EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 17, No. 4, 2024, 2467-2480
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

A Subclass of Bi-univalent Functions Defined by a Symmetric q-Derivative Operator and Gegenbauer Polynomials

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Abstract. This paper introduces a novel subclass of bi-univalent analytic functions by utilizing a symmetric q-derivative operator in conjunction with Gegenbauer polynomials. Within this newly defined subclass, we derive bounds for the first two Maclaurin coefficients and address the Fekete-Szegő problem. By varying the parameters in our results between 0 and 1, we obtain a range of new insights and rediscover some previously established results. This approach not only broadens the scope of bi-univalent function theory but also deepens the understanding of coefficient bounds and extremal problems within this context.

2020 Mathematics Subject Classifications: 30C45

Key Words and Phrases: Bi-univalent analytic functions, Gegenbauer (or Ultraspherical) polynomials, Fekete-Szegö functional

1. Definitions and Preliminaries

Let $\mathcal{A}$ denote the class of all analytic functions f defined in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ and normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. Thus each $f \in \mathcal{A}$ has a Taylor-Maclaurin series expansion of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (z \in \mathbb{U}). \tag{1}$$

Let $\mathcal{S}$ denote the class of all functions $f \in \mathcal{A}$ which are univalent in $\mathbb{U}$. In addition, Subordination, denoted as $f \prec g$, between functions f and g in $\mathcal{S}$ occurs when there exists

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v17i4.5408

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an analytic function $w(z)$ such that $w(0) = 0$, $|w(z)| < 1$ for $z \in \mathbb{U}$, and $f(z) = g(w(z))$. The inverse function of the function $f \in \mathcal{S}$ is given by:

$$f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots . \tag{2}$$

A function f is said to be bi-univalent in $\mathbb{U}$ if both $f(z)$ and $f^{-1}(z)$ are univalent in $\mathbb{U}$. Let Σ denote the class of bi-univalent functions in $\mathbb{U}$ given by (1).

Lewin [25], Brannan and Clunie [9], and Netanyahu [28] are known to be the first researchers who have studied the class Σ. Since then, the class Σ has attracted several researchers, see [2, 3, 13, 16, 20, 27, 30, 33, 34].

Orthogonal polynomials have been widely studied since their discovery by Legendre in 1784 [24]. They have been used as a mathematical approach to solve ordinary differential equations associated with model problems under certain conditions. The advantages of orthogonal polynomials in modern mathematics and their application in physics and engineering cannot be ignored. Orthogonal polynomials play a key role in approximation theory, differential integral equations, and mathematical statistics. Additionally, these polynomials have been instrumental in various applications, such as scattering theory, signal analysis [1, 5, 8, 10, 12, 14, 15, 17, 18, 31].

Let $C_n^{\alpha}(x)$ be the Gegenbauer polynomial of degree n defined using the following recurrence relation

$$C_n^{\alpha}(x) = \frac{1}{n}\left[2x(n+\alpha-1)C_{n-1}^{\alpha}(x) - (n+2\alpha-2)C_{n-1}^{\alpha}(x)\right],$$

with

$$\begin{aligned} &C_0^{\alpha}(x) = 1, \\ &C_1^{\alpha}(x) = 2\alpha x, \\ &C_2^{\alpha}(x) = 2\alpha(1+\alpha)x^2 - \alpha. \end{aligned} \tag{3}$$

The Gegenbauer polynomials generate Legendre polynomials and Chebyshev polynomials when taking α equaling 1/2 and 1, respectively.

Amourah *et al.* [6] were the first to investigate the polynomials generated by $H_\alpha(x, z)$, defining them as follows:

$$H_\alpha(x, z) = \frac{1}{(1 - 2xz + z^2)^{\alpha}}, (-1 \leq x \leq 1, and\ z \in \mathbb{U}).$$

Also, since H_α is an analytic function in $\mathbb{U}$, it can be expressed as follows:

$$H_\alpha(x, z) = \sum_{n=0}^{\infty} C_n^{\alpha}(x) z^n. \tag{4}$$

The theory of q-calculus operators has many applications in science and engineering. Notably, several researchers have made significant contributions to the study of q-calculus, see [4, 7, 23, 26, 29].

Definition 1. ([22]) *Let $f \in \mathcal{A}$, the Jackson's q-derivative is defined by*

$$\mathbb{D}_q f(z) = \begin{cases} \frac{f(z)-f(qz)}{(1-q)z} & \text{for} \quad z \neq 0, \\ f'(0) & \text{for} \quad z = 0 \end{cases} \tag{5}$$

where $0 < q < 1$. From (5), we can write

$$\mathbb{D}_q f(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1} \tag{6}$$

where $[n]_q$ denotes the basic number and given by

$$[n]_q = \frac{1-q^n}{1-q}, n \in \mathbb{N} = \{1, 2, \ldots\}.$$

Definition 2. *For a function f given by (1), the symmetric q-derivative is defined as:*

$$\left(\widetilde{\mathbb{D}}_q f\right)(z) = \begin{cases} \frac{f(qz)-f\left(q^{-1}z\right)}{(q-q^{-1})z} & z \neq 0 \\ f'(0) & z = 0 \end{cases}. \tag{7}$$

Equation (7) implies $\widetilde{\mathbb{D}}_q z^n = \widetilde{[n]}_q z^{n-1}$, and $\widetilde{\mathbb{D}}_q f$ of a function f given by (1) is defined as

$$\left(\widetilde{\mathbb{D}}_q f\right)(z) = 1 + \sum_{n=2}^{\infty} \widetilde{[n]}_q a_n z^{n-1}$$

where the symbol $\widetilde{[n]}_q$ is defined as

$$\widetilde{[n]}_q = \frac{q^n - q^{-n}}{q - q^{-1}}.$$

Using equations (2) and (7), we obtain

$$\begin{aligned} \left(\widetilde{\mathbb{D}}_q g\right)(w) &= \frac{g(qw) - g\left(q^{-1}w\right)}{\left(q-q^{-1}\right)w} \\ &= 1 - \widetilde{[2]}_q a_2 w + \widetilde{[3]}_q \left(2a_2^2 - a_3\right) w^2 - \widetilde{[4]}_q \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^3 + \cdots \end{aligned} \tag{8}$$

In recent times, numerous researchers have been investigating the concept of bi-univalent functions linked to Gegenbour polynomials. Some notable studies in this area include references [19] and [21]. In the present work, we propose the following novel subclasses.

2. The class $\mathcal{B}_{\Sigma}^{\alpha}(t, \gamma, \nu, \epsilon)$

Definition 3. ([32]) *For $\gamma \geq 1$, $\nu, \epsilon \geq 0$, $0 \leq \alpha \leq 1$, $\zeta = \frac{2\gamma+\nu}{2\gamma+1}$ and $t \in (1/2, 1]$, a function $f \in \Sigma$ given by (1) is in $\mathcal{M}_{\Sigma}^{\alpha}(\gamma, \nu, \epsilon)$ if for all $z, w \in \mathbb{D}$, it satisfies the following subordination:*

$$\mathfrak{Re}\left((1-\gamma)\left(\frac{f(z)}{z}\right)^{\nu} + \gamma f'(z)\left(\frac{f(z)}{z}\right)^{\nu-1} + \zeta\epsilon z f''(z)\right) > \alpha \tag{9}$$

and

$$\mathfrak{Re}\left((1-\gamma)\left(\frac{g(w)}{w}\right)^{\nu}+\gamma g'(w)\left(\frac{g(w)}{w}\right)^{\nu-1}+\zeta\epsilon z g''(w)\right)>\alpha \tag{10}$$

where $f \in \Sigma$ defined by (1), and $g = f^{-1}$ given by (2).

Definition 4. *Let $\alpha > 0$, $\gamma \geq 1$, $\nu \geq 0$, $\epsilon \geq 0$, $\zeta = \frac{2\gamma+\nu}{2\gamma+1}$, $t \in (1/2, 1]$, and $f \in \Sigma$ that is given by (4) is in $\widetilde{\mathcal{B}}^q_\Sigma(t,\gamma,\nu,\epsilon)$ if for all $z, w \in \mathbb{D}$, it satisfies the following subordination*

$$(1-\gamma)\left(\frac{f(z)}{z}\right)^{\nu}+\gamma\widetilde{\mathbb{D}}_q\left(f(z)\right)\left(\frac{f(z)}{z}\right)^{\nu-1}+\zeta\epsilon z\widetilde{\mathbb{D}}_q\left(\widetilde{\mathbb{D}}_q\left(f(z)\right)\right)\prec H_\alpha(t,z) \tag{11}$$

and

$$(1-\gamma)\left(\frac{g(w)}{w}\right)^{\nu}+\gamma\widetilde{\mathbb{D}}_q\left(g(w)\right)\left(\frac{g(w)}{w}\right)^{\nu-1}+\zeta\epsilon z\widetilde{\mathbb{D}}_q\left(\widetilde{\mathbb{D}}_q\left(g(w)\right)\right)\prec H_\alpha(t,w), \tag{12}$$

where $g = f^{-1}(w)$ is given by (2). given by (4).

Definition 5. *The function $f \in {}^1\widetilde{\mathcal{B}}^q_\Sigma(t,\gamma,\nu) := \widetilde{\mathcal{B}}^q_\Sigma(t,\gamma,\nu,0)$ iff it satisfies the following subordination*

$$(1-\gamma)\left(\frac{f(z)}{z}\right)^{\nu}+\gamma\widetilde{\mathbb{D}}_q\left(f(z)\right)\left(\frac{f(z)}{z}\right)^{\nu-1}\prec H_\alpha(t,z)$$

and

$$(1-\gamma)\left(\frac{g(w)}{w}\right)^{\nu}+\gamma\widetilde{\mathbb{D}}_q\left(g(w)\right)\left(\frac{g(w)}{w}\right)^{\nu-1}\prec H_\alpha(t,w).$$

Definition 6. *The function $f \in {}^2\widetilde{\mathcal{B}}^q_\Sigma(t,\gamma,\epsilon) := \widetilde{\mathcal{B}}^q_\Sigma(t,\gamma,1,\epsilon)$ iff it satisfies the following subordination*

$$(1-\gamma)\left(\frac{f(z)}{z}\right)+\gamma\widetilde{\mathbb{D}}_q\left(f(z)\right)+\zeta\epsilon z\widetilde{\mathbb{D}}_q\left(\widetilde{\mathbb{D}}_q\left(f(z)\right)\right)\prec H_\alpha(t,z)$$

and

$$(1-\gamma)\left(\frac{g(w)}{w}\right)+\gamma\widetilde{\mathbb{D}}_q\left(g(w)\right)+\zeta\epsilon z\widetilde{\mathbb{D}}_q\left(\widetilde{\mathbb{D}}_q\left(g(w)\right)\right)\prec H_\alpha(t,w).$$

Definition 7. *The function $f \in {}^3\widetilde{\mathcal{B}}^q_\Sigma(t,\gamma) := \widetilde{\mathcal{B}}^q_\Sigma(t,\gamma,1,0)$ iff it satisfies the following subordination:*

$$(1-\gamma)\left(\frac{f(z)}{z}\right)+\gamma\widetilde{\mathbb{D}}_q\left(f(z)\right)\prec H_\alpha(t,z)$$

and

$$(1-\gamma)\left(\frac{g(w)}{w}\right)+\gamma\widetilde{\mathbb{D}}_q\left(g(w)\right)\prec H_\alpha(t,w).$$

Definition 8. *The function* $f \in {}^{4}\widetilde{\mathcal{B}}_{\Sigma}^{q}(t) := \widetilde{\mathcal{B}}_{\Sigma}^{q}(t,1,1,0)$ *iff it satisfies the following subordination:*

$$\widetilde{\mathbb{D}}_q\left(f(z)\right) \prec H_\alpha(t,z)$$

and

$$\left(\widetilde{\mathbb{D}}_q g(w)\right) \prec H_\alpha(t,w).$$

Let $\mathcal{P} = \{p : \mathbb{U} \to \mathbb{C} \mid p(z) = 1 + \sum_{n=1}^{\infty} p_n\, z^n,$ is analytic function, and $\mathfrak{Re}(p) > 0\}$. The following lemma will be used when proofing our main results.

Lemma 1. ([11]) *If* $p \in \mathcal{P}$, *then*

$$|p_n| \leq 2, n \in \mathbb{N}. \tag{13}$$

Throughout the rest of the paper, we assume that $0 < q < 1,\ x \in \left(\frac{1}{2}, 1\right]$ and α is a nonzero real constant.

3. Main Results

Theorem 1. *Let* $f \in \widetilde{\mathcal{B}}_{\Sigma}^{q}(t,\gamma,\nu,\epsilon)$. *Then*

$$|a_2| \leq \frac{2\alpha x\sqrt{x}}{\sqrt{\left| x^2\left[\alpha\left(2\widetilde{[2]}_q\gamma(\nu-1) + 2\widetilde{[3]}_q\gamma + \nu(\nu-2\gamma+1) + 2\widetilde{[2]}_q\widetilde{[3]}_q\zeta\epsilon\right) - 2(1+\alpha)\Upsilon\right] + (1+2x)\Upsilon\right|}}$$

and

$$|a_3| \leq \frac{2\left[\widetilde{[3]}_q\gamma - \widetilde{[2]}_q\zeta\epsilon\right]x^2\alpha^2}{\Upsilon} - \frac{4\alpha x}{\left[\gamma\left(\widetilde{[3]}_q + \nu - 1\right) + (1-\gamma)\nu + \widetilde{[2]}_q\widetilde{[3]}_q\zeta\epsilon\right]},$$

where

$$\Upsilon := \left(\nu - \gamma + \widetilde{[2]}_q(\gamma + \zeta\epsilon)\right)^2$$

.

Proof. Let $f \in \widetilde{\mathcal{B}}_{\Sigma}^{q}(t,\gamma,\nu,\epsilon)$. By Definition 4, there exist u, v such that $u(0) = v(0) = 0$ and $|u(z)| < 1, |v(w)| < 1$ where $z, w \in \mathbb{U}$, then

$$(1-\gamma)\left(\frac{f(z)}{z}\right)^{\nu} + \gamma\widetilde{\mathbb{D}}_q\left(f(z)\right)\left(\frac{f(z)}{z}\right)^{\nu-1} + \zeta\epsilon z\widetilde{\mathbb{D}}_q\left(\widetilde{\mathbb{D}}_q\left(f(z)\right)\right) = H_\alpha(x,u(z)) \tag{14}$$

and

$$(1-\gamma)\left(\frac{g(w)}{w}\right)^{\nu}+\gamma\widetilde{\mathbb{D}}_q\left(g(w)\right)\left(\frac{g(w)}{w}\right)^{\nu-1}+\zeta\epsilon z\widetilde{\mathbb{D}}_q\left(\widetilde{\mathbb{D}}_q\left(g(w)\right)\right)=H_\alpha(x,v(w)) \tag{15}$$

Now, let $p, q \in \mathcal{P}$ given by

$$p(z)=\frac{1+u(z)}{1-u(z)}=1+c_1z+c_2z^2+\cdots$$

and

$$q(w)=\frac{1+v(w)}{1-v(w)}=1+d_1w+d_2w^2+\cdots.$$

Hence, we can write

$$u(z)=\frac{p(z)-1}{p(z)+1}=\frac{1}{2}c_1z+\frac{1}{2}\left(c_2-\frac{1}{2}c_1^2\right)z^2+\cdots \tag{16}$$

and

$$v(w)=\frac{q(w)-1}{q(w)+1}=\frac{1}{2}d_1w+\frac{1}{2}\left(d_2-\frac{1}{2}d_1^2\right)w^2+\cdots. \tag{17}$$

Now, using equations (14), (15), (16) and (17), we can write

$$H_\alpha(x,u(z))=1+\frac{1}{2}C_1^\alpha(x)c_1z+\left[\frac{1}{4}C_2^\alpha(x)c_1^2+\frac{1}{2}C_1^\alpha(x)\left(c_2-\frac{1}{2}c_1^2\right)\right]z^2+\cdots, \tag{18}$$

and

$$H_\alpha(x,v(w))=1+\frac{1}{2}C_1^\alpha(x)d_1w+\left[\frac{1}{4}C_2^\alpha(x)d_1^2+\frac{1}{2}C_1^\alpha(x)\left(d_2-\frac{1}{2}d_1^2\right)\right]w^2+\cdots. \tag{19}$$

Also, from equations (18) and (19), we get

$$\left(\nu-\gamma+\widetilde{[2]}_q(\gamma+\zeta\epsilon)\right)a_2=\frac{1}{2}C_1^\alpha(x)c_1, \tag{20}$$

$$\begin{aligned}(\nu-1)\left[\gamma\widetilde{[2]}_q+\frac{\gamma(\nu-2)}{2}+\frac{(1-\gamma)\nu}{2}\right]a_2^2+\left[\gamma\left(\widetilde{[3]}_q+\nu-1\right)+(1-\gamma)\nu+\widetilde{[2]}_q\widetilde{[3]}_q\zeta\epsilon\right]a_3\\=\frac{1}{2}C_1^\alpha(x)\left(c_2-\frac{c_1^2}{2}\right)+\frac{1}{4}C_2^\alpha(x)c_1^2,\end{aligned} \tag{21}$$

$$-\left(\nu-\gamma+\widetilde{[2]}_q(\gamma+\zeta\epsilon)\right)a_2=\frac{1}{2}C_1^\alpha(x)d_1, \tag{22}$$

and

$$\begin{aligned}&(\nu-1)\left[\widetilde{[2]}_q\gamma+2\widetilde{[3]}_q\gamma+\frac{(\nu+2)\gamma}{2}+\frac{\nu(\nu+3)(1-\gamma)}{2(\nu-1)}+\frac{2\widetilde{[2]}_q\widetilde{[3]}_q\zeta\epsilon}{(\nu-1)}\right]a_2^2\\&-\left[\gamma\left(\widetilde{[3]}_q+\nu-1\right)+(1-\gamma)\nu+\widetilde{[2]}_q\widetilde{[3]}_q\zeta\epsilon\right]a_3=\frac{1}{2}C_1^\alpha(x)\left(d_2-\frac{d_1^2}{2}\right)+\frac{1}{4}C_2^\alpha(x)d_1^2.\end{aligned} \tag{23}$$

Equations (20) and (22) implies

$$c_1 = -d_1 \tag{24}$$

and

$$8\left(\nu - \gamma + \widetilde{[2]}_q(\gamma + \zeta\epsilon)\right)^2 a_2^2 = \left(C_1^{\alpha}(x)\right)^2\left(c_1^2 + d_1^2\right). \tag{25}$$

Adding (21) and (23), we deduce

$$\left[2\widetilde{[2]}_q\gamma(\nu-1) + 2\widetilde{[3]}_q\gamma + \nu(\nu - 2\gamma + 1) + 2\widetilde{[2]}_q\widetilde{[3]}_q\zeta\epsilon\right]a_2^2 = \frac{1}{2}C_1^{\alpha}(x)(c_2 + d_2) + \frac{1}{4}\left(C_2^{\alpha}(x) - C_1^{\alpha}(x)\right)\left(c_1^2 + d_1^2\right). \tag{26}$$

Plugging $\left(c_1^2 + d_1^2\right)$ obtained from (25) into (26) yield

$$\left[2\widetilde{[2]}_q\gamma(\nu-1) + 2\widetilde{[3]}_q\gamma + \nu(\nu - 2\gamma + 1) + 2\widetilde{[2]}_q\widetilde{[3]}_q\zeta\epsilon - \frac{2\Upsilon\left[C_2^{\alpha}(x) - C_1^{\alpha}(x)\right]}{\left[C_1^{\alpha}(x)\right]^2}\right]a_2^2 = \frac{1}{2}C_1^{\alpha}(x)(c_2 + d_2), \tag{27}$$

where

$$\Upsilon := \left(\nu - \gamma + \widetilde{[2]}_q(\gamma + \zeta\epsilon)\right)^2.$$

Furthermore, from (13), (19) and (27), it follows that

$$|a_2| \leq \frac{2\alpha x\sqrt{x}}{\sqrt{\left|x^2\left[\alpha\left(2\widetilde{[2]}_q\gamma(\nu-1) + 2\widetilde{[3]}_q\gamma + \nu(\nu - 2\gamma + 1) + 2\widetilde{[2]}_q\widetilde{[3]}_q\zeta\epsilon\right) - 2(1+\alpha)\Upsilon\right] + (1+2x)\Upsilon\right|}}$$

Subtracting (21) from (23), we have

$$2\left[\gamma\left(\widetilde{[3]_q} + \nu - 1\right) + (1-\gamma)\nu + \widetilde{[2]_q}\widetilde{[3]_q}\zeta\epsilon\right]\left(a_3 - a_2^2\right) = \frac{1}{2}C_1^{\alpha}(x)(c_2 - d_2) + \frac{1}{4}\left(C_2^{\alpha}(x) - C_1^{\alpha}(x)\right)\left(c_1^2 - d_1^2\right). \tag{28}$$

Utilizing equations (3) and (25), we can write (28) as

$$a_3 = a_2^2 + \frac{C_1^{\alpha}(x)}{4\left[\gamma\widetilde{[3]}_q - (\gamma - \nu) + \widetilde{[2]_q}\widetilde{[3]_q}\zeta\epsilon\right]}(c_2 - d_2). \tag{29}$$

Now, using equations (3) and (13), we can write

$$a_3 = \frac{2\alpha^2 x^3 (c_2 + d_2)}{x^2 \left[4\gamma\widetilde{[2]}_q(\nu - 1) + 4\gamma\widetilde{[3]}_q + 2\nu(\nu - 2\gamma + 1) + 4\widetilde{[2]}_q\widetilde{[3]}_q\zeta\epsilon - 2(1+\alpha)\Upsilon\right] + (1+2x)\Upsilon} + \frac{\alpha x (c_2 - d_2)}{2\left[\gamma\widetilde{[3]}_q - (\gamma - \nu) + \widetilde{[2]}_q\widetilde{[3]}_q\zeta\epsilon\right]}. \tag{30}$$

This concludes the proof of Theorem 1.

4. The Fekete-Szegö Inequality $|a_3 - \varphi a_2^2|$

Theorem 2. *If* $f \in \widetilde{\mathcal{B}}_\Sigma^q(t, \gamma, \nu, \epsilon)$, *then*

$$\left|a_3 - \varphi a_2^2\right| \leq \begin{cases} \frac{\alpha x}{\gamma\widetilde{[3]}_q - (\gamma-\nu) + \widetilde{[2]}_q\widetilde{[3]}_q\zeta\epsilon} & \text{if} \quad 0 \leq |h(\varphi)| \leq \frac{1}{2\left[\gamma\widetilde{[3]}_q - (\gamma-\nu) + \widetilde{[2]}_q\widetilde{[3]}_q\zeta\epsilon\right]}, \\ 2\alpha x\,|h(\varphi)| & \text{if} \quad |h(\varphi)| \geq \frac{1}{2\left[\gamma\widetilde{[3]}_q - (\gamma-\nu) + \widetilde{[2]}_q\widetilde{[3]}_q\zeta\epsilon\right]}, \end{cases}$$

where

$$h(\varphi) = \frac{2(1-\varphi)\alpha x^2}{\alpha x^2 \left[4\gamma\widetilde{[2]}_q(\nu - 1) + 4\gamma\widetilde{[3]}_q + 2\nu(\nu - 2\gamma + 1) + 4\widetilde{[2]}_q\widetilde{[3]}_q\zeta\epsilon - 2(1 + (1/\alpha))\Upsilon\right] + (1+2x)\Upsilon},$$

and

$$\Upsilon := \left(\nu - \gamma + \widetilde{[2]}_q(\gamma + \zeta\epsilon)\right)^2.$$

Proof: Consider f in $\mathcal{B}_\Sigma^\alpha(x, \tau, \gamma, \nu, \epsilon)$, then by (29) we obtain

$$\begin{aligned} a_3 - \varphi a_2^2 &= a_2^2 + \frac{C_1^\alpha(x)}{4\left[\gamma\widetilde{[3]}_q - (\gamma - \nu) + \widetilde{[2]}_q\widetilde{[3]}_q\zeta\epsilon\right]}(c_2 - d_2) - \varphi a_2^2 \\ &= (1-\varphi)a_2^2 + \frac{C_1^\alpha(x)}{4\left[\gamma\widetilde{[3]}_q - (\gamma - \nu) + \widetilde{[2]}_q\widetilde{[3]}_q\zeta\epsilon\right]}(c_2 - d_2). \end{aligned}$$

Then, in view of (3), and the value of a^2 in equation (27), we have

$$a_3-\varphi a_2^2=\frac{2(1-\varphi)\alpha^2x^3(c_2+d_2)}{\alpha x^2\left[4\gamma\widetilde{[2]}_q(\nu-1)+4\gamma\widetilde{[3]}_q+2\nu(\nu-2\gamma+1)+4\widetilde{[2]}_q\widetilde{[3]}_q\zeta\epsilon-2(1+(1/\alpha))\Upsilon\right]+(1+2x)\Upsilon}$$

$$+\frac{\alpha x}{2\left[\gamma\widetilde{[3]}_q-(\gamma-\nu)+\widetilde{[2]}_q\widetilde{[3]}_q\zeta\epsilon\right]}(c_2-d_2)$$

$$=\alpha x\Bigg(\left[h(\varphi)+\frac{1}{2\left[\gamma\widetilde{[3]}_q-(\gamma-\nu)+\widetilde{[2]}_q\widetilde{[3]}_q\zeta\epsilon\right]}\right]c_2$$

$$+\left[h(\varphi)-\frac{1}{2\left[\gamma\widetilde{[3]}_q-(\gamma-\nu)+\widetilde{[2]}_q\widetilde{[3]}_q\zeta\epsilon\right]}\right]d_2\Bigg),$$

where

$$\Upsilon:=\Big(\nu-\gamma+\widetilde{[2]}_q(\gamma+\zeta\epsilon)\Big)^2,$$

and

$$h(\varphi)=\frac{2(1-\varphi)\alpha x^2}{\alpha x^2\left[4\gamma\widetilde{[2]}_q(\nu-1)+4\gamma\widetilde{[3]}_q+2\nu(\nu-2\gamma+1)+4\widetilde{[2]}_q\widetilde{[3]}_q\zeta\epsilon-2(1+(1/\alpha))\Upsilon\right]+(1+2x)\Upsilon}.$$

This completes the proof of Theorem 2.

5. Consequences and Corollaries

Corollary 1. *If* $f\in{}^1\widetilde{\mathcal{B}}^q_\Sigma(t,\gamma,\nu)$, *then*

$$|a_2|\le\frac{2\alpha x\sqrt{x}}{\sqrt{\left|x^2\left[\alpha\left(2\widetilde{[2]}_q\gamma(\nu-1)+2\widetilde{[3]}_q\gamma+\nu(\nu-2\gamma+1)\right)-2(1+\alpha)\left(\nu-\gamma+\widetilde{[2]}_q\gamma\right)^2\right]+(1+2x)\left(\nu-\gamma+\widetilde{[2]}_q\gamma\right)^2\right|}},$$

$$|a_3|\le\frac{2\widetilde{[3]}_q\gamma x^2\alpha^2}{\left(\nu-\gamma+\widetilde{[2]}_q\gamma\right)^2}-\frac{4x\alpha}{\left[\gamma\left(\widetilde{[3]}_q+\nu-1\right)+(1-\gamma)\nu\right]}$$

and

$$\left|a_3-\varphi a_2^2\right|\le\begin{cases}\frac{\alpha x}{\gamma\widetilde{[3]}_q-(\gamma-\nu)} & \text{if}\quad 0\le|h(\varphi)|\le\frac{1}{2\left[\gamma\widetilde{[3]}_q-(\gamma-\nu)\right]},\\ 2\alpha x\,|h(\varphi)| & \text{if}\quad |h(\varphi)|\ge\frac{1}{2\left[\gamma\widetilde{[3]}_q-(\gamma-\nu)\right]},\end{cases}$$

where

$$h(\varphi)=\frac{2(1-\varphi)\alpha x^2}{\begin{array}{c}\alpha x^2\left[4\gamma\widetilde{[2]}_q(\nu-1)+4\gamma\widetilde{[3]}_q+2\nu(\nu-2\gamma+1)-2(1+(1/\alpha))\left(\nu-\gamma+\widetilde{[2]}_q\gamma\right)^2\right]\\ +(1+2x)\left(\nu-\gamma+\widetilde{[2]}_q\gamma\right)^2\end{array}}.$$

Next, making $\nu=1$, yields.

Corollary 2. *If* $f\in{}^2\widetilde{\mathcal{B}}_\Sigma^q(t,\gamma,\epsilon)$, *then*

$$|a_2|\le\frac{2\alpha x\sqrt{x}}{\sqrt{\left|\begin{array}{c}x^2\left[\alpha\left(2\widetilde{[3]}_q\gamma+2(1-\gamma)+2\widetilde{[2]}_q\widetilde{[3]}_q\zeta\epsilon\right)-2(1+\alpha)\left(1-\gamma+\widetilde{[2]}_q(\gamma+\zeta\epsilon)\right)^2\right]\\ +(1+2x)\left(1-\gamma+\widetilde{[2]}_q(\gamma+\zeta\epsilon)\right)^2\end{array}\right|}},$$

$$|a_3|\le\frac{2\left[\widetilde{[3]}_q\gamma-\widetilde{[2]}_q\zeta\epsilon\right]x^2\alpha^2}{\left(1-\gamma+\widetilde{[2]}_q(\gamma+\zeta\epsilon)\right)^2}-\frac{4\alpha x}{\left[\widetilde{[3]}_q\gamma+1-\gamma+\widetilde{[2]}_q\widetilde{[3]}_q\zeta\epsilon\right]}$$

and

$$\left|a_3-\varphi a_2^2\right|\le\begin{cases}\frac{\alpha x}{\gamma\widetilde{[3]}_q-(\gamma-1)+\widetilde{[2]}_q\widetilde{[3]}_q\zeta\epsilon} & \text{if}\quad 0\le|h(\varphi)|\le\frac{1}{2\left[\gamma\widetilde{[3]}_q-(\gamma-1)+\widetilde{[2]}_q\widetilde{[3]}_q\zeta\epsilon\right]},\\ 2\alpha x\,|h(\varphi)| & \text{if}\quad |h(\varphi)|\ge\frac{1}{2\left[\gamma\widetilde{[3]}_q-(\gamma-1)+\widetilde{[2]}_q\widetilde{[3]}_q\zeta\epsilon\right]},\end{cases}$$

where

$$h(\varphi)=\frac{2(1-\varphi)\alpha x^2}{\begin{array}{c}\alpha x^2\left[4\gamma\widetilde{[3]}_q+4(1-\gamma)+4\widetilde{[2]}_q\widetilde{[3]}_q\zeta\epsilon-2(1+(1/\alpha))\left(1-\gamma+\widetilde{[2]}_q(\gamma+\zeta\epsilon)\right)^2\right]\\ +(1+2x)\left(1-\gamma+\widetilde{[2]}_q(\gamma+\zeta\epsilon)\right)^2\end{array}}.$$

Setting $\nu=1$ and $\epsilon=0$, we obtain the following corollary.

Corollary 3. *If* $f\in{}^3\widetilde{\mathcal{B}}_\Sigma^q(t,\gamma)$, *then*

$$|a_2|\le\frac{2\alpha x\sqrt{x}}{\sqrt{\left|x^2\left[\alpha\left(2\widetilde{[3]}_q\gamma+2(1-\gamma)\right)-2(1+\alpha)\left(1-\gamma+\widetilde{[2]}_q\gamma\right)^2\right]+(1+2x)\left(1-\gamma+\widetilde{[2]}_q\gamma\right)^2\right|}},$$

$$|a_3| \leq \frac{2\widetilde{[3]}_q \gamma x^2 \alpha^2}{\left(1-\gamma+\widetilde{[2]}_q \gamma\right)^2} - \frac{4\alpha x}{\left[\widetilde{[3]}_q \gamma + 1 - \gamma\right]}$$

and

$$\left|a_3 - \varphi a_2^2\right| \leq \begin{cases} \frac{\alpha x}{\gamma \widetilde{[3]}_q - (\gamma - 1)} & if \quad 0 \leq |h(\varphi)| \leq \frac{1}{2\left[\gamma \widetilde{[3]}_q - (\gamma-1)\right]}, \\ 2\alpha x \,|h(\varphi)| & if \quad |h(\varphi)| \geq \frac{1}{2\left[\gamma \widetilde{[3]}_q - (\gamma-1)\right]}, \end{cases}$$

where

$$h(\varphi) = \frac{2(1-\varphi)\alpha x^2}{\alpha x^2 \left[4\gamma \widetilde{[3]}_q + 4(1-\gamma) - 2(1+(1/\alpha))\left(1-\gamma+\widetilde{[2]}_q \gamma\right)^2\right] + (1+2x)\left(1-\gamma+\widetilde{[2]}_q \gamma\right)^2}.$$

Next, letting $\gamma = \nu = 1$ and $\epsilon = 0$, yields.

Corollary 4. *If* $f \in {}^4\widetilde{\mathcal{B}}_\Sigma^q(t)$, *then*

$$|a_2| \leq \frac{2\alpha x \sqrt{x}}{\sqrt{\left|x^2 \left[2\alpha \widetilde{[3]}_q - 2(1+\alpha)\left(\widetilde{[2]}_q\right)^2\right] + (1+2x)\left(\widetilde{[2]}_q\right)^2\right|}},$$

$$|a_3| \leq \frac{2\widetilde{[3]}_q x^2 \alpha^2}{\left(\widetilde{[2]}_q\right)^2} - \frac{4\alpha x}{\widetilde{[3]}_q}$$

and

$$\left|a_3 - \varphi a_2^2\right| \leq \begin{cases} \frac{\alpha x}{\widetilde{[3]}_q} & if \quad 0 \leq |h(\varphi)| \leq \frac{1}{2\widetilde{[3]}_q}, \\ 2\alpha x \,|h(\varphi)| & if \quad |h(\varphi)| \geq \frac{1}{2\widetilde{[3]}_q}, \end{cases}$$

where

$$h(\varphi) = \frac{2(1-\varphi)\alpha x^2}{\alpha x^2 \left[4\widetilde{[3]}_q - 2(1+(1/\alpha))\left(\widetilde{[2]}_q\right)^2\right] + (1+2x)\left(\widetilde{[2]}_q\right)^2}.$$

6. Conclusion

In our current investigation, a novel subclass $\widetilde{\mathcal{B}}_\Sigma^q(t,\gamma,\nu,\epsilon)$ of normalized bi-univalent analytic functions has been delineated. This subclass integrates Gegenbauer polynomials and a symmetric q-derivative operator series. Initially, we have furnished an estimate for the primary Taylor-Maclaurin coefficients, $|a_2|$ and $|a_3|$. Subsequently, we have successfully addressed the Fekete-Szegö inequality problem.
Furthermore, through substituting some values for the parameters ϵ, ν, and γ,we derived analogous outcomes for the following subclasses: ${}^1\widetilde{\mathcal{B}}_\Sigma^q(t,\gamma,\nu) := \widetilde{\mathcal{B}}_\Sigma^q(t,\gamma,\nu,0)$, ${}^2\widetilde{\mathcal{B}}_\Sigma^q(t,\gamma,\epsilon) := \widetilde{\mathcal{B}}_\Sigma^q(t,\gamma,1,\epsilon)$, ${}^3\widetilde{\mathcal{B}}_\Sigma^q(t,\gamma) := \widetilde{\mathcal{B}}_\Sigma^q(t,\gamma,1,0)$, and ${}^4\widetilde{\mathcal{B}}_\Sigma^q(t) := \widetilde{\mathcal{B}}_\Sigma^q(t,1,1,0)$.

Acknowledgements

This study was conducted during the third author's sabbatical leave from The University of Jordan in the USA. The authors would like to thank the editor and anonymous reviewers for their valuable comments, which have helped to improve the quality of this work.

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