



Applying Block Method for the Numerical Solutions of the Second Order n-Refined Neutrosophic ODE for n=2, 3

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Abstract

In this paper, we study the applications of block method to find the numerical solutions of some neutrosophic differential problems, where we discuss the approximated n-refined neutrosophic solutions and absolute n-refined neutrosophic errors in two special cases for n=2, and n=3. In addition, we list the numerical tables of our results.

Keywords: Weak Fuzzy Complex (WFC) Numbers; Weak Fuzzy Complex Functions;; Differential Equations (DE)

1. Introduction

Numerical analysis in its various branches aims to find approximate numerical solutions and errors resulting from the approximation process for many mathematical problems that are difficult to solve using traditional methods [1-3]. Block method is considered one of the most important methods used in finding numerical solutions too many differential problems, which has many applications in various sciences such as physics, economics, and even computer science [4-6].

In [7-9], we find the concept of refined neutrosophic structures a generalization of classical. They have been studied widely in [10-12]. Many authors [13-17] have studied Neutrosophic numerical analysis with its applications, where we can find some examples for numerical algorithms and numerical tables for neutrosophic differential problems in [18-19].

In this work, we are motivated to study the applications of block method to find the numerical solutions of some neutrosophic differential problems, where we discuss the approximated n-refined neutrosophic solutions and absolute n-refined neutrosophic errors in two special cases for n=2, and n=3. In addition, we list the numerical tables of our results.

2. Main discussion

Definition 2.1

Consider the following second-order refined neutrosophic equation:

$$y''_0 + y''_1 I_1 + y''_2 I_2 = f(x_0 + x_1 I_1 + x_2 I_2, y_0 + y_1 I_1 + y_2 I_2, y'_0 + y'_1 I_1 + y'_2 I_2),$$

$$\begin{cases} (y_0 + y_1 I_1 + y_2 I_2)[a_0 + a_1 I_1 + a_2 I_2] = y_0^* + y_1^* I_1 + y_2^* I_2, \\ ((y_0' + y_1' I_1 + y_2' I_2)[a_0 + a_1 I_1 + a_2 I_2] = (y_0^*)' + (y_1^*)' I_1 + (y_2^*)' I_2, \end{cases}$$

The Method:

The approximated solution of $y_{n+1}^{(0)} + y_{n+1}^{(1)} I_1 + y_{n+1}^{(2)} I_2, y_{n+2}^{(0)} + y_{n+2}^{(1)} I_1 + y_{n+2}^{(2)} I_2, \dots, y_{n+4}^{(0)} + y_{n+4}^{(1)} I_1 + y_{n+4}^{(2)} I_2$ are computed in the block at the points:

$$x_{n+1}^{(0)} + x_{n+1}^{(1)} I_1 + x_{n+1}^{(2)} I_2, x_{n+2}^{(0)} + x_{n+2}^{(1)} I_1 + x_{n+2}^{(2)} I_2, \dots, x_{n+4}^{(0)} + x_{n+4}^{(1)} I_1 + x_{n+4}^{(2)} I_2 \text{ in the } k\text{-block.}$$

The refined neutrosophic interval $[a_0 + a_1 I_1 + a_2 I_2, b_0 + b_1 I_1 + b_2 I_2]$ is divided into a series of blocks can be gotten from Euler's formula.

$(x_n^{(i)}, f_n^{(i)})$ are derived by Lagrange's interpolation polynomial, as follows:

$$(y_{n+1}^{(0)})^0 + (y_{n+1}^{(1)})^1 I_1 + (y_{n+1}^{(2)})^2 I_2 = (y_n^{(0)})^0 + y_n^{(1)} I_1 + y_n^{(2)} I_2 + \frac{h_0 + h_1 I_1 + h_2 I_2}{720} [A_1];$$

Where:

$$A_1 = 251(f_n^{(0)} + f_n^{(1)} I_1 + f_n^{(2)} I_2) + 646(f_{n+1}^{(0)} + f_{n+1}^{(1)} I_1 + f_{n+1}^{(2)} I_2) - 246(f_{n+2}^{(0)} + f_{n+2}^{(1)} I_1 + f_{n+2}^{(2)} I_2) + 106(f_{n+3}^{(0)} + f_{n+3}^{(1)} I_1 + f_{n+3}^{(2)} I_2) - 19(f_{n+4}^{(0)} + f_{n+4}^{(1)} I_1 + f_{n+4}^{(2)} I_2).$$

$$y_{n+1}^{(0)} + y_{n+1}^{(1)} I_1 + y_{n+1}^{(2)} I_2 = y_n^{(0)} + y_n^{(1)} I_1 + y_n^{(2)} I_2 + (h_0 + h_1 I_1 + h_2 I_2)(y_n^{(0)})^0 + y_n^{(1)} I_1 + y_n^{(2)} I_2 + \frac{(h_0 + h_1 I_1 + h_2 I_2)^2}{1440} A_2; \text{ where:}$$

$$A_2 = 367(f_n^{(0)} + f_n^{(1)} I_1 + f_n^{(2)} I_2) + 540(f_{n+1}^{(0)} + f_{n+1}^{(1)} I_1 + f_{n+1}^{(2)} I_2) - 282(f_{n+2}^{(0)} + f_{n+2}^{(1)} I_1 + f_{n+2}^{(2)} I_2) + 116(f_{n+3}^{(0)} + f_{n+3}^{(1)} I_1 + f_{n+3}^{(2)} I_2) - 21(f_{n+4}^{(0)} + f_{n+4}^{(1)} I_1 + f_{n+4}^{(2)} I_2),$$

$$y_{n+2}^{(0)} + y_{n+2}^{(1)} I_1 + y_{n+2}^{(2)} I_2 = y_{n+1}^{(0)} + y_{n+1}^{(1)} I_1 + y_{n+1}^{(2)} I_2 + \frac{h_0 + h_1 I_1 + h_2 I_2}{720} A_3,$$

Where:

$$A_3 = -19(f_n^{(0)} + f_n^{(1)} I_1 + f_n^{(2)} I_2) + 346(f_{n+1}^{(0)} + f_{n+1}^{(1)} I_1 + f_{n+1}^{(2)} I_2) + 456(f_{n+2}^{(0)} + f_{n+2}^{(1)} I_1 + f_{n+2}^{(2)} I_2) - 74(f_{n+3}^{(0)} + f_{n+3}^{(1)} I_1 + f_{n+3}^{(2)} I_2) + 11(f_{n+4}^{(0)} + f_{n+4}^{(1)} I_1 + f_{n+4}^{(2)} I_2),$$

$$y_{n+2}^{(0)} + y_{n+2}^{(1)} I_1 + y_{n+2}^{(2)} I_2 = y_{n+1}^{(0)} + y_{n+1}^{(1)} I_1 + y_{n+1}^{(2)} I_2 + (h_0 + h_1 I_1 + h_2 I_2)(y_{n+1}^{(0)})^0 + y_{n+1}^{(1)} I_1 + y_{n+1}^{(2)} I_2 + \frac{(h_0 + h_1 I_1 + h_2 I_2)^2}{1440} A_4; \text{ where:}$$

$$A_4 = -21(f_n^{(0)} + f_n^{(1)} I_1 + f_n^{(2)} I_2) + 472(f_{n+1}^{(0)} + f_{n+1}^{(1)} I_1 + f_{n+1}^{(2)} I_2) + 330(f_{n+2}^{(0)} + f_{n+2}^{(1)} I_1 + f_{n+2}^{(2)} I_2) - 72(f_{n+3}^{(0)} + f_{n+3}^{(1)} I_1 + f_{n+3}^{(2)} I_2) + 11(f_{n+4}^{(0)} + f_{n+4}^{(1)} I_1 + f_{n+4}^{(2)} I_2),$$

$$y_{n+3}^{(0)} + y_{n+3}^{(1)} I_1 + y_{n+3}^{(2)} I_2 = y_{n+2}^{(0)} + y_{n+2}^{(1)} I_1 + y_{n+2}^{(2)} I_2 + \frac{h_0 + h_1 I_1 + h_2 I_2}{720} A_5,$$

Where:

$$A_5 = 11(f_n^{(0)} + f_n^{(1)} I_1 + f_n^{(2)} I_2) - 74(f_{n+1}^{(0)} + f_{n+1}^{(1)} I_1 + f_{n+1}^{(2)} I_2) + 456(f_{n+2}^{(0)} + f_{n+2}^{(1)} I_1 + f_{n+2}^{(2)} I_2) + 346(f_{n+3}^{(0)} + f_{n+3}^{(1)} I_1 + f_{n+3}^{(2)} I_2) - 19(f_{n+4}^{(0)} + f_{n+4}^{(1)} I_1 + f_{n+4}^{(2)} I_2),$$

$$y_{n+3}^{(0)} + y_{n+3}^{(1)} I_1 + y_{n+3}^{(2)} I_2 = y_{n+2}^{(0)} + y_{n+2}^{(1)} I_1 + y_{n+2}^{(2)} I_2 + (h_0 + h_1 I_1 + h_2 I_2)(y_{n+2}^{(0)})^0 + y_{n+2}^{(1)} I_1 + y_{n+2}^{(2)} I_2 + \frac{(h_0 + h_1 I_1 + h_2 I_2)^2}{1440} A_6; \text{ where:}$$

$$A_6 = 11(f_n^{(0)} + f_n^{(1)} I_1 + f_n^{(2)} I_2) - 76(f_{n+1}^{(0)} + f_{n+1}^{(1)} I_1 + f_{n+1}^{(2)} I_2) + 582(f_{n+2}^{(0)} + f_{n+2}^{(1)} I_1 + f_{n+2}^{(2)} I_2) + 220(f_{n+3}^{(0)} + f_{n+3}^{(1)} I_1 + f_{n+3}^{(2)} I_2) - 17(f_{n+4}^{(0)} + f_{n+4}^{(1)} I_1 + f_{n+4}^{(2)} I_2),$$

$$y_{n+4}^{(0)} + y_{n+4}^{(1)} I_1 + y_{n+4}^{(2)} I_2 = y_{n+3}^{(0)} + y_{n+3}^{(1)} I_1 + y_{n+3}^{(2)} I_2 + \frac{h_0 + h_1 I_1 + h_2 I_2}{720} A_7,$$

$$A_7 = -19(f_n^{(0)} + f_n^{(1)} I_1 + f_n^{(2)} I_2) + 106(f_{n+1}^{(0)} + f_{n+1}^{(1)} I_1 + f_{n+1}^{(2)} I_2) - 264(f_{n+2}^{(0)} + f_{n+2}^{(1)} I_1 + f_{n+2}^{(2)} I_2) + 646(f_{n+3}^{(0)} + f_{n+3}^{(1)} I_1 + f_{n+3}^{(2)} I_2) + 251(f_{n+4}^{(0)} + f_{n+4}^{(1)} I_1 + f_{n+4}^{(2)} I_2),$$

$$\begin{aligned}
& y_{n+4}^{(0)} + y_{n+4}^{(1)} I_1 + y_{n+4}^{(2)} I_2 \\
& = y_{n+3}^{(0)} + y_{n+3}^{(1)} I_1 + y_{n+3}^{(2)} I_2 + (h_0 + h_1 I_1 + h_2 I_2) (y_{n+3}'^{(0)} + y_{n+3}'^{(1)} I_1 + y_{n+3}'^{(2)} I_2) \\
& + \frac{(h_0 + h_1 I_1 + h_2 I_2)^2}{1440} A_8;
\end{aligned}$$

$$\begin{aligned}
A_8 = & -17(f_n^{(0)} + f_n^{(1)} I_1 + f_n^{(2)} I_2) + 96(f_{n+1}^{(0)} + f_{n+1}^{(1)} I_1 + f_{n+1}^{(2)} I_2) - 246(f_{n+2}^{(0)} + f_{n+2}^{(1)} I_1 + f_{n+2}^{(2)} I_2) + \\
& 752(f_{n+3}^{(0)} + f_{n+3}^{(1)} I_1 + f_{n+3}^{(2)} I_2) + 135(f_{n+4}^{(0)} + f_{n+4}^{(1)} I_1 + f_{n+4}^{(2)} I_2),
\end{aligned}$$

Implementation:

For making the selection of the next step size, we use:

$$y_{n+i}'^{(0)} + y_{n+i}'^{(1)} I_1 + y_{n+i}'^{(2)} I_2 = y_{n+(i-1)}'^{(0)} + y_{n+(i-1)}'^{(1)} I_1 + y_{n+(i-1)}'^{(2)} I_2 + (h_0 + h_1 I_1 + h_2 I_2) (f_{n+(i-1)}^{(0)} + f_{n+(i-1)}^{(1)} I_1 + f_{n+(i-1)}^{(2)} I_2),$$

$$\begin{aligned}
& y_{n+i}'^{(0)} + y_{n+i}'^{(1)} I_1 + y_{n+i}'^{(2)} I_2 = y_{n+(i-1)}'^{(0)} + y_{n+(i-1)}'^{(1)} I_1 + y_{n+(i-1)}'^{(2)} I_2 + (h_0 + h_1 I_1 + h_2 I_2) (y_{n+(i-1)}'^{(0)} + \\
& y_{n+(i-1)}'^{(1)} I_1 + y_{n+(i-1)}'^{(2)} I_2) + \frac{(h_0 + h_1 I_1 + h_2 I_2)^2}{2} [f_{n+(i-1)}^{(0)} + f_{n+(i-1)}^{(1)} I_1 + f_{n+(i-1)}^{(2)} I_2]; i = 1, 2, 3, 4, \dots
\end{aligned}$$

The corrector iteration converges by:

$$|y_{n+4,r+1}^{c(0)} + y_{n+4,r+1}^{c(1)} I_1 + y_{n+4,r+1}^{c(2)} I_2 - y_{n+4,r}^{c(0)} + y_{n+4,r}^{c(1)} I_1 + y_{n+4,r}^{c(2)} I_2| < 0.1 + 0.1 I_1 + 0.1 I_2.$$

(r) is the number of iterations.

Numerical tests: (Refined neutrosophic two-body orbit)

$$\begin{cases}
y_1^{(0)} + y_1^{(1)} I_1 + y_1^{(2)} I_2 = \frac{-1}{(r_0 + r_1 I_1 + r I_2)^3} (y_1^{(0)} + y_1^{(1)} I_1 + y_1^{(2)} I_2) \\
y_2^{(0)} + y_2^{(1)} I_1 + y_2^{(2)} I_2 = \frac{-1}{(r_0 + r_1 I_1 + r I_2)^3} (y_2^{(0)} + y_2^{(1)} I_1 + y_2^{(2)} I_2)
\end{cases}$$

$$\text{With: } \begin{cases} (y_1^{(0)} + y_1^{(1)} I_1 + y_1^{(2)} I_2)(0) = 1 \\ (y_1'^{(0)} + y_1'^{(1)} I_1 + y_1'^{(2)} I_2)(0) = 0 \end{cases} \text{ and:}$$

$$\begin{cases}
(y_2^{(0)} + y_2^{(1)} I_1 + y_2^{(2)} I_2)(0) = 0 \\
(y_2'^{(0)} + y_2'^{(1)} I_1 + y_2'^{(2)} I_2)(0) = 1 \\
r_0 + r_1 I_1 + r I_2 = \sqrt{(y_1^{(0)} + y_1^{(1)} I_1 + y_1^{(2)} I_2)^2 + (y_2^{(0)} + y_2^{(1)} I_1 + y_2^{(2)} I_2)^2}
\end{cases}$$

Refined neutrosophic first order systems:

$$\begin{cases}
y_1'^{(0)} + y_1'^{(1)} I_1 + y_1'^{(2)} I_2 = y_3^{(0)} + y_3^{(1)} I_1 + y_3^{(2)} I_2 \\
(y_1^{(0)} + y_1^{(1)} I_1 + y_1^{(2)} I_2)(0) = 1
\end{cases}$$

$$\begin{cases}
y_2'^{(0)} + y_2'^{(1)} I_1 + y_2'^{(2)} I_2 = y_4^{(0)} + y_4^{(1)} I_1 + y_4^{(2)} I_2 \\
(y_2^{(0)} + y_2^{(1)} I_1 + y_2^{(2)} I_2)(0) = 0
\end{cases}$$

$$\begin{cases}
y_3'^{(0)} + y_3'^{(1)} I_1 + y_3'^{(2)} I_2 = \frac{-1}{(r_0 + r_1 I_1 + r I_2)^3} (y_1^{(0)} + y_1^{(1)} I_1 + y_1^{(2)} I_2) \\
(y_3^{(0)} + y_3^{(1)} I_1 + y_3^{(2)} I_2)(0) = 0
\end{cases}$$

$$\begin{cases} y_4^{(0)} + y_4^{(1)}I_1 + y_4^{(2)}I_2 = \frac{-1}{(r_0 + r_1I_1 + r_2I_2)^3} (y_2^{(0)} + y_2^{(1)}I_1 + y_2^{(2)}I_2) \\ (y_4^{(0)} + y_4^{(1)}I_1 + y_4^{(2)}I_2)(0) = 1 \end{cases}$$

Problem (2):

Refined neutrosophic van der pol oscillator:

$(y_0 + y_1I_1 + y_2I_2)'' - 2m [1 - (y_0 + y_1I_1 + y_2I_2)^2](y_0 + y_1I_1 + y_2I_2)' + (y_0 + y_1I_1 + y_2I_2) = 0$, with:

$$\begin{cases} (y_0 + y_1I_1 + y_2I_2)(0) = 0 \\ (y_0 + y_1I_1 + y_2I_2)'(0) = \frac{1}{2} \end{cases}$$

The first order system:

$$\begin{cases} (y_0^{(1)} + y_1^{(1)}I_1 + y_2^{(1)}I_2)' = y_0^{(2)} + y_1^{(2)}I_1 + y_2^{(2)}I_2, \\ (y_0^{(1)} + y_1^{(1)}I_1 + y_2^{(1)}I_2)(0) = 0 \\ \begin{cases} y_0^{(2)} + y_1^{(2)}I_1 + y_2^{(2)}I_2 = 2m [1 - (y_0^{(1)} + y_1^{(1)}I_1 + y_2^{(1)}I_2)^2] (y_0^{(1)} + y_1^{(1)}I_1 + y_2^{(1)}I_2)' - (y_0^{(1)} + y_1^{(1)}I_1 + y_2^{(1)}I_2); \\ m = \frac{25}{1000} (y_0^{(2)} + y_1^{(2)}I_1 + y_2^{(2)}I_2)(0) = \frac{1}{2} \end{cases} \end{cases}$$

Table 1: Numerical results for the first problem (A)

Tol	MTD	Ts	MaxE	AVE	FCN
10⁻⁴	4 PrED 4 PosB	43+12I ₁ +6I ₂ 27+4I ₁ +6I ₂	6.03214e-04+4.0983I ₁ +I ₂ 2.9181e-05+2.09987I ₁ +I ₂	7.1337e-05+0.765I ₁ +I ₂ 9.0450e-06+0.4432I ₁ +I ₂	1017+11I ₁ +12I ₂ 529+22I ₁ +6I ₂
10⁻⁶	4 PrED 4 PosB	125+I ₁ +I ₂ 57+3I ₁ +I ₂	1.2363e-06+123I ₁ +I ₂ 6.4539e-06+I ₁ +I ₂	2.1477e-07+13I ₁ +I ₂ 1.4363e-06+12I ₁ +I ₂	2749+56I ₁ +I ₂ 909+45I ₁ +I ₂
10⁻⁸	4 PrED 4 PosB	207+67I ₁ +I ₂ 121+89I ₁ +I ₂	5.9493e-09+I ₁ +I ₂ 3.0989e-08+I ₁ +I ₂	8.8773e-10+I ₁ +I ₂ 6.6848e-09+I ₁ +I ₂	5317+I ₁ +I ₂ 1937+I ₁ +I ₂
10⁻¹⁰	4 PrED 4 PosB	637+21I ₁ +I ₂ 261+14I ₁ +I ₂	2.1775e-11+17I ₁ +I ₂ 1.3901e-10+12I ₁ +I ₂	4.6349e-12+I ₁ +I ₂ 2.9203e-11+I ₁ +I ₂	14861+8I ₁ +I ₂ 4177+16I ₁ +I ₂
10⁻¹²	4 PrED 4 PosB	911+I ₁ +I ₂ 312+I ₁ +I ₂	2.172375e-11+17I ₁ +I ₂ 1.3901e-10+12I ₁ +I ₂	4.6349e-12+I ₁ +I ₂ 2.9203e-11+I ₁ +I ₂	15861+8I ₁ +I ₂ 5177+16I ₁ +I ₂
10⁻¹⁴	4 PrED 4 PosB	1013+I ₁ +I ₂ 402+I ₁ +I ₂	2.172375e-11+17I ₁ +I ₂ 1.3901e-10+12I ₁ +I ₂	4.6349e-12+I ₁ +I ₂ 2.9203e-11+I ₁ +I ₂	16861+8I ₁ +I ₂ 4777+16I ₁ +I ₂
10⁻¹⁶	4 PrED 4 PosB	1201+I ₁ +I ₂ 572+I ₁ +I ₂	2.172375e-11+17I ₁ +I ₂ 1.3901e-10+12I ₁ +I ₂	4.6349e-12+I ₁ +I ₂ 2.9203e-11+I ₁ +I ₂	18861+8I ₁ +I ₂ 7177+16I ₁ +I ₂

Table 2: Problem (2) (A)

Tol	MTD	Ts	Absolutes error	
			$y(x)$	$y'(x)$
10^{-4}	RK 45 4 PosB	$285+I_1+I_2$ $43+I_1+I_2$	$7.016927e-0410+12I_1+I_2$ $9.610276e-0410+12I_1+I_2$	$4.157618e-04+16I_1+I_2$ $4.888968e-04+16I_1+I_2$
10^{-6}	RK 45 4 PosB	$713+15I_1+21I_2$ $62+15I_1+33I_2$	$6.567087e-0611+I_1+I_2$ $4.267339e-011+I_1+I_26$	$5.359170e-07+I_1+I_2$ $5.667588e-06+I_1+I_2$
10^{-8}	RK 45 4 PosB	$1789+17I_1+5I_2$ $122+57I_1+43I_2$	$5.764219e-0810+12I_1+I_2$ $4.399870e-0810+12I_1+I_2$	$2.270758e-08+16I_1+I_2$ $6.230616e-08+16I_1+I_2$
10^{-10}	RK 45 4 PosB	$4489+13I_1+5I_2$ $267+21I_1+5I_2$	$5.331400e-1011+I_1+I_2$ $4.851652e-111+I_1+I_20$	$2.876250e-10+I_1+I_2$ $5.869338e-10+I_1+I_2$

Definition 2.2 Consider the following second-order 3-refined neutrosophic equation:

$$y_0'' + y_1''I_1 + y_2''I_2 + y_3''I_3 = f(x_0 + x_1I_1 + x_2I_2 + x_3I_3, y_0 + y_1I_1 + y_2I_2 + y_3I_3, y_0' + y_1'I_1 + y_2'I_2 + y_3'I_3),$$

$$\begin{cases} (y_0 + y_1I_1 + y_2I_2 + y_3I_3)[a_0 + a_1I_1 + a_2I_2 + a_3I_3] = y_0^* + y_1^*I_1 + y_2^*I_2 + y_3^*I_3, \\ (y_0' + y_1'I_1 + y_2'I_2 + y_3'I_3)[a_0 + a_1I_1 + a_2I_2 + a_3I_3] = (y_0^*)' + (y_1^*)'I_1 + (y_2^*)'I_2 + (y_3^*)'I_3, \end{cases}$$

The Method:

The approximated solution of $y_{n+1}^{(0)} + y_{n+1}^{(1)}I_1 + y_{n+1}^{(2)}I_2 + y_{n+1}^{(3)}I_3, y_{n+2}^{(0)} + y_{n+2}^{(1)}I_1 + y_{n+2}^{(2)}I_2 + y_{n+2}^{(3)}I_3, \dots, y_{n+4}^{(0)} + y_{n+4}^{(1)}I_1 + y_{n+4}^{(2)}I_2 + y_{n+4}^{(3)}I_3$ are computed in the block at the points:

$x_{n+1}^{(0)} + x_{n+1}^{(1)}I_1 + x_{n+1}^{(2)}I_2 + x_{n+1}^{(3)}I_3, x_{n+2}^{(0)} + x_{n+2}^{(1)}I_1 + x_{n+2}^{(2)}I_2 + x_{n+2}^{(3)}I_3, \dots, x_{n+4}^{(0)} + x_{n+4}^{(1)}I_1 + x_{n+4}^{(2)}I_2 + x_{n+4}^{(3)}I_3$ in the k-block.

The refined neutrosophic interval $[a_0 + a_1I_1 + a_2I_2 + a_3I_3, b_0 + b_1I_1 + b_2I_2 + b_3I_3]$ is divided into a series of blocks can be gotten from Euler's formula.

$(x_n^{(i)}, f_n^{(i)})$ are derived by Lagrange's interpolation polynomial, as follows:

$$\begin{aligned} & (y_{n+1}')^0 + (y_{n+1}')^1I_1 + (y_{n+1}')^2I_2 + (y_{n+1}')^3I_3 \\ &= (y_n'^{(0)} + y_n'^{(1)}I_1 + y_n'^{(2)}I_2 + y_n'^{(3)}I_3) + \frac{h_0 + h_1I_1 + h_2I_2 + h_3I_3}{720} [A_1]; \end{aligned}$$

Where:

$$A_1 = 251(f_n^{(0)} + f_n^{(1)}I_1 + f_n^{(2)}I_2 + f_n^{(3)}I_3) + 646(f_{n+1}^{(0)} + f_{n+1}^{(1)}I_1 + f_{n+1}^{(2)}I_2 + f_{n+1}^{(3)}I_3) - 246(f_{n+2}^{(0)} + f_{n+2}^{(1)}I_1 + f_{n+2}^{(2)}I_2 + f_{n+2}^{(3)}I_3) + 106(f_{n+3}^{(0)} + f_{n+3}^{(1)}I_1 + f_{n+3}^{(2)}I_2 + f_{n+3}^{(3)}I_3) - 19(f_{n+4}^{(0)} + f_{n+4}^{(1)}I_1 + f_{n+4}^{(2)}I_2 + f_{n+4}^{(3)}I_3).$$

$$y_{n+1}^{(0)} + y_{n+1}^{(1)}I_1 + y_{n+1}^{(2)}I_2 + y_{n+1}^{(3)}I_3 = y_n^{(0)} + y_n^{(1)}I_1 + y_n^{(2)}I_2 + y_n^{(3)}I_3 + (h_0 + h_1I_1 + h_2I_2 + h_3I_3)(y_n'^{(0)} + y_n'^{(1)}I_1 + y_n'^{(2)}I_2 + y_n'^{(3)}I_3) + \frac{(h_0 + h_1I_1 + h_2I_2 + h_3I_3)^2}{1440} A_2; \text{ where:}$$

$$A_2 = 367(f_n^{(0)} + f_n^{(1)}I_1 + f_n^{(2)}I_2 + f_n^{(3)}I_3) + 540(f_{n+1}^{(0)} + f_{n+1}^{(1)}I_1 + f_{n+1}^{(2)}I_2 + f_{n+1}^{(3)}I_3) - 282(f_{n+2}^{(0)} + f_{n+2}^{(1)}I_1 + f_{n+2}^{(2)}I_2 + f_{n+2}^{(3)}I_3) + 116(f_{n+3}^{(0)} + f_{n+3}^{(1)}I_1 + f_{n+3}^{(2)}I_2 + f_{n+3}^{(3)}I_3) - 21(f_{n+4}^{(0)} + f_{n+4}^{(1)}I_1 + f_{n+4}^{(2)}I_2 + f_{n+4}^{(3)}I_3),$$

$$y'_{n+2}^{(0)} + y'_{n+2}^{(1)}I_1 + y'_{n+2}^{(2)}I_2 + y'_{n+2}^{(3)}I_3 = y'_{n+1}^{(0)} + y'_{n+1}^{(1)}I_1 + y'_{n+1}^{(2)}I_2 + y'_{n+1}^{(3)}I_3 + \frac{h_0+h_1I_1+h_2I_2+h_3I_3}{720}A_3,$$

Where:

$$A_3 = -19(f_n^{(0)} + f_n^{(1)}I_1 + f_n^{(2)}I_2 + f_n^{(3)}I_3) + 346(f_{n+1}^{(0)} + f_{n+1}^{(1)}I_1 + f_{n+1}^{(2)}I_2 + f_{n+1}^{(3)}I_3) + 456(f_{n+2}^{(0)} + f_{n+2}^{(1)}I_1 + f_{n+2}^{(2)}I_2 + f_{n+2}^{(3)}I_3) - 74(f_{n+3}^{(0)} + f_{n+3}^{(1)}I_1 + f_{n+3}^{(2)}I_2 + f_{n+3}^{(3)}I_3) + 11(f_{n+4}^{(0)} + f_{n+4}^{(1)}I_1 + f_{n+4}^{(2)}I_2 + f_{n+4}^{(3)}I_3),$$

$$y_{n+2}^{(0)} + y_{n+2}^{(1)}I_1 + y_{n+2}^{(2)}I_2 + y_{n+2}^{(3)}I_3 = y_{n+1}^{(0)} + y_{n+1}^{(1)}I_1 + y_{n+1}^{(2)}I_2 + y_{n+1}^{(3)}I_3 + (h_0 + h_1I_1 + h_2I_2 + h_3I_3)(y'_{n+1}^{(0)} + y'_{n+1}^{(1)}I_1 + y'_{n+1}^{(2)}I_2 + y'_{n+1}^{(3)}I_3) + \frac{(h_0+h_1I_1+h_2I_2+h_3I_3)^2}{1440}A_4; \text{ where:}$$

$$A_4 = -21(f_n^{(0)} + f_n^{(1)}I_1 + f_n^{(2)}I_2 + f_n^{(3)}I_3) + 472(f_{n+1}^{(0)} + f_{n+1}^{(1)}I_1 + f_{n+1}^{(2)}I_2 + f_{n+1}^{(3)}I_3) + 330(f_{n+2}^{(0)} + f_{n+2}^{(1)}I_1 + f_{n+2}^{(2)}I_2 + f_{n+2}^{(3)}I_3) - 72(f_{n+3}^{(0)} + f_{n+3}^{(1)}I_1 + f_{n+3}^{(2)}I_2 + f_{n+3}^{(3)}I_3) + 11(f_{n+4}^{(0)} + f_{n+4}^{(1)}I_1 + f_{n+4}^{(2)}I_2 + f_{n+4}^{(3)}I_3),$$

$$y'_{n+3}^{(0)} + y'_{n+3}^{(1)}I_1 + y'_{n+3}^{(2)}I_2 + y'_{n+3}^{(3)}I_3 = y'_{n+2}^{(0)} + y'_{n+2}^{(1)}I_1 + y'_{n+2}^{(2)}I_2 + y'_{n+2}^{(3)}I_3 + \frac{h_0+h_1I_1+h_2I_2+h_3I_3}{720}A_5,$$

Where:

$$A_5 = 11(f_n^{(0)} + f_n^{(1)}I_1 + f_n^{(2)}I_2 + f_n^{(3)}I_3) - 74(f_{n+1}^{(0)} + f_{n+1}^{(1)}I_1 + f_{n+1}^{(2)}I_2 + f_{n+1}^{(3)}I_3) + 456(f_{n+2}^{(0)} + f_{n+2}^{(1)}I_1 + f_{n+2}^{(2)}I_2 + f_{n+2}^{(3)}I_3) + 346(f_{n+3}^{(0)} + f_{n+3}^{(1)}I_1 + f_{n+3}^{(2)}I_2 + f_{n+3}^{(3)}I_3) - 19(f_{n+4}^{(0)} + f_{n+4}^{(1)}I_1 + f_{n+4}^{(2)}I_2 + f_{n+4}^{(3)}I_3),$$

$$y_{n+3}^{(0)} + y_{n+3}^{(1)}I_1 + y_{n+3}^{(2)}I_2 + y_{n+3}^{(3)}I_3 = y_{n+2}^{(0)} + y_{n+2}^{(1)}I_1 + y_{n+2}^{(2)}I_2 + y_{n+2}^{(3)}I_3 + (h_0 + h_1I_1 + h_2I_2 + h_3I_3)(y'_{n+2}^{(0)} + y'_{n+2}^{(1)}I_1 + y'_{n+2}^{(2)}I_2 + y'_{n+2}^{(3)}I_3) + \frac{(h_0+h_1I_1+h_2I_2+h_3I_3)^2}{1440}A_6; \text{ where:}$$

$$A_6 = 11(f_n^{(0)} + f_n^{(1)}I_1 + f_n^{(2)}I_2 + f_n^{(3)}I_3) - 76(f_{n+1}^{(0)} + f_{n+1}^{(1)}I_1 + f_{n+1}^{(2)}I_2 + f_{n+1}^{(3)}I_3) + 582(f_{n+2}^{(0)} + f_{n+2}^{(1)}I_1 + f_{n+2}^{(2)}I_2 + f_{n+2}^{(3)}I_3) + 220(f_{n+3}^{(0)} + f_{n+3}^{(1)}I_1 + f_{n+3}^{(2)}I_2 + f_{n+3}^{(3)}I_3) - 17(f_{n+4}^{(0)} + f_{n+4}^{(1)}I_1 + f_{n+4}^{(2)}I_2 + f_{n+4}^{(3)}I_3),$$

$$y'_{n+4}^{(0)} + y'_{n+4}^{(1)}I_1 + y'_{n+4}^{(2)}I_2 + y'_{n+4}^{(3)}I_3 = y'_{n+3}^{(0)} + y'_{n+3}^{(1)}I_1 + y'_{n+3}^{(2)}I_2 + y'_{n+3}^{(3)}I_3 + \frac{h_0+h_1I_1+h_2I_2+h_3I_3}{720}A_7,$$

$$A_7 = -19(f_n^{(0)} + f_n^{(1)}I_1 + f_n^{(2)}I_2 + f_n^{(3)}I_3) + 106(f_{n+1}^{(0)} + f_{n+1}^{(1)}I_1 + f_{n+1}^{(2)}I_2 + f_{n+1}^{(3)}I_3) - 264(f_{n+2}^{(0)} + f_{n+2}^{(1)}I_1 + f_{n+2}^{(2)}I_2 + f_{n+2}^{(3)}I_3) + 646(f_{n+3}^{(0)} + f_{n+3}^{(1)}I_1 + f_{n+3}^{(2)}I_2 + f_{n+3}^{(3)}I_3) + 251(f_{n+4}^{(0)} + f_{n+4}^{(1)}I_1 + f_{n+4}^{(2)}I_2 + f_{n+4}^{(3)}I_3),$$

$$y_{n+4}^{(0)} + y_{n+4}^{(1)}I_1 + y_{n+4}^{(2)}I_2 + y_{n+4}^{(3)}I_3 = y_{n+3}^{(0)} + y_{n+3}^{(1)}I_1 + y_{n+3}^{(2)}I_2 + y_{n+3}^{(3)}I_3 + (h_0 + h_1I_1 + h_2I_2 + h_3I_3)(y'_{n+3}^{(0)} + y'_{n+3}^{(1)}I_1 + y'_{n+3}^{(2)}I_2 + y'_{n+3}^{(3)}I_3) + \frac{(h_0+h_1I_1+h_2I_2+h_3I_3)^2}{1440}A_8;$$

$$A_8 = -17(f_n^{(0)} + f_n^{(1)}I_1 + f_n^{(2)}I_2 + f_n^{(3)}I_3) + 96(f_{n+1}^{(0)} + f_{n+1}^{(1)}I_1 + f_{n+1}^{(2)}I_2 + f_{n+1}^{(3)}I_3) - 246(f_{n+2}^{(0)} + f_{n+2}^{(1)}I_1 + f_{n+2}^{(2)}I_2 + f_{n+2}^{(3)}I_3) + 752(f_{n+3}^{(0)} + f_{n+3}^{(1)}I_1 + f_{n+3}^{(2)}I_2 + f_{n+3}^{(3)}I_3) + 135(f_{n+4}^{(0)} + f_{n+4}^{(1)}I_1 + f_{n+4}^{(2)}I_2 + f_{n+4}^{(3)}I_3),$$

Implementation:

For making the selection of the next step size, we use:

$$y'_{n+i}^{(0)} + y'_{n+i}^{(1)}I_1 + y'_{n+i}^{(2)}I_2 + y'_{n+i}^{(3)}I_3 = y'_{n+(i-1)}^{(0)} + y'_{n+(i-1)}^{(1)}I_1 + y'_{n+(i-1)}^{(2)}I_2 + y'_{n+(i-1)}^{(3)}I_3 + (h_0 + h_1I_1 + h_2I_2 + h_3I_3)(f_{n+(i-1)}^{(0)} + f_{n+(i-1)}^{(1)}I_1 + f_{n+(i-1)}^{(2)}I_2 + f_{n+(i-1)}^{(3)}I_3),$$

$$y'_{n+i}^{(0)} + y'_{n+i}^{(1)}I_1 + y'_{n+i}^{(2)}I_2 + y'_{n+i}^{(3)}I_3 = y'_{n+(i-1)}^{(0)} + y'_{n+(i-1)}^{(1)}I_1 + y'_{n+(i-1)}^{(2)}I_2 + y'_{n+(i-1)}^{(3)}I_3 + (h_0 + h_1I_1 + h_2I_2 + h_3I_3)(y'_{n+(i-1)}^{(0)} + y'_{n+(i-1)}^{(1)}I_1 + y'_{n+(i-1)}^{(2)}I_2 + y'_{n+(i-1)}^{(3)}I_3) + \frac{(h_0+h_1I_1+h_2I_2+h_3I_3)^2}{2}[f_{n+(i-1)}^{(0)} + f_{n+(i-1)}^{(1)}I_1 + f_{n+(i-1)}^{(2)}I_2 + f_{n+(i-1)}^{(3)}I_3]; i = 1, 2, 3, 4, \dots$$

The corrector iteration converges by:

$$\left| y_{n+4,r+1}^{c(0)} + y_{n+4,r+1}^{c(1)}I_1 + y_{n+4,r+1}^{c(2)}I_2 + y_{n+4,r+1}^{c(3)}I_3 - y_{n+4,r}^{c(0)} - y_{n+4,r}^{c(1)}I_1 - y_{n+4,r}^{c(2)}I_2 - y_{n+4,r}^{c(3)}I_3 \right| < 0.1 + 0.1I_1 + 0.1I_2 + 0.1I_3.$$

(r) is the number of iterations.

Numerical tests: (Refined neutrosophic two-body orbit)

$$\begin{cases} y_1^{(0)} + y_1^{(1)}I_1 + y_1^{(2)}I_2 + y_1^{(3)}I_3 = \frac{-1}{(r_0 + r_1I_1 + r_2I_2 + r_3I_3)^3} (y_1^{(0)} + y_1^{(1)}I_1 + y_1^{(2)}I_2 + y_1^{(3)}I_3) \\ y_2^{(0)} + y_2^{(1)}I_1 + y_2^{(2)}I_2 + y_2^{(3)}I_3 = \frac{-1}{(r_0 + r_1I_1 + r_2I_2 + r_3I_3)^3} (y_2^{(0)} + y_2^{(1)}I_1 + y_2^{(2)}I_2 + y_2^{(3)}I_3) \end{cases}$$

With: $\begin{cases} (y_1^{(0)} + y_1^{(1)}I_1 + y_1^{(2)}I_2 + y_1^{(3)}I_3)(0) = 1 \\ (y_1^{(0)} + y_1^{(1)}I_1 + y_1^{(2)}I_2 + y_1^{(3)}I_3)'(0) = 0 \end{cases}$ and:

$$\begin{cases} (y_2^{(0)} + y_2^{(1)}I_1 + y_2^{(2)}I_2 + y_2^{(3)}I_3)(0) = 0 \\ (y_2^{(0)} + y_2^{(1)}I_1 + y_2^{(2)}I_2 + y_2^{(3)}I_3)'(0) = 1 \\ r_0 + r_1I_1 + r_2I_2 + r_3I_3 = \sqrt{(y_1^{(0)} + y_1^{(1)}I_1 + y_1^{(2)}I_2 + y_1^{(3)}I_3)^2 + (y_2^{(0)} + y_2^{(1)}I_1 + y_2^{(2)}I_2 + y_2^{(3)}I_3)^2} \end{cases}$$

Refined neutrosophic first order systems:

$$\begin{cases} y_1^{(0)} + y_1^{(1)}I_1 + y_1^{(2)}I_2 + y_1^{(3)}I_3 = y_3^{(0)} + y_3^{(1)}I_1 + y_3^{(2)}I_2 + y_3^{(3)}I_3 \\ (y_1^{(0)} + y_1^{(1)}I_1 + y_1^{(2)}I_2 + y_1^{(3)}I_3)(0) = 1 \end{cases}$$

$$\begin{cases} y_2^{(0)} + y_2^{(1)}I_1 + y_2^{(2)}I_2 + y_2^{(3)}I_3 = y_4^{(0)} + y_4^{(1)}I_1 + y_4^{(2)}I_2 + y_4^{(3)}I_3 \\ (y_2^{(0)} + y_2^{(1)}I_1 + y_2^{(2)}I_2 + y_2^{(3)}I_3)(0) = 0 \end{cases}$$

$$\begin{cases} y_3^{(0)} + y_3^{(1)}I_1 + y_3^{(2)}I_2 + y_3^{(3)}I_3 = \frac{-1}{(r_0 + r_1I_1 + r_2I_2 + r_3I_3)^3} (y_1^{(0)} + y_1^{(1)}I_1 + y_1^{(2)}I_2 + y_1^{(3)}I_3) \\ (y_3^{(0)} + y_3^{(1)}I_1 + y_3^{(2)}I_2 + y_3^{(3)}I_3)(0) = 0 \end{cases}$$

$$\begin{cases} y_4^{(0)} + y_4^{(1)}I_1 + y_4^{(2)}I_2 + y_4^{(3)}I_3 = \frac{-1}{(r_0 + r_1I_1 + r_2I_2 + r_3I_3)^3} (y_2^{(0)} + y_2^{(1)}I_1 + y_2^{(2)}I_2 + y_2^{(3)}I_3) \\ (y_4^{(0)} + y_4^{(1)}I_1 + y_4^{(2)}I_2 + y_4^{(3)}I_3)(0) = 1 \end{cases}$$

Problem (2):

Refined neutrosophic van der pol oscillator:

$(y_0 + y_1I_1 + y_2I_2 + y_3I_3)'' - 2m[1 - (y_0 + y_1I_1 + y_2I_2 + y_3I_3)^2](y_0 + y_1I_1 + y_2I_2 + y_3I_3)' + (y_0 + y_1I_1 + y_2I_2 + y_3I_3) = 0$, with:

$$\begin{cases} (y_0 + y_1I_1 + y_2I_2 + y_3I_3)(0) = 0 \\ (y_0 + y_1I_1 + y_2I_2 + y_3I_3)'(0) = \frac{1}{2} \end{cases}$$

The first order system:

$$\begin{cases} (y_0^{(1)} + y_1^{(1)}I_1 + y_2^{(1)}I_2 + y_3^{(1)}I_3)' = y_0^{(2)} + y_1^{(2)}I_1 + y_2^{(2)}I_2 + y_3^{(2)}I_3, \\ (y_0^{(1)} + y_1^{(1)}I_1 + y_2^{(1)}I_2 + y_3^{(1)}I_3)(0) = 0 \\ \begin{cases} y_0^{(2)} + y_1^{(2)}I_1 + y_2^{(2)}I_2 + y_3^{(2)}I_3 = 2m[1 - (y_0^{(1)} + y_1^{(1)}I_1 + y_2^{(1)}I_2 + y_3^{(1)}I_3)^2] \\ (y_0^{(2)} + y_1^{(2)}I_1 + y_2^{(2)}I_2 + y_3^{(2)}I_3)' - (y_0^{(1)} + y_1^{(1)}I_1 + y_2^{(1)}I_2 + y_3^{(1)}I_3) = \frac{25}{1000} \\ (y_0^{(2)} + y_1^{(2)}I_1 + y_2^{(2)}I_2 + y_3^{(2)}I_3)(0) = \frac{1}{2} \end{cases} \end{cases}$$

Table 3: Numerical results for the first problem (B)

Tol	MTD	Ts	Max E	AVE	FCN
10^{-4}	4 PrED 4 PosB	$43+12I_1+6I_2 + 53I_3$ $27+4I_1+6I_2 + 53I_3$	$6.03214e-04+4.0983I_1+I_2 + 53I_3$ $2.9181e-05+2.09987I_1+I_2$	$7.1337e-05+0.765I_1+I_2$ $9.0450e-+53I_3$ $06+0.4432I_1+I_2 + 53I_3$	$1017+11I_1+12I_2 + 53I_3$ $529+22I_1+6I_2 + 53I_3$
10^{-6}	4 PrED 4 PosB	$125+I_1+I_2 + 53I_3$ $57+3I_1+I_2 + 53I_3$	$1.2363e-06+123I_1+I_2 + 53I_3$ $6.4539e-06+I_1+I_2 + 53I_3$	$2.1477e-07+13I_1+I_2$ $1.4363e-+53I_3$ $06+12I_1+I_2 + 53I_3$	$2749+56I_1+I_2 + 53I_3$ $909+45I_1+I_2 + 53I_3$
10^{-8}	4 PrED 4 PosB	$207+67I_1+I_2 + 53I_3$ $121+89I_1+I_2 + 53I_3$	$5.9493e-09+I_1+I_2$ $3.0989e-08+I_1+I_2$	$8.8773e-10+I_1+I_2$ $6.6848e-09+I_1+I_2$	$5317+I_1+I_2$ $1937+I_1+I_2$
10^{-10}	4 PrED 4 PosB	$637+21I_1+I_2 + 53I_3$ $261+14I_1+I_2 + 53I_3$	$2.1775e-11+17I_1+I_2$ $1.3901e-+53I_3$ $10+12I_1+I_2 + 53I_3$	$4.6349e-12+I_1+I_2 + 53I_3$ $2.9203e-11+I_1+I_2 + 53I_3$	$14861+8I_1+I_2 + 53I_3$ $4177+16I_1+I_2 + 53I_3$
10^{-12}	4 PrED 4 PosB	$911+I_1+I_2 + 53I_3$ $312+I_1+I_2 + 53I_3$	$2.172375e-11+17I_1+I_2$ $1.3901e-+53I_3$ $10+12I_1+I_2 + 53I_3$	$4.6349e-12+I_1+I_2 + 53I_3$ $2.9203e-11+I_1+I_2 + 53I_3$	$15861+8I_1+I_2 + 53I_3$ $5177+16I_1+I_2 + 53I_3$
10^{-14}	4 PrED 4 PosB	$1013+I_1+I_2 + 53I_3$ $402+I_1+I_2 + 53I_3$	$2.172375e-11+17I_1+I_2 + 53I_3$ $1.3901e-10+12I_1+I_2$	$4.6349e-12+I_1+I_2 + 53I_3$ $2.9203e-11+I_1+I_2$	$16861+8I_1+I_2 + 53I_3$ $4777+16I_1+I_2$
10^{-16}	4 PrED 4 PosB	$1201+I_1+I_2 + 53I_3$ $572+I_1+I_2 + 53I_3$	$2.172375e-11+17I_1+I_2 + 53I_3$ $1.3901e-10+12I_1+I_2 + 53I_3$	$4.6349e-12+I_1+I_2 + 53I_3$ $2.9203e-11+I_1+I_2 + 53I_3$	$18861+8I_1+I_2 + 53I_3$ $7177+16I_1+I_2 + 53I_3$

Table 4: Problem (2) (B)

Tol	MTD	Ts	Absolutes error	
			$y(x)$	$y'(x)$
10^{-4}	RK 45 4 PosB	$285+I_1+I_2 + 22I_3$ $43+I_1+I_2 + 22I_3$	$7.016927e-0410+12I_1+I_2 + 22I_3$	$4.157618e-04+16I_1+I_2 + 22I_3$
			$9.610276e-0410+12I_1+I_2 + 22I_3$	$4.888968e-04+16I_1+I_2 + 22I_3$
10^{-6}	RK 45 4 PosB	$713+15I_1+21I_2 + 22I_3$ $62+15I_1+33I_2 + 22I_3$	$6.567087e-0611+I_1+I_2 + 22I_3$	$5.359170e-07+I_1+I_2 + 22I_3$
			$4.267339e-011+I_1+I_2 + 22I_3$	$5.667588e-06+I_1+I_2 + 22I_3$
10^{-8}	RK 45 4 PosB	$1789+17I_1+5I_2 + 22I_3$ $122+57I_1+43I_2 + 22I_3$	$5.764219e-0810+12I_1+I_2 + 22I_3$	$2.270758e-08+16I_1+I_2 + 22I_3$
			$4.399870e-0810+12I_1+I_2 + 22I_3$	$6.230616e-08+16I_1+I_2 + 22I_3$
10^{-10}	RK 45 4 PosB	$4489+13I_1+5I_2 + 22I_3$ $267+21I_1+5I_2 + 22I_3$	$5.331400e-1011+I_1+I_2 + 22I_3$	$2.876250e-10+I_1+I_2 + 22I_3$
			$4.851652e-111+I_1+I_2 + 22I_3$	$5.869338e-10+I_1+I_2 + 22I_3$

3. Conclusion

In this paper, we have studied the applications of block method to find the numerical solutions of some neutrosophic differential problems, where we discussed the approximated n-refined neutrosophic solutions and absolute n-refined neutrosophic errors in two special cases for $n=2$, and $n=3$. In addition, we listed the numerical tables of our results.

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