor & if\ n \equiv 1 (mod\ 4). \end{cases}$

Proof. For $m = 5$ we have:

$$Q_1 = \{(1,1), (1,n), (5,1), (5,n)\};$$

$$Q_2 = \{(1,j), (5,j) : 2 \le j \le n-1\} \cup \{(2,1), (2,n), (3,1), (3,n), (4,1), (4,n)\};$$

$$Q_3 = \{(2,j), (3,j), (4,j) : 2 \le j \le n-1\}.$$

Since $k = 3$, all vertices of $Q_1 \cup Q_2$ must be included in S_0, it is obvious that $|Q_1 \cup Q_2| = 2n + 6$. We define the sets $U_j : 2 \le j \le n-3$ as $U_j = \{(3,j), (2,j+1), (4,j+1), (3,j+2)\}$. We notice that for any j if $U_j \cap S_0 = \emptyset$ then U_j is 3-unconvertable because each vertex of $u \in U_j$ is of degree 4 and is adjacent to two vertices of U_j. Figure 5 shows that U_3 is 3-unconvertable on $P_5 \times P_5$ if $U_3 \cap S_0 = \emptyset$ even when $S_0 = V - U_3$.

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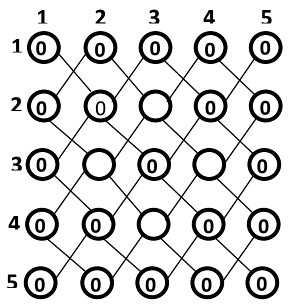

Figure 5. 3-unconvertable U_3 on $P_5 \times P_5$.

This means every set $\{(2,j),(3,j),(4,j),(2,j+1),(3,j+1),(4,j+1),(2,j+2),(3,j+2),(4,j+2),(2,j+3),(3,j+3),(4,j+3): 2 \le j \le n-4\}$ must contain at least two vertices of S_0, otherwise at least one version of $U_j \cap S_0 = \emptyset$ will be left on $P_5 \times P_n$ and the process will fail. To avoid that we need to include at least $\left\lfloor\frac{n-2}{4}\right\rfloor$ vertices of Q_3 in S_0 if $n \equiv 0,2,3 (\text{mod } 4)$ while we need to include $\left\lfloor\frac{n-2}{4}\right\rfloor + 1$ vertices of Q_3 in S_0 if $n \equiv 1 (\text{mod } 4)$. We conclude that:

$$c_3(P_5 \times P_n) \ge \begin{cases} 2n+6+\left\lfloor\frac{n-2}{4}\right\rfloor \; if \; n \equiv 0,2,3 (mod\ 4); \\ 2n+7+\left\lfloor\frac{n-2}{4}\right\rfloor if \; n \equiv 1 (mod\ 4). \end{cases} \quad (4)$$

We consider the following cases for n:

Case 1. $n \equiv 0 (mod\ 4)$.

Let the seed set be $S_0 = Q_1 \cup Q_2 \cup \{(3,4l+4),(3,4l+5): 0 \le l \le \left\lfloor\frac{n-2}{4}\right\rfloor - 1\}$ which is of cardinality $2n+6+\left\lfloor\frac{n-2}{4}\right\rfloor$. The process goes as follows:

$t = 0: S_0 = Q_1 \cup Q_2 \cup \{(3,4l+4),(3,4l+5): 0 \le l \le \left\lfloor\frac{n-2}{4}\right\rfloor - 1\}$.

$t = 1: S_1 = S_0 \cup \{(2,l),(4,l): 2 \le l \le n-1\}$.

$t = 2: S_2 = S_1 \cup \{(3,4l+2),(3,4l+3): 0 \le l \le \left\lfloor\frac{n-2}{4}\right\rfloor - 1\} = V(P_5 \times P_n)$ which means S_0 is an I3CS on $P_5 \times P_n$. Therefore, $c_3(P_5 \times P_n) \le 2n+6+\left\lfloor\frac{n-2}{4}\right\rfloor$. From (4) we conclude that $c_3(P_5 \times P_n) = 2n+6+\left\lfloor\frac{n-2}{4}\right\rfloor$ if $n \equiv 0 (mod\ 4)$

Case 2. $n \equiv 1 (mod\ 4)$.

We choose $S_0 = Q_1 \cup Q_2 \cup \{(3,4l+4),(3,4l+5): 0 \le l \le \left\lfloor\frac{n-2}{4}\right\rfloor - 1\} \cup \{(3,n-1)\}$ which is of cardinality $2n+7+\left\lfloor\frac{n-2}{4}\right\rfloor$. The process as follows:

$t = 0: S_0 = Q_1 \cup Q_2 \cup \{(3,4l+4),(3,4l+5): 0 \le l \le \left\lfloor\frac{n-2}{4}\right\rfloor - 1\}$.

$t = 1: S_1 = S_0 \cup \{(2,l),(4,l): 2 \le l \le n-1\}$.

$t = 2: S_2 = S_1 \cup \{(3,4l+2),(3,4l+3): 0 \le l \le \left\lfloor\frac{n-2}{4}\right\rfloor - 1\} = V(P_5 \times P_n)$.

Case 3. $n \equiv 2 (mod\ 4)$.

$t = 0: S_0 = Q_1 \cup Q_2 \cup \{(3,4l+4),(3,4l+5): 0 \le l \le \left\lfloor\frac{n-2}{4}\right\rfloor - 1\}$.

$t = 1: S_1 = S_0 \cup \{(2,l),(4,l): 2 \le l \le n-1\}$.

$t = 2: S_2 = S_1 \cup \{(3,4l+2),(3,4l+3): 0 \le l \le \left\lfloor\frac{n-2}{4}\right\rfloor - 1\} = V(P_5 \times P_n)$.

Case 4. $n \equiv 3 (mod\ 4)$.

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$t=0: S_0 = Q_1 \cup Q_2 \cup \{(3,4l+4),(3,4l+5): 0 \leq l \leq \left\lfloor \frac{n-2}{4} \right\rfloor - 1\}$.

$t=1: S_1 = S_0 \cup \{(2,l),(4,l): 2 \leq l \leq n-1\}$.

$t=2: S_2 = S_1 \cup \{(3,4l+2),(3,4l+3): 0 \leq l \leq \left\lfloor \frac{n-2}{4} \right\rfloor - 1\} \cup \{(3,n-1)\} = V(P_5 \times P_n)$.

From all the cases we conclude the requested.□

Theorem 2.8. For $m,n \geq 3$;

$$c_4(P_m \times P_n) = \begin{cases} nm - \max\{(n-2)\left\lceil \frac{m-2}{2} \right\rceil, (m-2)\left\lceil \frac{n-2}{2} \right\rceil\} & \text{if m or n is odd;} \\ \frac{mn+2m+2n-4}{2} & \text{if m and n are even.} \end{cases}.$$

Proof. Since $k=4$, all vertices of $Q_1 \cup Q_2$ must be included in S_0. Otherwise, the process automatically fails. Since every $u \in Q_3$ is of degree 4, there cannot be two adjacent unconverted vertices of Q_3 at $t=0$ or else neither one of these two vertices will satisfy the conversion rule at any step of the process, therefore the process fails. To avoid that, $Q_3 - S_0$ must be independent. In order to make S_0 as small as possible, we try to make $Q_3 - S_0$ as large as possible, thus $Q_3 - S_0$ must be the largest independent set of the graph G_{Q_3} which is induced by Q_3 on $P_m \times P_n$, which means $|Q_3 - S_0| = \alpha(G_{Q_3})$. We notice that G_{Q_3} represents a $P_{m-2} \times P_{n-2}$ graph. Therefore, due to Proposition 1.2, we have $\alpha(G_{Q_3}) = \alpha(P_{m-2} \times P_{n-2}) = \max\{\alpha(P_{m-2})|P_{n-2}|, \alpha(P_{n-2})|P_{m-2}|\}$ and the smallest seed set S_0 on $P_m \times P_n$ that contains $Q_1 \cup Q_2$ and guarantees not leaving two adjacent unconverted vertices from Q_3 is of cardinality:

$$|S_0| = |Q_1| + |Q_2| + |Q_3| - \alpha(P_{m-2} \times P_{n-2}).$$

This means:

$c_4(P_m \times P_n) = nm - \max\{(n-2)\left\lceil \frac{m-2}{2} \right\rceil, (m-2)\left\lceil \frac{n-2}{2} \right\rceil\}$. However, in case m,n are even, then $\left\lceil \frac{m-2}{2} \right\rceil = \frac{m-2}{2}$ and $\left\lceil \frac{n-2}{2} \right\rceil = \frac{n-2}{2}$, then $\max\{(n-2)\left\lceil \frac{m-2}{2} \right\rceil, (m-2)\left\lceil \frac{n-2}{2} \right\rceil\} = \frac{(m-2)(n-2)}{2}$ which means $c_4(P_m \times P_n) = nm - \frac{(m-2)(n-2)}{2} = \frac{mn+2m+2n-4}{2}$ and we prove the requested.□

3. Recommendations and conclusion

In this paper, we study IkCSs of toroidal grids and the tensor product of two paths. We determine $c_2(C_3 \times C_n)$ and we present upper and lower bounds for $c_2(C_m \times C_n)$ for $m,n \geq 3$. We also determine $c_2(P_2 \times P_n)$, $c_2(P_3 \times P_n)$ and present an upper bound for $c_2(P_m \times P_n)$ when $m,n > 3$. Then we determine $c_3(P_m \times P_n)$ for $m = 2,3,4$ and arbitrary n. Finally, we determine $c_4(P_m \times P_n)$ for arbitrary m,n.

In the future, we recommend the interested researchers in graph theory generally, and in neutrosophic graph theory as a special case, to try to generalize our results from classical graph products to direct neutrosophic graph products. In addition, the same result can be applied on fuzzy graphs products to obtain similar results.

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References

[1] Adams S.S,BrassZ.,StokesC.,and Troxell D.S.,(2015).Irreversible k-threshold and majority conversion processes on complete multipartite graphs and graph products, Australasian Journal of Combinatorics. (6)1156- 174. https://ajc.maths.uq.edu.au/pdf/61/ajc_v61_p156.pdf.

[2] Centeno C. C.,Dourado M. C., Penso L. D., Rautenbach D and Szwarcfiter J L, (2011) . Irreversible conversion of graphs, Theoretical Computer Science. 412 3693-3700. (DOI: https://doi.org/10.1016/j.tcs.2011.03.029).

[3] Dreyer, Jr P A. and Roberts F S., (2009). Irreversible k-threshold processes: Graph theoretical threshold models of the spread of disease and of opinion, Discrete Applied Mathematics. 157(7) 615-1627.(DOI: https://doi.org/10.1016/j.dam.2008.09.012).

DOI: https://doi.org/10.54216/IJNS.250216
Received: February 15, 2024 Revised: May 04, 2024 Accepted: August 10, 2024

[4] Frances M D, Mynhardt C M and Wodlinger J L, (2019). Subgraph-avoiding minimum decycling sets and k-conversion sets in graphs, Australasian Journal of Combinatorics. 74(3) 288-304. https://ajc.maths.uq.edu.au/pdf/74/ajc_v74_p288.pdf.
[5] Harary F, Graph Theory, Addison-Wesley, Reading Mass (1969).
[6] Jha P K and Klavžar S., (1998). Independence in direct-product graphs, Ars Combin. 50 53-63.
[7] Hatamleh, R. (2003). On the Form of Correlation Function for a Class of Nonstationary Field with a Zero Spectrum. Rocky Mountain Journal of Mathematics, 33(1)159-173. https://doi.org/10.1216/rmjm/1181069991.
[8] Kynčl J, Lidicky B. and Vyskočil T., (2017). Irreversible 2-conversion set in graphs of bounded degree, Discrete Mathematics & Theoretical Computer Science, 19(3) 81-89. (DOI: https://doi.org/10.23638/DMTCS-19-3-5).
[9] Hatamleh, R., Zolotarev,V.A. (2014). On Two-Dimensional Model Representations of One Class of Commuting Operators. Ukrainian Mathematical Journal, 66(1), 122–144. (Doi:https://doi.org/10.1007/s11253-014-0916-9).
[10] Mynhardt C M and Wodlinger J L,(2019).Alower bound on the k-conversion number of graphs of maximum degree k + 1, Transactions on Combinatorics. 9(3) 1-12.(DOI: http://dx.doi.org/10.22108/toc.2019.112258.1579).
[11] Hatamleh, R., Zolotarev,V.A.(2016). Triangular Models of Commutative Systems of Linear Operators Close to Unitary Operators. Ukrainian Mathematical Journal, 68(5), 791–811. (Doi: https://doi.org/10.1007/s11253-016-1258-6).
[12] Mynhardt C M and Wodlinger J L, (2020). The k-conversion number of regular graphs, AKCE International Journal of Graphs and Combinatorics. 17(3) 955-965, (DOI:https://doi.org/10.1016/j.akcej.2019.12.016).
[13] Shaheen R, Mahfud S and Kassem A, Irreversible k-Threshold Conversion Number of Circulant Graphs. Submitted.
[14] Takaoka A and Ueno S., ((2015).A note on irreversible 2-conversion sets in subcubic graphs, IEICE Transactions on Information and Systems. E98.D (8) 1589–1591. (Doi : https://doi.org/10.1007/a13278-012-0067-7).
[15] Wang T M., (2005). Toroidal grids are anti-magic, Computing and Combinatorics, Lecture Notes in Comput. Sci, 3595, Springer, Berlin, 671-679. (Doi: https://link.springer.com/chapter/10.1007/11533719_68).
[16] Heilat, A.S., Batiha,B ,T.Qawasmeh ,Hatamleh R.(2023). Hybrid Cubic B-spline Method for Solving A Class of Singular Boundary Value Problems. European Journal of Pure and Applied Mathematics, 16(2), 751-762. (DOI: https://doi.org/10.29020/nybg.ejpam.v16i2.4725).
[17] T.Qawasmeh,R.Hatamleh,(2023).A new contraction based on H-simulation functions in the frame of extended b-metric spaces and application,International Journal of Electrical and Computer Engineering 13 (4),4212- 4221.
[18] Ayman Hazaymeh, Rania Saadeh, Raed Hatamleh, Mohammad W Alomari, Ahmad Qazza. (2023). A Perturbed Milne’s Quadrature Rule for n-Times Differentiable Functions with Lp-Error Estimates, Axioms-MDPI,12(9),803.
[19] Ayman Hazaymeh, Ahmad Qazza, Raed Hatamleh,Mohammad W. Alomari,Rania Saadeh.(2023). On Further Refinements of Numerical Radius Inequalities. Axiom-MDPI,12(9),807.
[20] Batiha, B., Ghanim, F., Alayed, O., Hatamleh, R., Heilat, A. S., Zureigat, H., & Bazighifan, O. (2022). Solving Multispecies Lotka–Volterra Equations by the Daftardar-Gejji and Jafari Method. International Journal of Mathematics and Mathematical Sciences, 2022, 1–7. (Doi: https://doi.org/10.1155/2022/1839796).
[21] Al-Husban, A., Al-Qadri, M. O., Saadeh, R., Qazza, A., & Almomani, H. H. (2022). Multi-fuzzy rings. WSEAS Trans. Math, 21, 701-706.
[22] Hamadneh, T., Hioual, A., Alsayyed, O., Al-Khassawneh, Y. A., Al-Husban, A., & Ouannas, A. (2023). The FitzHugh–Nagumo Model Described by Fractional Difference Equations: Stability and Numerical Simulation. Axioms, 12(9), 806.
[23] Al-Husban, A. B. D. U. L. L. A. H., Al-Sharoa, D. O. A. A., Al-Kaseasbeh, M. O. H. A. M. M. A. D., & Mahmood, R. M. S. (2022). Structures of fibers of groups actions on graphs. WSEAS Trans. Math, 7, 650-658.
[24] AL-HUSBAN, A. B. D. A. L. L. A. H., AL-SHAROA, D. O. A. A., SAADEH, R., QAZZA, A., & MAHMOOD, R. (2023). ON THE FIBERS OF THE TREE PRODUCTS OF GROUPS WITH AMALGAMATION SUBGROUPS. Journal of applied mathematics & informatics, 41(6), 1237-1256.

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