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Superlinear Problem with Inverse Coefficient for a Time-Fractional Parabolic Equation with Integral Over-Determination Condition

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Abstract: The inverse problem of finding the right-hand side of a nonlinear fractional parabolic equation with an integral over-determination supplementary condition is examined in this study. The functional analysis method, which is based on energy inequality and the density of the range of the operator created by the problem addressed, is used to demonstrate the existence, uniqueness, and continuous dependence on the data of the direct problem. The existence theorem is then obtained from the solution of the given problem, starting with the uniqueness theorem, making the energy inequality method, also known as the method of a priori estimates, a higher character method. The hardest part of this approach is figuring out which functional spaces to use, E and F, and if the inverse problem can be solved uniquely under the right circumstances. The existence and uniqueness of the solution to the inverse problem, which arises frequently in engineering and physics modeling of diverse processes, are established using the fixed point theorem.

Keywords: *fixed point theorem; nonlocal integral condition; inverse nonlinear problem.*

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1 Introduction

The purpose of this paper is to examine if it is possible to solve the following two functions: $\{u(x,t), f(t)\}$ fulfilling the fractional parabolic equation that follows:

$$^{C}D_t^{\alpha}u - \Delta u + \beta u + u^3 = f(t)g(x,t),\ x \in \Omega, t \in (0,T), \tag{1}$$

with the initial condition

$$u(x,0) = 0,\ x \in \Omega, \tag{2}$$

the boundary condition

$$u(x,t) = 0,\ (x,t) \in \partial\Omega \times [0,T], \tag{3}$$

and the non-local condition

$$\int_{\Omega} v(x)u(x,t)dx = E(t),\ t \in [0,\ T], \tag{4}$$

where $\partial\Omega$ is taken to be a regular boundary of a bounded domain Ω in $\mathbb{R}^n$. There are two known functions, g and E, and a positive constant, β.

Numerous real-world situations naturally give rise to inverse issues for the heat equations, see [1–3]. The integral condition (4) provides further information about how to solve the inverse problem in this case. The integral condition is a crucial modeling tool in the theory of PDEs in physics and engineering [4–9]. It is important to remember that integral over-determination processes may not always be successful in addressing non-local challenges [10, 11]. Numerous approaches to solving issues brought on by non-local issues have been put out thus far. The type of the involved non-local boundary value dictates the chosen method [12–14]. Numerous authors have studied the inverse parabolic problem and its unique solvability, focusing on conditions of type (4), see for example, [2, 3, 15–18]. The existence and uniqueness of inverse problem solutions for different parabolic equations with unknown source functions have also been the subject of several studies. Reversing problems with a parabolic equation's determination term and over-determination condition were also considered in [19, 20].

Fractional differential equations (FDEs) are created by generating differential equations to any desired order [21–25]. Because fractional differential equations are used to simulate complicated phenomena, they are significant in the fields of engineering, physics, and applied mathematics [26, 27]. Because of this, engineers and scientists have shown a growing interest in them in recent years. Since FDEs have memory and non-local relations in space and time, they can be used to simulate complex phenomena [28–33]. Herein, the tools, which will be used in our investigation, are the energy inequality method and the fixed point theorem. The structure of the energy inequality approach can be summed up as follows:

- First, we write the problem in the form of an operational equation

  $$Lu = F,\ u \in D(L),$$

  where a Banach space E is considered, and the operator L is studied from it to a suitable Hilbert space F.

- The a priori estimate for the operator L is then established.

- Next, we establish the density of this operator's collection of values in space F.

The results of the previous procedure will help us in investigating the existence, uniqueness and continuous dependence of the problem at hand.

2 Functional Spaces

When it comes to inverse coefficient problems for time fractional parabolic equations under integral over-determination conditions, functional spaces are essential tools for delving into intricate mathematical difficulties. Unknown coefficient identification becomes a difficult challenge when studying dynamic systems governed by partial differential equations, which gives rise to these problems. Function spaces are used in this study to give the inverse coefficient superlinear problem a comprehensive framework for analysis and solution. We shall include some definitions and lemmas pertaining to our study in the sections that follow. Let us clarify the conventions and notations we will use:

$$g^*(t) = \int_\Omega g(x,t) \cdot v(x)dx, \; Q = \Omega \times (0,T). \tag{5}$$

- The left Caputo derivative is given by

$${}^cD_t^\alpha u := \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x,\tau)}{\partial \tau} \frac{1}{(t-\tau)^\alpha} d\tau. \tag{6}$$

- The left Riemann-Liouville derivative is given by

$${}^RD_t^\alpha u := \frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial t} \int_0^t \frac{u(x,\tau)}{(t-\tau)^\alpha} d\tau. \tag{7}$$

- The right Riemann-Liouville derivative is given by

$${}_t^RD^\alpha u(x,t) := \frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial t} \int_t^T \frac{u(x,\tau)}{(t-\tau)^\alpha} d\tau. \tag{8}$$

Because the Caputo version is easier to handle under homogenous initial conditions, several authors contend that it is more natural. A direct calculation can confirm the link between the two concepts (6) and (7) as follows:

$${}^rD_t^\alpha u = {}^cD_t^\alpha u + \frac{u(x,0)}{\Gamma(1-\alpha)t^\alpha}. \tag{9}$$

Definition 2.1 [34] For each real $\alpha > 0$, we define the space ${}^lH_0^\alpha(I)$ as the closure of $C_0^\infty(I)$ with regard to the norm $\|u\|_{{}^lH_0^\alpha(I)}$ as follows:

$$\|u\|_{{}^lH^\alpha(I)} := \left(\|u\|_{L^2(I)}^2 + |u|_{{}^lH_0^\alpha(I)}^2\right)^{\frac{1}{2}}, \tag{10}$$

where

$$|u|^2{}_{{}^lH^\alpha(I)} = \left\|{}_0^RD_t^\alpha u\right\|_{L^2(I)}.$$

Definition 2.2 For each real $\alpha > 0$, the space ${}^rH_0^\alpha(I)$ is defined as the closure of $C_0^\infty(I)$ with regard to the norm $\|u\|_{{}^rH_0^\alpha(I)}$ as follows:

$$\|u\|_{{}^rH_0^\alpha(I)} := \left(\|u\|_{L^2(I)}^2 + |u|^2{}_{{}^rH_0^\alpha(I)}\right)^{\frac{1}{2}}, \tag{11}$$

where

$$|u|_{{}^rH_0^\alpha(I)}^2 := \left\|{}_t^R\partial_T^\alpha u\right\|_{L^2(I)}^2.$$

Lemma 2.1 *[8, 34] If* $u \in {}^{l}H^{\alpha}(I)$ *and* $v \in C_0^{\infty}(I)$ *for any real* $\alpha \in \mathbb{R}_+$*, then*

$$\left({}^{R}D_t^{\alpha}u(t), v(t)\right)_{L^2(I)} = \left(u(t), {}_{t}^{R}D^{\alpha}v(t)\right)_{L^2(I)}.$$

Lemma 2.2 *[8, 34] For* $0 < \alpha < 2, \alpha \neq 1, u \in H_0^{\frac{\alpha}{2}}(I)$*, we have*

$${}^{R}D_t^{\alpha}u(t) = {}^{R}D_t^{\frac{\alpha}{2}}\,{}^{R}D_t^{\frac{\alpha}{2}}u(t).$$

Lemma 2.3 *[8, 34] For* $\alpha \in \mathbb{R}_+$*, the semi-norms* $|.|_{{}^{l}H^{\alpha}(I)}, |.|_{{}^{r}H^{\alpha}(I)}$ *and* $|.|_{{}^{c}H^{\alpha}(I)}$ *are equivalent, for which* $\alpha \neq n + \frac{1}{2}$*. Thus, we have*

$$|u|_{{}^{l}H^{\alpha}(I)} \cong |.|_{{}^{r}H^{\alpha}(I)} \cong |.|_{{}^{c}H^{\alpha}(I)}.$$

Lemma 2.4 *The space* ${}^{l}H_0^{\alpha}(I)$ *is complete for every real* $\alpha > 0$ *with respect to the norm (10).*

Definition 2.3 The space of square functions, in the Bochner sense, integrated with the scalar product is represented by $L_2\left(0, T, L_2(0, d)\right)$, and it is given by

$$(u, w)_{L_2(0,T,L_2(0,d))} = \int_0^T (u, w)_{L_2(0,d)}dt. \tag{12}$$

3 The Direct Fractional Parabolic Problem's Solvability

One of the fundamental aspects of a more general study of inverse coefficient super-linear problems related to time fractional parabolic equations under integral over-determination conditions is the solvability of the direct fractional parabolic problem. Understanding the forward dynamics regulated by partial differential equations is essential for comprehending the behavior of the underlying systems, and this is achieved through the study of direct fractional parabolic problems.

3.1 Problem setting

In the rectangular domain $Q = (0,\ d) \times (0,\ T)$, where $d, T < \infty$ and $0 < \alpha < 1$, we will examine the existence and uniqueness of solution $u = u(x, t)$ to the following fractional parabolic problem:

$$\begin{cases} {}^{c}D_t^{\alpha}u - \left(\frac{\partial^2 u(x,t)}{\partial x^2}\right) + \beta u + u^3 = \tilde{f}(x, t) \ \text{ in } Q, \\ u(x, 0) = 0,\ \forall x \in (0, d), \\ u(0, t) = u(d, t) = 0,\ \forall t \in (0, T), \end{cases}$$

whose fractional parabolic equation is nonlinear and provided as follows:

$$\mathcal{L}u = {}^{c}D_t^{\alpha}u - \frac{\partial^2 u}{\partial x^2} + \beta u + u^3 = \tilde{f}$$

with the initial condition

$$\ell u = u(x, 0) = 0,\ \forall x \in (0, d)$$

and

$$u(0, t) = u(d, t) = 0,\ \forall t \in (0, T),$$

where $\tilde{f}$ is a known function and $b \in \mathbb{R}_*^+$.

Within this segment, we exhibit the existence and uniqueness of the solution for the problem (1)-(3) as a resolution of the subsequent operator equation

$$Lu = \mathcal{F}, \tag{13}$$

for which $L = (\mathcal{L}, \ell)$, and the domain of definition $D(L) = B$ that can be outlined as

$$D(L) = \left\{ u \backslash u \in L^2(Q) \cap L^4(Q), {}^cD_t^\alpha u, \frac{\partial u}{\partial x} \in L^2(Q) \right\}.$$

The operator L is defined in the space between B and F, where B is the Banach space containing all functions $u(x,t)$ with a finite norm of the form

$$\|u\|_B^2 = \left\| {}^cD_t^{\frac{\alpha}{2}} u \right\|_{L^2(Q)}^2 + \left\| \frac{du}{dx} \right\|_{L^2(Q)}^2 + \|u\|_{L^2(Q)}^2 + \|u\|_{L^4(Q)}^4,$$

and F is the Hilbert space consisting of all Fourier elements $(f, 0)$ such that the norm $L^2(Q)$ is finite.

Theorem 3.1 *For each function $u \in B$, we have the inequality*

$$\|u\|_B \leq C \|Lu\|_{L^2(Q)}, \tag{14}$$

where C is a positive constant independent of u.

Proof. We now employ the function $Mu = u(x,t)$ together with the scalar product in $L^2(Q)$ of (1), where $Q = (0,d)x(0,T)$. Consequently, we can have

$$\begin{aligned} \int_Q \mathcal{L}u \cdot Mu \, dxdt &= \int_Q {}^cD_t^\alpha u(x,t) \cdot u(x,t) \, dxdt - \int_Q \left(\frac{\partial^2 u(x,t)}{\partial x^2} \right) \cdot u(x,t) \, dxdt \\ &+ b \int_Q u^2(x,t) \, dxdt + \int u^4(x,t) \, dxdt \\ &= \int_Q f(x,t) u(x,t) \, dxdt. \end{aligned} \tag{15}$$

Due to $u(x,0) = 0$, and by using Lemmas 2.1, 2.2 and 2.3, we get

$$\begin{aligned} \int_Q {}^cD_t^\alpha u(x,t) \cdot u(x,t) dxdt &= \left({}^cD_t^\alpha u(x,t), u(x,t) \right)_{L^2(Q)} \\ &= \left({}^RD_t^{\frac{\alpha}{2}} \, {}^RD_t^{\frac{\alpha}{2}} u(x,t), u(x,t) \right)_{L^2(Q)} \\ &= \left({}^RD_t^{\frac{\alpha}{2}} u(x,t), {}_t^RD^{\frac{\alpha}{2}} u(x,t) \right)_{L^2(Q)} \\ &= |u|^2{}_{{}^cH^\alpha(Q)} \cong |u|^2{}_{{}^lH^\alpha(Q)} = \left\| {}^CD_t^{\frac{\alpha}{2}} u \right\|_{L^2(Q)}^2. \end{aligned}$$

With the use of the relationship ($|ab| \leq \frac{\varepsilon a^2}{2} + \frac{b^2}{2\varepsilon}$) coupled with the integral by parts, we obtain

$$\|{}^C D_t^{\frac{\alpha}{2}} u\|^2_{L^2(Q)} + \|\frac{du}{dx}\|^2_{L^2(Q)} + (b - \frac{\varepsilon}{2})\|u\|^2_{L^2(Q)} + \|u\|^4_{L^4(Q)} \leq \frac{1}{2\varepsilon}\|f\|^2_{L^2(Q)}.$$

So, for $\varepsilon \leq 2b$, we can have

$$\|{}^c D_t^{\frac{\alpha}{2}} u\|^2_{L^2(Q)} + \|\frac{du}{dx}\|^2_{L^2(Q)} + \|u\|^2_{L^2(Q)} + \|u\|^4_{L^4(Q)} \leq c\|f\|^2_{L^2(Q)}$$

with

$$c = \frac{1}{2\varepsilon \min\left(1, b - \frac{\varepsilon}{2}\right)}.$$

Consequently, we get

$$\|u\|_B \leq C\|Lu\|_{L^2(Q)},$$

where $C = \sqrt{c}$.

Proposition 3.1 *There is a closure for the operator L from B to F.*

Proof. Consider $(u_n)_{n\in\mathbb{N}} \subset D(L)$ is a sequence in which $u_{n\to 0}$ in B, and $Lu_{n\to\mathcal{F}}$ in F. Herein, we should show $f \equiv 0$. To this end, we notice that in B, the convergence of u_n to 0 causes

$$u_{n\to 0} \ \text{ in } \ (C_0^\infty(Q))'. \tag{16}$$

Given the continuity of the fractional derivative, the continuity distribution of the function u^2, and 162 derivation of the first order of $(C_0^\infty(Q))'$ in $(C_0^\infty(Q))'$ as a special case of the fractional derivative, the relationship (16) involves

$$\mathcal{L}u_{n\to 0} \ \text{ in } \ (C_0^\infty(Q))'. \tag{17}$$

Furthermore, in $L^2(Q)$, the convergence of Lv_n to f yields

$$\mathcal{L}v_{n\to f} \ \text{ in } \ (C_0^\infty(Q))'. \tag{18}$$

Due to the limit in $(C_0^\infty(Q))'$ is unique, we may infer from (17) and (18) that $f \equiv 0$. Therefore, the operator L is closeable.

We will define $D(\bar{L})$ as the domain of definition of $\bar{L}$ and let $\bar{L}$ be the closure of L in the material that follows.

Definition 3.1 Problem (1)–(3) has a strong solution, which is the operator equation

$$\bar{L}u = \mathcal{F}.$$

Furthermore, we may expand the previous estimate to a strong solution, meaning we would get the estimate

$$\|u\|_B \leq C\|\bar{L}u\|_F, \ \forall u \in D(\bar{L}). \tag{19}$$

Corollary 3.1 *Problem (1)–(3) has a unique strong solution that is constantly dependent on $f \in F$.*

Corollary 3.2 *The closure of $R(L)$ and the range $R(\bar{L})$ of the operator $\bar{L}$ in F are equal, i.e.,*

$$R(\bar{L}) = \overline{R(L)}.$$

Proof. First, if the solution exists, we will show that it is unique. For this purpose, we assume that u_1 and u_2 are two solutions such that $\eta = u_1 - u_2$. So, η will satisfy

$$\begin{cases} {}^cD_t^\alpha \eta(x,t) - \left(\frac{\partial^2 \eta(x,t)}{\partial x^2}\right) + b\eta(x,t) + u_1^3 - u_2^3 = 0, \text{ in } Q, \\ \eta(x,0) = 0, \ \forall x \in (0,d), \\ \eta(x,t) = 0, \ \forall (x,t) \in \partial\Omega \times (0,T) \\ . \end{cases},$$

for which

$$ {}^cD_t^\alpha \eta(x,t) - \left(\frac{\partial^2 \eta(x,t)}{\partial x^2}\right) + b\eta(x,t) + u_1^3 - u_2^3 = 0, \text{ in } Q. \tag{20}$$

Using the scalar product of (20) and η in $L^2(\Omega)$, we obtain

$$\int_\Omega {}^cD_t^\alpha \eta(x,t) \cdot \eta(x,t) dx - \int_\Omega \left(\frac{\partial^2 \eta(x,t)}{\partial x^2}\right) \cdot \eta(x,t)\, dx + b \int_\Omega \eta^2(x,t)\, dx \\ + \int_\Omega (u_1^3 - u_2^3)(u_1 - u_2)\, dx = 0.$$

Due to $\eta(x,0) = 0$, with the use of Lemmas 2.1, 2.2 and 2.3 together with integrating by parts, we obtain

$$\|{}^cD_t^{\frac{\alpha}{2}} \eta\|^2_{L^2(\Omega)} + \|\frac{d\eta}{dx}\|^2_{L^2(\Omega)} + \|\eta\|^2_{L^2(\Omega)} + \int_\Omega (u_1^3 - u_2^3)(u_1 - u_2) dx = 0. \tag{21}$$

The last item on the left-hand side of equation (21) is positive since λ^3 is a monotone function in λ (on $\Omega = (0,d)$), and this leads to the following conclusion from equation (20):

$$\|\eta\|^2_{L^2(\Omega)} \leq 0,$$

which implies $u_1 = u_2$ for all $t\epsilon(0,T)$. We will now go back and illustrate the result we discuss. To this end, we let $z \in \overline{R(l)}$. Thus, there exists a sequence $(z_n)_{n\in\mathbb{N}}$ in $R(L)$, for which $\lim_n z_n = z$. So, just as $(z_n)_{n\in\mathbb{N}}$ in $R(L)$, $\exists\, (u_n)_{n\in\mathbb{N}}$ in $D(L)$, for which $Lu_n = z_n$. Assume that $\mathcal{E}, n \geq n_0$, and $m, m' \in \mathbb{N}$, $m \geq m'$, for which u_m and $u_{m\prime}$ satisfy

$$Lu_m = f \text{ and } Lu_{m'} = f.$$

We put $y = u_m - u_m\prime$, then y satisfies

$$\begin{cases} {}^cD_t^\alpha y(x,t) - \left(\frac{\partial^2 y(x,t)}{\partial x^2}\right) + by(x,t) + u_m^3 - u_{m\prime}^3 = 0, \text{ in } Q, \\ y(x,0) = 0, \ \forall x \in (0,d), \\ y(x,t) = 0, \ \forall (x,t) \in \partial\Omega \times (0,T) \end{cases}.$$

Using the same method we employed to demonstrate the solution's uniqueness, we can now obtain $y = 0$. This suggests that for every $t \in (0,\ T)$, we obtain

$$0 \leq \|u_m - u_{m'}\| \leq 0,$$

i.e.,

$$\forall \varepsilon \geq 0, \ \exists n_0 \in \mathbb{N}, \ \forall m, m' \geq n_0, \ \|u_m - u_{m,}\| \leq \varepsilon.$$

Therefore, since E is a Banach space and $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, there exists $u \in E$ such that $\lim_n u_n = u$. With the use of the definition of $\bar{L}$ ($\lim_n u_n = u$ in

E; given that $\lim_n u_n = \lim_n z_n = u$, $\lim_n \bar{L}u_n = z$ since $\bar{L}$ is closed, implying that $\bar{L}u = z$), we find that the function u satisfies $u \in D(\bar{L})$, for which $\bar{L}u = z$. Thus, we have $z \in R(\bar{L})$, and so we obtain

$$\overline{R(L)} \subset R(\bar{L}).$$

We also conclude that it is closed since $R(\bar{L})$ is Banach. It remains to show that this is not the case. For this purpose, we assume that $z \in R(\bar{L})$. Then, given the elements of the set $R(\bar{L})$, there exists a sequence of $(z_n)_n$ in F such that

$$\lim_n z_n = z.$$

Therefore, a matching sequence $(u_n)_{n\in\mathbb{N}}$ exists such that

$$\lim_n \bar{L}u_n = z_n.$$

However, we have a Cauchy sequence in F, which is $(u_n)_{n\in\mathbb{N}}$. Thus, $u \in E$ exists such that

$$\lim_n u_n = u \quad \text{in } E.$$

Consequently, we have $\lim_n \bar{L}u_n = z$. As a consequence, $z \in \overline{R(L)}$, and then we obtain

$$\overline{R(L)} = R(\bar{L}).$$

4 Existence of Solution

In order to prove that the solution exists, we show that for every $u \in B$ and for any arbitrary $\mathcal{F} = (f, 0) \in F$, $R(L)$ is dense in F.

Theorem 4.1 *The problem (1)-(3) has a solution.*

Proof. The definition of F's scalar product is

$$(Lv, W)_F = \int_Q \mathcal{L}v \cdot w dx dt, \tag{22}$$

where $W = (w, 0)$ in $D(L)$. Set $w \in (R(L))^\perp$, and the result is

$$\int_Q {}^cD_t^\alpha u(x,t)w(x,t)dxdt - \int_Q \left(\frac{\partial^2 u(x,t)}{\partial x^2}\right) w(x,t)dxdt + b\int_Q u(x,t)w(x,t)dxdt + \int_Q u^3(x,t)\cdot w(x,t)dxdt = 0.$$

Letting $w = u$ yields

$$\int_Q {}^cD_t^\alpha u(x,t)\cdot u(x,t)\,dx\,dt - \int_Q \left(\frac{\partial^2 u(x,t)}{\partial x^2}\right)\cdot u(x,t)\,dx\,dt + b\int_Q u^2(x,t)\,dx\,dt + \int_Q u^4(x,t)\,dx\,dt = 0.$$

After accounting for the condition of u and integrating by parts each term of (4), we get

$$\left\| {}^{c}D_t^{\frac{\alpha}{2}} u \right\|^2_{L^2(Q)} + \left\| \frac{du}{dx} \right\|^2_{L^2(Q)} + b\|u\|^2_{L^2(Q)} + \|u\|^4_{L^4(Q)} = 0.$$

So, we get

$$\left\| {}^{c}D_t^{\frac{\alpha}{2}} u \right\|^2_{L^2(Q)} + b\|u\|^2_{L^2(Q)} + \|u\|^4_{L^4(Q)} = -\left\| \frac{du}{dx} \right\|^2_{L^2(Q)} \leq 0.$$

Then, we have

$$\|u\|^2_{L^2(Q)} \leq 0.$$

Consequently, $u = 0$ in Q, providing $w = 0$ within Q, and this completes the proof.

5 Solvability of the Main Problem

We assume that the functions that show up in the problem's data are quantifiable and meet the following conditions:

$$\left\{ \begin{array}{c} g \in C\left((0,T), L^2(\Omega)\right), v \in W_2^1(\Omega) \cap L^4(\Omega), E \in W_2^2(0,T), \\ \|g(x,t)\| \leq m, |g^*(t)| \geq r > 0, \text{ for } r \in \mathbb{R}, \ (x,t) \in Q \end{array} \right. .$$

The following linear operator provides the relationship between f and u:

$$A : L^2(0,T) \to L^2(0,) \tag{23}$$

such that

$$(\mathrm{Af}(t)) = \frac{1}{g^*} \left\{ \int_\Omega \frac{du}{dx}\frac{dv}{dx} dx + \int_\Omega u^3(x,t) \cdot v(x) dx \right\}. \tag{24}$$

Consequently, for the function f over $L^2(0,T)$, the previous relationship between f and u may be expressed as a second-order linear equation. In other words, we have

$$f = Af + W, \tag{25}$$

where

$$W = \frac{D_t^{\alpha\,\alpha} + \beta E}{g^*}, \tag{26}$$

and $E(0) = 0$.

Theorem 5.1 *Presume that the condition* (H) *is validated by the data functions (1)-(4) of the inverse problem. Then we have the equivalent of the following statement:*

1. *If the inverse problem (1)-(4) can be solved, then equation (25) can be solved as well.*

2. *The inverse problem (1)-(4) has a solution if equation (25) has a solution and the compatibility requirement* $E(0) = 0$ *holds.*

Proof. Assume that problem (1)-(4) can be solved. We denote its solution as $\{u, f\}$ in this instance. Now, after integrating the outcomes over Ω and multiplying both sides of (1) by v, the following is obtained:

$$\begin{aligned} {}^{c}D_t^{\alpha} \int_{\Omega} u(x,t) \cdot v(x)dx + \int_{\Omega} \frac{du}{dx}\frac{dv}{dx}dx + \beta \int_{\Omega} u(x,t) \cdot v(x)dx \\ + \int_{\Omega} u^3(x,t) \cdot v(x)dx = f(t)g^*(t). \end{aligned} \tag{27}$$

By applying (4) and (24), we obtain

$$f = Af + \frac{\beta E + {}^{c}D_t^{\alpha} E}{g^*}.$$

It is still necessary to demonstrate that u fulfills the integral over-determination condition (4). The function u is subject to the following relation by equation (27):

$${}^{c}D_t^{\alpha} E + \int_{\Omega} \frac{du}{dx}\frac{dv}{dx}dx + \beta E + \int_{\Omega} u^3(x,t) \cdot v(x)dx = f(t)g^*(t). \tag{28}$$

Equation (27) is subtracted from equation (28) to obtain

$${}^{c}D_t^{\alpha} \int_{\Omega} u(x,t) \cdot v(x)dx + \beta \int_{\Omega} u(x,t) \cdot v(x)dx = {}^{c}D_t^{\alpha} E + \beta E. \tag{29}$$

We determine that u meets the integral condition (4) by integrating the preceding equation and accounting for the compatibility constraint $E(0) = 0$. Consequently, we may infer that the solution to the inverse problem (1)-(4) is $\{u, f\}$.

Lemma 5.1 *If (H) is true, then A is a contracting operator in $L^2(0,T)$ for some positive δ.*

Proof. Based on (24), the following estimate can be inferred

$$|Af(t)|^2 \le \frac{2}{r^2}\left[\left\|\frac{du}{dx}\right\|^2_{L^2(\Omega)}\left\|\frac{dv}{dx}\right\|^2_{L^2(\Omega)} + \|u\|^6_{L^4(\Omega)}\|v\|^2_{L^4(\Omega)}\right].$$

Now, we suppose $\|u\|^2_{L^{\infty}(0,T,L^4(\Omega))} = \Upsilon \ge 0$. Then we obtain

$$|Af(t)|^2 \le \frac{2}{r^2}\left[\left\|\frac{du}{dx}\right\|^2_{L^2(\Omega)}\left\|\frac{dv}{dx}\right\|^2_{L^2(\Omega)} + \Upsilon\|u\|^4_{L^4(\Omega)}\|v\|^2_{L^4(\Omega)}\right].$$

Now, integrating the previous inequality over $(0,\ T)$ yields

$$\begin{aligned} &\int_0^T |Af(t)|^2 dt \\ &\le \frac{2}{r^2}\max\left(\left\|\frac{dv}{dx}\right\|^2_{L^2(\Omega)}, \gamma\|v\|^2_{L^4(\Omega)}\right)\left[\int_0^T \left\|\frac{du}{dx}\right\|^2_{L^2(\Omega)} dt + \int_0^T \|u\|^4_{L^4(\Omega)}\right) dt\right]. \end{aligned} \tag{30}$$

Consequently, we get

$$\|Af\|_{L^2(0,T)} \leq K \left[\int_0^T \left\| \frac{du}{dx} \right\|^2_{L^2(\Omega)} dt + \int_0^T \|u\|^4_{L^4(\Omega)} \Big) dt \right]^{\frac{1}{2}},$$

for which

$$K = \sqrt{\frac{2}{r^2} \max \left(\left\| \frac{dv}{dx} \right\|^2_{L^2(\Omega)}, Y\|v\|^2_{L^4(\Omega)} \right)}.$$

After removing a few terms and applying the a priori estimate, we now have

$$\|\frac{du}{dx}\|^2_{L^2(Q)} + \|u\|^4_{L^4(Q)} \leq C\|f\|^2_{L^2(Q)}.$$

Thus, we have

$$\|Af\|_{L^2(0,T)} \leq \delta\|f\|_{L^2(0,T)}, \tag{31}$$

where $\delta = K\sqrt{C}$. The previous relation indicates that there exists a positive δ such that $\delta \leq 1$. Hence, the operator A is a contracting mapping on $L^2(0,T)$, as shown by inequality (31).

Theorem 5.2 *If the compatibility condition and assumption* (H) *are met, then there is only one solution* $\{u, f\}$ *to the inverse problems (1)-(4).*

Proof. It is evident that there is only one solution f for equation (25) in $L^2(0,T)$. It is established by Lemma 2.3 that there is a solution to the inverse problem (1)-(4). We still need to prove that this approach is unique. However, suppose that the inverse problem under consideration has two distinct solutions, $\{u_1, f_1\}$ and $\{u_2, f_2\}$. Now, the theorem on the uniqueness of the solution of the main direct problem (1)-(3) produces $z_1 = z_2$ if the linear operator A contracts on $L^2(0,\ T)$ from Lemma 5.1, resulting in $f_1 = f_2$.

Corollary 5.1 *The solution* f *to equation (25) is continuously dependent on the data* W*, under the presumptions of Theorem 5.1.*

Proof. Let us assume two data sets that meet the conditions of Theorem 5.1: ω and v. For each set of data, ω and v, let f and g represent the solutions to equation (25), respectively. Now, based on (25), we can have

$$f = Af + v, \qquad g = Ag + \omega.$$

In this regard, it is necessary to compute $f - g$. When utilizing (31), it is evident that

$$\|f - g\|_{L^2(0,T)} = \|(Af + v) - (Ag + \omega)\|_{L^2(0,T)} \leq \delta\|f - g\|_{L^2(0,T)} + \|v - \omega\|_{L^2(0,T)}.$$

Consequently, we get

$$\|f - g\|_{L^2(0,T)} \leq \frac{1}{1-\delta}\|v - \omega\|_{L^2(0,T)}.$$

6 Discussion

When the initial condition is homogeneous, the inverse problem of finding the right-hand side of a nonlinear fractional parabolic equation with an integral over-determination condition has been examined. Theoretical analysis has been conducted for this inverse problem. This study has established the conditions for the problem's existence, uniqueness, and continuous dependence on data. The work done in this paper can therefore be continued from a variety of intriguing angles in numerical analysis, particularly with regard to creating efficient numerical techniques that are compliant with integrative type non-local conditions and considering how to solve the same problem but with incompatible initial conditions.

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