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Article

A Bi-Starlike Class in a Leaf-like Domain Defined through Subordination via q-Calculus

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Abstract: Bi-univalent functions associated with the leaf-like domain within the open unit disk are investigated and a new subclass is introduced and studied in the research presented here. This is achieved by applying the subordination principle for analytic functions in conjunction with q-calculus. The class is proved to be not empty. By proving its existence, generalizations can be given to other sets of functions. In addition, coefficient bounds are examined with a particular focus on $|\alpha_2|$ and $|\alpha_3|$ coefficients, and Fekete–Szegö inequalities are estimated for the functions in this new class. To support the conclusions, previous works are cited for confirmation.

Keywords: analytic functions; convolution; fractional derivatives; bi-univalent functions; starlike class; q-calculus; leaf-like domain; Fekete–Szegö problem; subordination

MSC: 30C45; 30C50

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1. Introduction, Definitions, and Motivation

Let $\psi(\zeta)$ be an analytic function defined in the open unit disk $\mathfrak{U} = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$. We can classify this function as a member of a specific class $\mathcal{A}$ if it can be represented as:

$$\psi(\zeta) = \zeta + \sum_{n=2}^{\infty} \alpha_n \zeta^n. \tag{1}$$

Furthermore, the class $\mathcal{S}$ consists of normalized analytic and univalent functions that satisfy the following normalization conditions:

$$\psi'(0) = 1 \quad \text{and} \quad \psi(0) = 0. \tag{2}$$

An analytic function η that satisfies $\eta(z) < 1$ and $\eta(0) = 0$ within the domain $\mathfrak{U}$ is called a Schwarz function. When considering two functions ψ_1 and ψ_2 from $\mathcal{A}$, ψ_1 is referred to as subordinate to ψ_2, denoted by $\psi_1 \prec \psi_2$, if a Schwarz function η exists such that $\psi_1(\zeta) = \psi_2(\eta(\zeta))$ for all $\zeta \in \mathfrak{U}$.

The class P is associated with Carathodory functions, as characterized by Miller [1]. These functions satisfy:

$$\varphi(0) = 1 \quad \text{and} \quad \Re e\{\varphi(\zeta)\} > 0, \ \forall\, \zeta \in \mathfrak{U}.$$

A Taylor series expansion offers an exact representation for any polynomial function $\varphi(\zeta)$ in the family P:

$$\varphi(\zeta) = 1 + \sum_{n=1}^{\infty} \varphi_n \zeta^n, \quad (\zeta \in \text{Ų}). \tag{3}$$

According to the Carathodory's Lemma (as referenced in [2]), it is established that

$$|\varphi_n| \leq 2, \quad \text{for all} \quad n \geq 1. \tag{4}$$

In essence, $\varphi \in \mathrm{P}$ if, and only if,

$$\varphi(\zeta) \prec (1+\zeta)(1-\zeta)^{-1}, \quad (\zeta \in \text{Ų}).$$

As the foundation upon which many important subclasses of analytic functions are built, the class P is crucial to the study of analytic functions.

For any function ψ in the subfamily of normalized analytic and univalent $\mathcal{S}$, there exists an inverse function denoted as ψ^{-1} and defined by

$$\zeta = \psi^{-1}(\psi(\zeta)), \quad \xi = \psi(\psi^{-1}(\xi)) \qquad \left(r_0(\psi) \geq \frac{1}{4};\ |\xi| < r_0(\psi); \zeta \in \text{Ų} \right),$$

where

$$\psi^{-1}(\xi) = \xi\Big(1 - \alpha_2\xi + \xi^2(-\alpha_3 + 2\alpha_2^2) - \xi^3(\alpha_4 + 5\alpha_2^3 - 5\alpha_3\alpha_2) + \cdots\Big). \tag{5}$$

A function $\psi(\zeta) \in S$ is said to be bi-univalent if the inverse function $\psi^{-1}(\xi) \in S$. The subclass of $\mathcal{S}$ denoted by Σ contains all bi-univalent functions in Ų.

A table illustrating certain functions within class Σ and their inverse functions is provided in Table 1.

Table 1. List of some of the functions in class Σ along with their inverses.

The Function	The Corresponding Inverse
$\psi_1(\zeta) = \frac{\zeta}{1+\zeta}$	$\psi_1^{-1}(\xi) = \frac{\xi}{1-\xi}$
$\psi_2(\zeta) = \frac{e^{2\zeta}-1}{e^{2\zeta}+1}$	$\psi_2^{-1}(\xi) = -\log(1-\xi)$
$\psi_3(\zeta) = \frac{1}{2}\log\left(\frac{1+\zeta}{1-\zeta}\right)$	$\psi_3^{-1}(\xi) = e^{-\xi}(e^{\xi} - 1)$

Analytic functions and their subclasses have been the subject of extensive research in the field of complex analysis, especially from the geometric function theory point of view. The class $\mathcal{S}^*$ is one of these subclasses that has attracted a lot of interest. Here is how the class of starlike functions is defined:

$$\mathcal{S}^* = \left\{ \psi \in \mathcal{S} \quad \text{and} \quad \mathrm{Re}\left\{ \frac{\zeta\,\psi'(\zeta)}{\psi(\zeta)} \right\} > 0, \quad (\zeta \in \text{Ų}) \right\}.$$

The class $\mathcal{S}^*$ comprises functions ψ that not only belong to $\mathcal{S}$ but also satisfy the additional condition that the real part of the derivative ratio $\frac{\zeta\,\psi'(\zeta)}{\psi(\zeta)}$ is strictly positive for all ζ in the unit disk.

The study of $\mathcal{S}^*$ and its properties is fundamental. Scholars work to comprehend the geometric and analytic characteristics of functions in this class, investigating mappings, singularities, and other properties. Through exploring the challenges of $\mathcal{S}^*$, researchers hope to gain a better understanding of the structure and behavior of analytic functions, which will enhance their knowledge of complex analysis and its uses.

In 1992, Ma and Minda [3] introduced the set $S^*(\Omega)$ by employing the concept of subordination, outlined as follows:

$$S^*(\Omega) = \left\{ \psi \in \mathcal{A} : \quad \frac{\zeta \psi'(\zeta)}{\psi(\zeta)} \prec \Omega(\zeta) \right\}.$$

Here, Ω represents an analytic function with $\Re e\{\Omega(\zeta)\} > 0$ $(\zeta \in Ų)$ and is normalized according to conditions (2). Table 2 shows how different authors approached the problem of defining additional subclasses of starlike functions by choosing particular expressions for Ω.

Table 2. Some subclasses of starlike functions defined by subordination.

Author/s	$\mathbf{\Omega(\zeta)}$	**Year**	**Reference**
Sokól and Stankiewicz	$\sqrt{1+\zeta}$	1996	[4]
Raina and Sokól	$\zeta + \sqrt{1+\zeta^2}$	2015	[5]
Priya and Sharma	$\zeta + \sqrt[3]{1+\zeta^2}$	2018	[6]
Rath et al.	$1/(1-\zeta)$	2022	[7]
Mahmoud et al.	e^{ζ}	2019	[8]
Ullah et al.	$1 + \tanh\zeta$	2021	[9]
Shi et al.	$1 + \sin\zeta$	2022	[10]

In 2015, Raina and Sokól [5] examined a new family of starlike functions that are subordinate to the analytic function $\Omega(\zeta) = \zeta + \sqrt{1+\zeta^2}$. These functions are associated with a shell-shaped region. The authors established coefficient inequalities for this family of functions [5].

Taking inspiration from their work, Priya and Sharma [6] introduced a particular class of functions that are subordinate to the function $\Omega(\zeta) = \zeta + \sqrt[3]{1+\zeta^2}$, which is related to a leaf-like domain. Additionally, they presented another class of functions that are subordinate to $\Omega(\zeta) = \zeta + \sqrt[3]{1+\zeta^3}$, which is also associated with the leaf-like domain. This is illustrated in Figure 1.

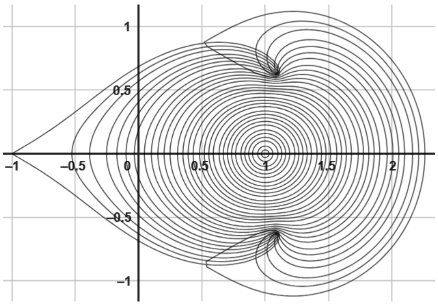


Figure 1. Leaf-shaped region, displays the image of $\Omega(Ų)$, where $\Omega(\zeta) = \zeta + \sqrt[3]{1+\zeta^3}$.

Quantum calculus, or q-difference calculus, does not use the idea of a derivative as the limit of a ratio as the increment tends to zero. Instead, it relies on the q-difference operator, which is crucial for our discussion. This calculus expands the traditional concepts of mathematical analysis by introducing the parameter $q \in (0,1)$. For a detailed exploration of this topic, readers are advised to check the comprehensive treatise by Gasper and Rahman [11], which offers in-depth explanations and practical applications of q-difference calculus in a variety of disciplines, including number theory, physics, and combinatorics.

A q-analytic function is a specific type of analytic function that relies on a parameter called q. This parameter can either be a complex number or a formal variable. The definition of this function falls within the scope of q-calculus, a generalization of calculus that incorporates the concepts of the q-binomial coefficient and the q-derivative [12].

Definition 1 ([13]). *The q-bracket represented by* $[\Bbbk]_q$ *is defined explicitly for* $(0 < q < 1)$ *as follows:*

$$[\Bbbk]_q = \begin{cases} \frac{1-q^{\Bbbk}}{1-q} & , \quad \text{if } \Bbbk \in \mathbb{C}\backslash\{0\} \\ q^{n-1} + q^{n-2} + \cdots + q + 1 = \sum_{j=0}^{n-1} q^j & , \quad \text{if } \Bbbk = n \in \mathbb{N} \\ 1 & , \quad \text{if} \quad q \to 0^+, \ \Bbbk \in \mathbb{C}\backslash\{0\} \\ \Bbbk & , \quad \text{if} \quad q \to 1^-, \ \Bbbk \in \mathbb{C}\backslash\{0\} \end{cases}.$$

with the useful identity $[n+1]_q - [n]_q = q^n$.

Table 3 displays the initial terms of the sequence $[n]_q, n \in \mathbb{N}$, along with its overarching formula.

Table 3. For $n \in \mathbb{N}$, the first few terms of $[n]_q$ are.

n	$[n]_q$
$n = 1$	$[1]_q = 1$
$n = 2$	$[2]_q = 1 + q$
$n = 3$	$[3]_q = 1 + q + q^2$
$n = 4$	$[4]_q = 1 + q + q^2 + q^3$
$\vdots$	$\vdots$
$n \in \mathbb{N}$	$[n]_q = 1 + q + q^2 + q^3 + \cdots + q^{n-1}$

Definition 2 ([13,14]). *The q-difference operator, or q-derivative, of a function* ψ *is defined for* $0 < q < 1$ *by:*

$$\partial_q \psi(\zeta) = \begin{cases} \frac{\psi(\zeta) - \psi(q\zeta)}{\zeta - q\zeta}, & \text{if } \zeta \neq 0, \\ \psi'(0), & \text{if} \quad \zeta = 0, \\ \psi'(\zeta), & \text{if} \quad q \to 1^-, \ \zeta \neq 0. \end{cases}.$$

Remark 1. *For* $\psi \in \mathcal{A}$ *of the form* (1)*, it can easily be seen that:*

$$\partial_q \psi(\zeta) = \partial_q \left\{ \zeta + \sum_{n=2}^{\infty} \alpha_n \zeta^n \right\} = 1 + \sum_{n=2}^{\infty} [n]_q \alpha_n \eta^{n-1}, \qquad (\zeta \in \text{Џ})$$

and for ψ^{-1} *of the form* (5)*, we have*

$$\partial_q \left(\psi^{-1}(\xi) \right) = 1 - [2]_q \, \alpha_2 \, \xi + [3]_q \, (2\alpha_2^3 - \alpha_3) \, \xi^2 + \cdots, \qquad (\xi \in \text{Џ}).$$

Because of its numerous applications in physics, quantum mechanics and mathematics, particularly in the field of geometric function theory, researchers are still drawn to the study of q-calculus. A significant aspect of q-calculus is the operator ∂_q, which is important for the analysis of different classes of analytic functions. In 1990, Ismail et al. [15] made a significant breakthrough by introducing the concept of q-extension for starlike functions in the unit disk. This breakthrough opened the door for further investigations in geometric function theory. For example, in [16], Srivastava and his colleagues explored q-starlike functions in conic domains and conducted studies on the upper bounds of the Fekete–Szegö functional. Recently, Srivastava provided a comprehensive survey that explains the mathematical foundations and practical applications of fractional q-derivative operators, within the context of geometric function theory [17]. For those interested in delving deeper into q-calculus and its implications in this field, an abundance of research is at one's disposal, starting with classical publications [18,19], continuing with studies like [20–23], and considering very recent research outcomes on the topic like [24–32].

Definition 3 ([15]). *A function $\psi \in \mathcal{A}$ of the form (1) is said to belong to the class S_q^* if it satisfies the condition given by*

$$\left| \frac{\zeta(\partial_q \psi(\zeta))}{\psi(\zeta)} - (1-q)^{-1} \right| \leq (1-q)^{-1}, \qquad (\zeta \in \mathcal{U}). \tag{6}$$

Remark 2. *As a result, it is evident from the previous inequality that as q approaches 1^-, the inequality simplifies to:*

$$\left| \mathsf{w} - (1-q)^{-1} \right| \leq (1-q)^{-1}, \quad q \in (0,1).$$

The closed disk above represents only the right-half plane, and the class $\mathcal{S}_q^*$ of q-starlike functions transforms into the known class $\mathcal{S}^*$.

Similarly, the relationship in Equation (6) can be expressed as follows (see [21]) using the idea of subordination:

$$\frac{\zeta \partial_q \psi(\zeta)}{\psi(\zeta)} \prec (1+\zeta)(1-q\,\zeta)^{-1}, \quad (0 < q < 1). \tag{7}$$

Applying the principles mentioned earlier and employing the concept of subordination by utilizing q-calculus, we establish a novel subclass of analytic functions that are associated with a particular leaf-like domain.

Definition 4. *A function $\psi \in \mathcal{S}$ is said to belong to the class $\mathcal{S}^*(q;\Omega(\zeta))$ if it satisfies the condition given by*

$$\frac{\zeta \partial_q \psi(\zeta)}{\psi(\zeta)} \prec \Omega(\zeta), \qquad (\zeta \in \mathcal{U}), \tag{8}$$

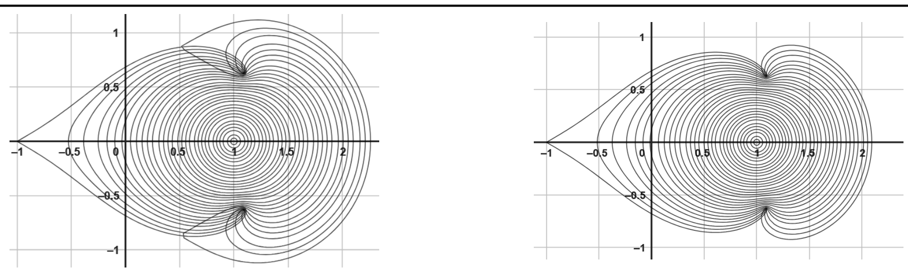
where

$$\Omega(\zeta) = \frac{(1+q)\zeta}{2+(1-q)\zeta} + \sqrt[3]{1 + \left(\frac{(1+q)\zeta}{2+(1-q)\zeta} \right)^3}.$$

Alsoboh and Oros [13] conducted a study on a particular category of bi-univalent functions related to the leaf-like domain within the open unit disk using q-calculus.

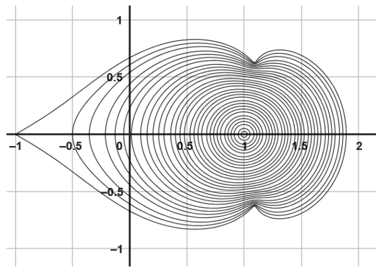
Remark 3. *The unit disk is mapped onto a leaf-shaped region via the analytic and univalent function $\Omega(\zeta)$. With regard to the real axis, it is symmetric. The function $\Omega(\zeta)$ satisfies $\Omega(0) = \partial_q \Omega(0) = 1$ and has a positive real part, see Table 4.*

Table 4. Figure illustrating the leaf-shaped region $\Omega(\mathcal{U})$, which is bounded and symmetric with respect to the real axis. The figure was generated using GeoGebra Classic 6. For more details, see [13].

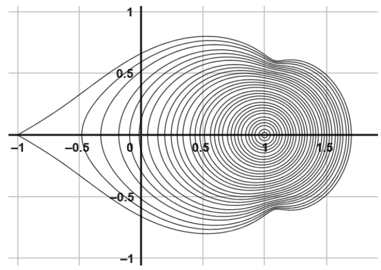


(**a**) Depicts the image of $\Omega(\mathcal{U})$ with $q \to 1^-$ (**b**) Displays the image of $\Omega(\mathcal{U})$ with $q = 0.89$

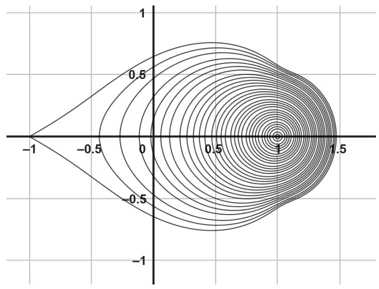
Table 4. *Cont.*



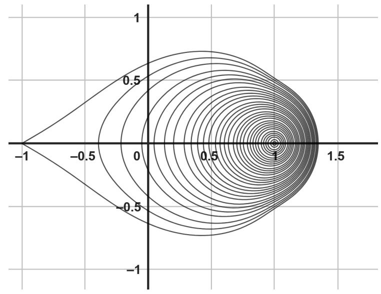
(**c**) The image of $\Omega(Ų)$ with $q = 0.73$



(**d**) The image of $\Omega(Ų)$ with $q = 0.54$



(**e**) The image of $\Omega(Ų)$ with $q = 0.23$



(**f**) The image of $\Omega(Ų)$ as $q \to 0^{+}$.

The main goal of this work is to investigate the properties of bi-univalent functions associated with the leaf-like domain in Ų. In the section that follows, the definition of the investigated class and examples to assist with achieving this goal will be given. In Section 3, coefficient estimates are obtained for the new class and the Fekete–Szegö functional is evaluated in Section 4. Corollaries that correspond to the examples provided, generated by the theorems proved in the previous sections, are stated in Sections 3 and 4.

2. Definition and Example

We will use the q-calculus theory and the previously mentioned subordination principle among analytic functions to give an exact mathematical description of the newly defined class $\Sigma_{\mathcal{S}^*}(q;\Omega(\zeta))$ of bi-univalent functions related to a leaf-like domain.

Definition 5. *A bi-univalent function* ψ *of the type* (1) *belongs to the class* $\Sigma_{\mathcal{S}^*}(q;\Omega(\zeta))$ *if it fulfills the following subordinations:*

$$\Phi(\zeta) = \frac{\zeta \partial_q \psi(\zeta)}{\psi(\zeta)} \prec \Omega(\zeta), \quad (\zeta \in Ų), \tag{9}$$

and

$$\Psi(\xi) = \frac{\xi \partial_q^{-1} \psi(\xi)}{\psi^{-1}(\xi)} \prec \Omega(\xi) \quad (\xi \in Ų), \tag{10}$$

where

$$\Omega(\zeta) = \frac{(1+q)\zeta}{2+(1-q)\zeta} + \sqrt[3]{1+\left(\frac{(1+q)\zeta}{2+(1-q)\zeta}\right)^3}, \tag{11}$$

with $\Omega(0) = 1$.

Example 1. *If* $q \to 1^{-}$, *then* $\Sigma_{\mathcal{S}^*}(q;\Omega(\zeta))$ *is reduced to* $\Sigma_{\mathcal{S}^*}(\zeta + \sqrt[3]{1+\zeta^3})$ *defined by*

$$\left\{ \psi \in \Sigma : \frac{\zeta\, \psi'(\zeta)}{\psi(\zeta)} \prec \zeta + \sqrt[3]{1+\zeta^3}, \quad (\zeta \in Ų) \right\} \tag{12}$$

and

$$\left\{\psi \in \Sigma : \frac{\xi\,(\psi^{-1})'(\xi)}{\psi(\xi)} \prec \xi + \sqrt[3]{1+\xi^3}, \quad (\xi \in \text{Ų})\right\}. \tag{13}$$

Remark 4. *We want to emphasize that the class $\Sigma_{\mathcal{S}^*}(q;\Omega(\zeta))$ is not empty. In particular, consider the functions defined by:*

$$\psi_*(\zeta) = \frac{\zeta}{1-\vartheta\zeta}, \quad |\vartheta| < 0.49. \tag{14}$$

It is easy to see that $\psi_* \in \mathcal{S}$ and, moreover, $\psi_* \in \Sigma$ with its inverse:

$$\psi_*^{-1}(\xi) = \frac{\xi}{1+\vartheta\xi}. \quad |\vartheta| < 0.49. \tag{15}$$

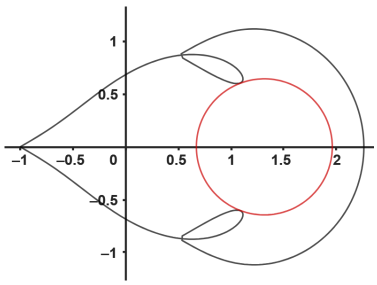
By utilizing the notations provided in Equations (9) and (10), we can easily demonstrate through a straightforward calculation that:

$$\begin{aligned} \Phi(\psi_*(\zeta)) &= \frac{1}{(1-q\,\vartheta\zeta)}, \quad |\vartheta| < 0.49 \\ \Psi(\psi_*^{-1}(\xi)) &= \frac{1}{(1+q\,\vartheta\xi)}, \quad |\vartheta| < 0.49. \end{aligned} \tag{16}$$

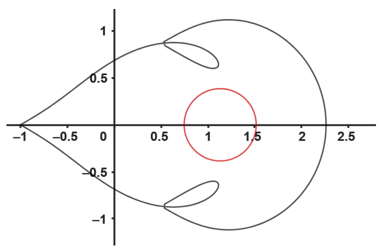
Also, for all $\zeta \in \text{Ų}$, $\Phi(-\vartheta\zeta) = \Psi(\vartheta\zeta)$, which implies that $\Phi(\text{Ų}) = \Psi(\text{Ų})$.

We employ the GeoGebra Classic 6 to generate the visual representations of the boundary ∂Ų using the functions Φ and Ω, as illustrated in Table 5. This is applicable to various scenarios where the conditions $|\vartheta| < 0.44$, $\chi \geq 0$, and $q \in (0,1)$ are satisfied. It is worth noting that Ω is a univalent function within Ų. As a consequence, the relationships $\Phi(\zeta) \prec \Omega(\zeta)$ and $\Psi(\zeta) \prec \Omega(\zeta)$ are valid. These relationships can be justified by the facts that $\Phi(0) = \Psi(0) = \Omega(0) = 1$, $\Phi(\text{Ų}) \subset \Omega(\text{Ų})$, and $\Psi(\text{Ų}) \subset \Omega(\text{Ų})$. Please refer to Table 5 for further clarification.

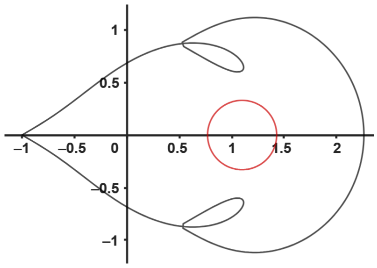
Table 5. The image of $\Phi(e^{i\theta})$ (red color) and $\Omega(e^{i\theta})$ (black color) with several values of q, ϑ and $\theta \in [0, 2\pi)$.



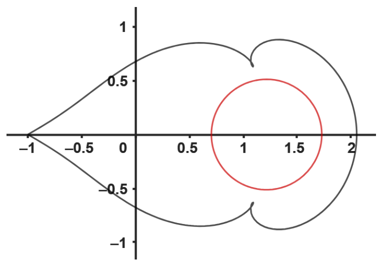
(**a**) The image of $\Phi(e^{i\theta})$ (red color) and $\Omega(e^{i\theta})$ (black color) with $q \to 1^-$, $\vartheta = 0.49$ and $\theta \in [0, 2\pi)$.



(**b**) The image of $\Phi(e^{i\theta})$ (red color) and $\Omega(e^{i\theta})$ (black color) with $q \to 1^-$ and $\vartheta = 0.34$.

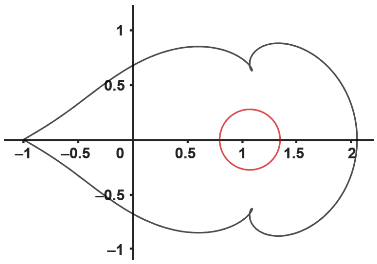


(**c**) The image of $\Phi(e^{i\theta})$ (red color) and $\Omega(e^{i\theta})$ (black color) with $q \to 1^-$ and $\vartheta = -0.3$.

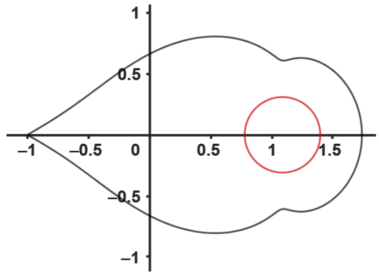


(**d**) The image of $\Phi(e^{i\theta})$ (red color) and $\Omega(e^{i\theta})$ (black color) with $q = 0.86$ and $\vartheta = 0.49$.

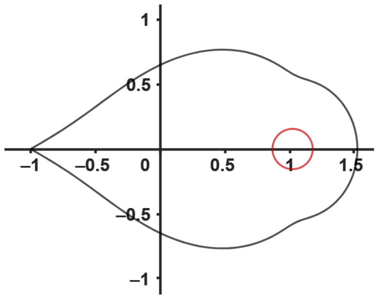
Table 5. *Cont.*



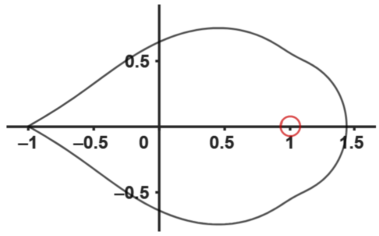
(**e**) The image of $\Phi(e^{i\theta})$ (red color) and $\Omega(e^{i\theta})$ (black color) with $q = 0.86$, $\vartheta = 0.3$ and $\theta \in [0, 2\pi)$.



(**f**) The image of $\Phi(e^{i\theta})$ (red color) and $\Omega(e^{i\theta})$ (black color) with $q = 0.86$ and $\vartheta = 0.49$.



(**g**) The image of $\Phi(e^{i\theta})$ (red color) and $\Omega(e^{i\theta})$ (black color) with $q = 0.31$, $\vartheta = 0.49$ and $\theta \in [0, 2\pi)$.



(**h**) The image of $\Phi(e^{i\theta})$ (red color) and $\Omega(e^{i\theta})$ (black color) $q = 0.17$ and $\vartheta = 0.44$.

Up to this moment, there has been a scarcity of academic study on the numerous parameters that influence the functional classification of a leaf-like domain. The fundamental purpose of this work is to examine the initial Taylor–Maclaurin coefficients of functions ψ, as given by Equation (1), which are significant for the class $\Sigma_{\mathcal{S}^*}(q; \Omega(\zeta))$ related to a leaf-like domain. Furthermore, we seek to study the estimated value of the Fekete–Szegö functional.

3. The Bounds of the Coefficients within the $\Sigma_{\mathbf{q}}(\chi, \Omega(\zeta))$ Class

Initially, the estimates for the coefficients of the class $\Sigma_{\mathcal{S}^*}(q; \Omega(\zeta))$, as defined in Definition 5, are provided.

Theorem 1. *Let* $\psi \in \Sigma$ *of the type* (1) *belong to the class* $\Sigma_{\mathcal{S}^*}(q; \Omega(\zeta))$. *Then,*

$$|\alpha_2| \leq \frac{(1+q)}{2q\sqrt{(3q-1)}},$$

and

$$|\alpha_3| \leq \frac{(1+q)^2}{16q^2} + \frac{1}{4q}.$$

Proof. If $\psi \in S^*(q, \Omega(\zeta))$. As per Definition 5, the presence of certain analytic functions η_1 and η_2 can be established, satisfying the conditions $\eta_1(0) = \eta_2(0) = 0$, and $|\eta_1(\zeta)| < 1$, $|\eta_2(\xi)| < 1$ for all $\zeta, \xi \in U$. Setting

$$\varphi_1(\zeta) = \frac{1+\eta_1(\zeta)}{1-\eta_1(\zeta)} = 1 + \ell_1\zeta + \ell_2\zeta^2 + \cdots, (\zeta \in \mathbb{U}),$$

and

$$\varphi_2(\xi) = \frac{1+\eta_2(\xi)}{1-\eta_2(\xi)} = 1 + \jmath_1\xi + \jmath_2\xi^2 + \cdots, (\xi \in \mathbb{U}),$$

then $\varphi_1, \varphi_2 \in \mathrm{P}$. From the above relations, we obtain

$$\eta_1(\zeta) = \frac{\varphi_1(\zeta) - 1}{\varphi_1(\zeta) + 1} \qquad (\zeta \in \mathfrak{U}),$$

and

$$\eta_2(\xi) = \frac{\varphi_2(\xi) - 1}{\varphi_2(\xi) + 1} \qquad (\xi \in U).$$

From (9) and (10), it follows that

$$\begin{aligned} \Omega(\eta_1(\zeta)) &= \frac{(1+q)(\varphi_1(\zeta)-1)}{1+3\varphi_1(\zeta)+q(1-\varphi_1(\zeta))} + \sqrt[3]{1+\left(\frac{(1+q)(\varphi_1(\zeta)-1)}{1+3\varphi_1(\zeta)+q(1-\varphi_1(\zeta))}\right)^3} \\ &= 1 + \frac{1+q}{4}\ell_1\zeta + \frac{1+q}{4}\left(\ell_2 - \frac{(3-q)}{4}\ell_1^2\right)\zeta^2 + \cdots, \qquad (\zeta \in \mathfrak{U}) \end{aligned} \tag{17}$$

and

$$\begin{aligned} \Omega(\eta_2(\xi)) &= \frac{(1+q)(\varphi_2(\xi)-1)}{1+3\varphi_2(\xi)+q(1-\varphi_2(\xi))} + \sqrt[3]{1+\left(\frac{(1+q)(\varphi_2(\xi)-1)}{1+3\varphi_2(\xi)+q(1-\varphi_2(\xi))}\right)^3} \\ &= 1 + \frac{1+q}{4}\jmath_1\xi + \frac{1+q}{4}\left(\jmath_2 - \frac{(3-q)}{4}\jmath_1^2\right)\xi^2 + \cdots, \qquad (\xi \in \mathfrak{U}). \end{aligned} \tag{18}$$

Also,

$$\frac{\zeta\partial_q\psi(\zeta)}{\psi(\zeta)} = 1 + (1+q\chi)\alpha_2\zeta + (1+q[2]_q\chi)\alpha_3^2\zeta^2 + \cdots, \tag{19}$$

and

$$\frac{\xi\partial_q\psi^{-1}(\xi)}{\psi^{-1}(\xi)} = 1 - (1+q\chi)\alpha_2\xi + (1+q[2]_q\chi)\left(2\alpha_2^2 - \alpha_3\right)\xi^2 + \cdots. \tag{20}$$

By comparing the pertinent coefficients in (19) and (20), we arrive at the following:

$$q\alpha_2 = \frac{1+q}{4}\ell_1, \tag{21}$$

$$-q\alpha_2 = \frac{1+q}{4}\jmath_1, \tag{22}$$

$$q(1+q)\alpha_3 - q\alpha_2^2 = \frac{1+q}{4}\left(\ell_2 - \frac{(3-q)}{4}\ell_1^2\right), \tag{23}$$

and

$$q(1+2q)\alpha_2^2 - q(1+q)\alpha_3 = \frac{1+q}{4}\left(\jmath_2 - \frac{(3-q)}{4}\jmath_1^2\right). \tag{24}$$

It follows from (21) and (22) that

$$\ell_1 = -\jmath_1 \quad \text{and} \quad \ell_1^2 = \jmath_1^2, \tag{25}$$

and

$$2q^2\alpha_2^2 = \frac{(1+q)^2}{16}\left(\ell_1^2 + \jmath_1^2\right)\alpha_2^2 = \frac{(1+q)^2}{32q^2}\left(\ell_1^2 + \jmath_1^2\right) \iff \ell_1^2 + \jmath_1^2 = \frac{32q^2}{(1+q)^2}\alpha_2^2. \tag{26}$$

Adding (23) and (24), performing some calculations, we obtain

$$2q^2\alpha_2^2 = \frac{1+q}{8}\left[(\ell_2 + \jmath_2) + \frac{(3-q)}{4}\left(\ell_1^2 + \jmath_1^2\right)\right].$$

Substituting the value of $\left(\ell_1^2 + \jmath_1^2\right)$ from (26), we obtain

$$q^2\left[2 - \frac{(3-q)}{(1+q)}\right]\alpha_2^2 = \frac{1+q}{8}(\ell_2 + \jmath_2).$$

Moreover,

$$\alpha_2^2 = \frac{(1+q)^2}{8q^2(3q-1)}(\ell_2 + \jmath_2). \tag{27}$$

Applying (4) for the coefficients ℓ_2 and $\jmath_2$, we obtain

$$|a_2| \leq \frac{(1+q)}{2q\sqrt{(3q-1)}},$$

By subtracting (24) from (23), and using $\ell_1^2 = \jmath_1^2$, we obtain

$$\alpha_3 - \alpha_2^2 = \frac{1}{8q}(\ell_2 - \jmath_2). \tag{28}$$

Then, in view of (26), Equation (28) becomes

$$\alpha_3 = \frac{(1+q)^2}{32q^2}\left(\ell_1^2 + \jmath_1^2\right) + \frac{1}{8q}(\ell_2 - \jmath_2).$$

Thus, applying (4), we conclude that

$$|\alpha_3| \leq \frac{(1+q)^2}{16q^2} + \frac{1}{4q}.$$

The proof of the theorem has been successfully concluded. □

Corollary 1. *If* ψ *is an element of* Σ *defined by* (1) *and belongs to the class* $\Sigma_{\mathcal{S}^*}(\zeta + \sqrt[3]{1+\zeta^3})$, *then we can state the following:*

$$|\alpha_2| \leq \frac{1}{\sqrt{2}} \qquad \textit{and} \qquad |\alpha_3| \leq \frac{1}{2}.$$

4. The Fekete–Szegö Functional

Both Fekete and Szegö published their work in 1933, establishing a precise limit for the functional $\mu a_2^2 - a_3$ [33]. This limit, known as the classical Fekete–Szegö inequality, was derived using real values of μ $(0 \leq \mu \leq 1)$. It is a challenging task to establish precise boundaries for a given function within a compact family of functions $\psi \in \mathcal{A}$, for a real parameter μ. In this context, the Fekete–Szegö inequality for functions belonging to the class $\Sigma_{\mathcal{S}^*}(q; \Omega(\zeta))$ is examined, considering the findings of Zaprawa [34].

Theorem 2. *Let* ψ *be an element of* Σ *defined by* (1) *and belonging to the class* $\Sigma_{\mathcal{S}^*}(q; \Omega(\zeta))$ *and* $\mu \in \mathbb{R}$. *Then, we have*

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} \frac{1}{4q}, & |1-\mu| \leq \left|\frac{q(3q-1)}{(1+q)^2}\right|, \\ \frac{1}{4q}|\mathcal{Y}(\mu)|, & |1-\mu| \geq \left|\frac{q(3q-1)}{(1+q)^2}\right|, \end{cases}$$

where

$$\mathcal{Y}(\mu) = \frac{(1-\mu)(1+q)^2}{q(3q-1)}.$$

Proof. If $\Sigma_{\mathcal{S}^*}(q; \Omega(\zeta))$ is given by (1), from (27) and (28), we have

$$
\begin{aligned}
a_3 - \mu a_2^2 = & \frac{(1-\mu)(1+q)^2}{8q^2(3q-1)}(\ell_2 + \jmath_2) + \frac{1}{8q}(\ell_2 - \jmath_2) \\
= & \frac{1}{8q}[(\mathcal{Y}(\mu)+1)\ell_2 + (\mathcal{Y}(\mu)-1)\jmath_2],
\end{aligned}
$$

where

$$
\mathcal{Y}(\mu) = \frac{(1-\mu)(1+q)^2}{q(3q-1)}
$$

Then, we conclude that

$$
\left|a_3 - \mu a_2^2\right| \le \begin{cases} \frac{1}{4q}, & |\mathcal{Y}(\mu)| \le 1, \\ \\ \frac{1}{4q}|\mathcal{Y}(\mu)|, & |\mathcal{Y}(\mu)| \ge 1. \end{cases}
$$

Which completes the proof of Theorem 2. □

Corollary 2. *Let* ψ *be an element of* Σ *defined by* (1) *and belonging to the class* $\Sigma_{\mathcal{S}^*}(\zeta + \sqrt[3]{1+\zeta^3})$ *and* $\mu \in \mathbb{R}$. *Then, we have*

$$
\left|a_3 - \mu a_2^2\right| \le \begin{cases} \frac{1}{4}, & |1-\mu| \le \frac{1}{2}, \\ \\ \frac{1}{2}|1-\mu|, & |1-\mu| \ge \frac{1}{2}. \end{cases}
$$

5. Conclusions

In this study, we have conducted an investigation on coefficient problems related to recently defined subclasses of bi-univalent functions in Ų given in Definition 5. The investigated subclasses are $\Sigma_{\mathcal{S}^*}(q;\Omega(\zeta))$ and $\Sigma_{\mathcal{S}^*}(\zeta + \sqrt[3]{1+\zeta^3})$. We have computed the Taylor–Maclaurin coefficients $|\alpha_2|$ and $|\alpha_3|$, along with estimates for the Fekete–Szegö functional problem, for functions belonging to each of these bi-univalent function classes.

In future research, the exploration of upper bounds for the Zaclman conjecture and the investigation of the Hankel determinants of orders two and three within the aforementioned subclasses show potential for new avenues of research and exploration.

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