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Innovative rough set approaches using novel initial-neighborhood systems: Applications in medical diagnosis of Covid-19 variants

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ABSTRACT

The theory of rough sets produces a potent framework for administrating uncertainty and ambiguity in data, which is crucial for effective decision-making. However, the reliance on equivalence relations within this framework has led to the exploration of various generalizations and extensions. In this paper, we introduce eight new types of initial neighborhoods, expanding on the idea of initial neighborhoods, and examine the relationships and properties of twelve distinct types of neighborhoods derived from binary relations. We define initial-minimal and initial-maximal neighborhoods and develop eight types of rough approximations ($\mathbb{I}_\jmath$-approximations) that generalize Pawlak's theory. These new approximations significantly improve upon previous methods, achieving accuracy rates of up to 100%. Furthermore, we implement Generalized Nano-topological frameworks in conjunction with our novel methodologies to address clinical applications, particularly focusing on advancing diagnostic strategies for Covid-19. By employing a universal binary relation, we clarify the effectiveness for our methodology per enhancing decision-making processes and pinpointing significant risk factors associated with Covid-19. Additionally, we introduce two algorithms for decision-making problems in information systems, emphasizing the broader applicability and significance of our approach across various fields.

1. Introduction

Literature review

The theory of rough sets offers a powerful framework for managing uncertainty and vagueness in data, alongside facilitating knowledge analysis and extraction. Initially proposed by Pawlak [1] in 1982, this theory has established itself as a cornerstone in addressing decision-making challenges. However, its reliance on equivalence relations has presented certain limitations, prompting researchers to develop generalizations and extensions to expand its applicability. Examples of these efforts include using general binary relations [2,3] and neighborhood-based rough sets [4–6].

In 1996, Yao [7] significantly broadened rough set theory by introducing binary relation-based neighborhoods into inductive set theory without imposing specific conditions on the relations. This approach led to the development of neighborhood systems defining

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distinct right-neighborhoods and left-neighborhoods, which align with after-sets and fore-sets [8]. Nevertheless, Yao noted that these neighborhood-based approximations do not fully satisfy Pawlak's axioms, indicating the need for additional conditions on relations to ensure compatibility with Pawlak's properties. Following Yao's foundation, several researchers have introduced extensions using various kinds of the binary relations, including a relation of (tolerance [9], similarity [10,11], quasi-order [12]), and general binary relations [13–15]. Numerous researchers [16–19] have proposed several generalizations of rough sets in various directions, such as those presented in [20,21].

Allam et al. [22,23] furthered this line of research by defining minimal-right and minimal-left neighborhoods, which are derived from right-neighborhoods (resp. left-neighborhoods), providing new methods within rough set theory. Building on the right and left neighborhood concepts, Abo Khadra et al. [24] introduced a topological perspective in 2007. They developed a novel method for generating topology directly from binary relations, eliminating the need for a base or sub base. This broadened rough set applications within a topological context, especially for non-specialists. This framework inspired El-Bably's Master's Thesis [25], where he introduced and analyzed near-open concepts in rough set theory, further enriching the topological tools available for rough set applications. This approach defines a topology $\mathcal{T}$ on a universe $\mathcal{U}$ as:

$$\mathcal{T} = \{\mathcal{M} \subseteq \mathcal{U} : n(x) \subseteq \mathcal{M},\ \forall x \in \mathcal{M}\}, \tag{1}$$

where $n(x)$ denotes the neighborhood for each element x in $\mathcal{U}$. This technique has been used in various papers to extend topological frameworks within the domain of rough set theory; see, for example, [26,27]. In 2014, Abd El-Monsef et al. [28] built upon the framework established by Abo Khadra et al. by proposing the notion of a $\jmath$-neighborhood space (denoted by $\jmath$-***NS***), thereby broadening the scope of neighborhood-based rough set theory. They introduced novel neighborhood categories constructed through the intersection and union of right and left neighborhoods, as initially conceptualized by Yao, alongside the minimal neighborhoods pioneered by Allam et al. This structure yielded eight distinct neighborhood types, enabling a versatile approach to generalizing Pawlak's model without restrictions on the relations. The concepts of intersection and union neighborhoods from [28] have been widely adapted, resulting in additional neighborhood types, such as generalized covering approximation spaces by ideal G_n-CAS [29], topologies generated via extended $\jmath$-neighborhoods [30], and graphs generated by $\jmath$-neighborhoods [31]. These advancements extend Pawlak's rough set model using different topologies derived from $\jmath$-***NS***.

The idea of 'initial neighborhoods,' derived from right neighborhoods, was introduced for the first time in 2021 by El-Sayed et al. [32] and formulated as:

$$n_i(x) = \{y \in \mathcal{U} : n_r(x) \subseteq n_r(y)\}. \tag{2}$$

Here $n_r(x)$ refers to the right neighborhood for x. Applied within rough set theory and Covid-19 studies, these initial neighborhoods expanded rough set capabilities within generalized nano-topology. Building on this, Al-Shami and Ciucci [33] introduced "subset neighborhoods," creating additional neighborhood types within $\jmath$-***NS***.

Motivation

The motivation behind this study stems from the limitations of traditional rough set theory and the need for advanced generalizations to enhance decision-making processes, particularly in complex applications like medical diagnostics. Several key factors drive this research:

- Addressing the limitations of traditional rough set theory by introducing generalizations based on El-Sayed et al. [32].
- Defining and analyzing eight new types of initial neighborhoods, exploring their properties through examples, counterexamples, and theorems.
- Introducing eight types of rough approximations that generalize existing rough set methods and their extensions [34–40].
- Enhancing decision-making processes in complex scenarios, particularly in medical applications such as Covid-19 [41], by introducing a topological reduction technique for identifying critical risk factors.
- Contributing new theoretical insights and practical applications, broadening the applicability of rough set methods in various fields.

Objective

This paper aims to extend the notion of initial neighborhoods by developing new generalizations using "minimal" and "maximal" neighborhoods. The specific objectives of this study include:

- Introducing eight new types of initial neighborhoods, classifying them into initial-minimal and initial-maximal types, and analyzing their relationships with existing neighborhood types.
- Defining eight types of rough approximations (termed initial $\jmath$-approximations for each $\jmath \in \mathcal{J}$).
- Examining the features of these approximations and their connections with existing methods through theoretical results and illustrative examples.
- Applying these new generalizations to medical diagnostics, specifically in developing a precise diagnostic framework for Covid-19.
- Expanding the theoretical and practical foundations of nano-topology [42] within the framework of generalized rough sets to establish novel analytical constructs.

Paper organization

The organization of this manuscript proceeds systematically as follows:

- **Section 2** presents the fundamental concepts used in this research.
- **Section 3** examines the theoretical underpinnings of rough set theory and their extensions.
- **Section 4** introduces new initial neighborhoods and rough approximations.
- **Section 5** conducts a rigorous comparative evaluation of the proposed methodology, benchmarking against existing approaches.
- **Section 6** explores the concept of Generalized Nano-Topology in relation to rough sets. Furthermore, it proposes two distinct medical applications along with two algorithms.
- **Section 7** presents a summary and future research directions.

2. $\jmath$-neighborhood space and different neighborhoods induced from a binary relation

This section is dedicated to discussing the fundamental concepts and key findings from previous studies that are essential for this paper. Additionally, we introduce three new neighborhoods derived from maximal neighborhoods [11], thoroughly examining their properties and relationships through proven results and counterexamples.

Definition 2.1. Consider a binary relation $\mathcal{R}$ defined over an arbitrary non-empty universe $\mathcal{U}$. For any member x from $\mathcal{U}$, the after set (also called the right neighborhood) and the fore set (also called the left neighborhood) are given by

$$x\mathcal{R} = \{y \in \mathcal{U} : x\mathcal{R}y\} \quad \text{and} \quad \mathcal{R}x = \{y \in \mathcal{U} : y\mathcal{R}x\},$$

respectively.

Definition 2.2. A binary relation $\mathcal{R}$ on $\mathcal{U}$ is named as:

1. **Inverse serial**: For each element x in $\mathcal{U}$, there exists a member y from $\mathcal{U}$ s.t. $y\mathcal{R}x$.
2. **Reflexive**: For each member y from $\mathcal{U}$, the relation gratifies $x\mathcal{R}x$.
3. **Symmetric**: For all $x, y \in \mathcal{U}$, if $x\mathcal{R}y$, then $y\mathcal{R}x$.
4. **Transitive**: For all $x, y, z \in \mathcal{U}$, if $x\mathcal{R}y$ and $y\mathcal{R}z$, then $x\mathcal{R}z$.
5. **Similarity**: $\mathcal{R}$ characterized by concurrent reflexivity and symmetry over the universe $\mathcal{U}$.
6. **Preorder**: $\mathcal{R}$ characterized by concurrent reflexivity and transitivity.
7. **Equivalence**: A relation that satisfies reflexivity, symmetry, and transitivity.

Note that: Let $\mathcal{R}$ be an equivalence relation defined on a non-empty universe $\mathcal{U}$. The partition induced by $\mathcal{R}$, denoted as $\mathcal{U}/\mathcal{R}$, is formally characterized by:

$$\mathcal{U}/\mathcal{R} = \left\{[x]_{\mathcal{R}} : [x]_{\mathcal{R}} = x\mathcal{R},\ \forall x \in \mathcal{U}\right\},$$

where $[x]_{\mathcal{R}}$ represents the equivalence class containing an arbitrary element $x \in \mathcal{U}$.

Definition 2.3. Let $\mathcal{R}$ be a binary relation defined on a non-empty universe $\mathcal{U}$. For an element $x \in \mathcal{U}$, the *fundamental $\jmath$-neighborhood* (where $\jmath \in \boldsymbol{J} = \{r, \ell, \wedge, \curlyvee, \langle r\rangle, \langle \ell\rangle, \langle\wedge\rangle, \langle\curlyvee\rangle\}$) are formally defined as:

- **$\jmath$-neighborhoods:**
  1. r-neighborhood [7,8]: $n_r(x) = x\mathcal{R}$.
  2. ℓ-neighborhood [7,8]: $n_\ell(x) = \mathcal{R}x$.
  3. $\wedge$-neighborhood [28]: $n_\wedge(x) = n_r(x) \cap n_\ell(x)$.
  4. $\curlyvee$-neighborhood [28]: $n_\curlyvee(x) = n_r(x) \cup n_\ell(x)$.
- **Minimal $\jmath$-neighborhoods:**
  1. $\langle r\rangle$-neighborhood [22,23]: $n_{\langle r\rangle}(x) = \cap\left\{n_r(y)\ : x \in n_r(y)\right\}$.
  2. $\langle \ell\rangle$-neighborhood [22,23]:$n_{\langle \ell\rangle}(x) = \cap\left\{n_\ell(y)\ : x \in n_\ell(y)\right\}$.
  3. $\langle\wedge\rangle$-neighborhood [28]: $n_{\langle\wedge\rangle}(x) = n_{\langle r\rangle}(x) \cap n_{\langle \ell\rangle}(x)$.
  4. $\langle\curlyvee\rangle$-neighborhood [28]: $n_{\langle\curlyvee\rangle}(x) = n_{\langle r\rangle}(x) \cup n_{\langle \ell\rangle}(x)$.

Definition 2.4. [28] Let $\mathcal{R}$ be a binary relation defined on a non-empty universe $\mathcal{U}$. For every selector $\jmath \in \{r, \ell, \wedge, \curlyvee, \langle r\rangle, \langle \ell\rangle, \langle\wedge\rangle, \langle\curlyvee\rangle\}$, we construct a set-valued function $\xi_\jmath : \mathcal{U} \longrightarrow \wp(\mathcal{U})$ that associates each element x of $\mathcal{U}$ with its corresponding $\jmath$-neighborhood in $\wp(\mathcal{U})$, the set of all subsets of $\mathcal{U}$. The structure $\left(\mathcal{U}, \mathcal{R}, \xi_\jmath\right)$ is termed a *$\jmath$-Neighborhood-Space* ($\jmath$-***NS***).

Now, we define new sorts of neighborhoods based on the above neighborhoods as follows:

Definition 2.5. Consider $(\mathcal{U}, \mathcal{R}, \xi_J)$ being a J-**NS**. For every x in $\mathcal{U}$, we constitute the following neighborhoods:

- **Maximal J-neighborhoods:**
  1. (r)-neighborhood [11]: $n_{(r)}(x) = \cup \{n_r(y) \ : x \in n_r(y)\}$.
  2. (ℓ)-neighborhood: $n_{(\ell)}(x) = \cup \{n_\ell(y) : x \in n_\ell(y)\}$.
  3. $(\wedge)$-neighborhood: $n_{(\wedge)}(x) = n_{(r)}(x) \cap n_{(\ell)}(x)$.
  4. $(\vee)$-neighborhood: $n_{(\vee)}(x) = n_{(r)}(x) \cup n_{(\ell)}(x)$.
- **Initial J-neighborhoods:**
  1. initial r-neighborhood [32]: $n^{\mathrm{i}}_r(x) = \{y \in \mathcal{U} \ : n_r(x) \subseteq n_r(y)\}$.
  2. initial ℓ-neighborhood [43]: $n^{\mathrm{i}}_\ell(x) = \{y \in \mathcal{U} \ : n_\ell(x) \subseteq n_\ell(y)\}$.
  3. initial $\wedge$-neighborhood [43]: $n^{\mathrm{i}}_\wedge(x) = n^{\mathrm{i}}_r(x) \cap n^{\mathrm{i}}_\ell(x)$.
  4. initial $\vee$-neighborhood [43]: $n^{\mathrm{i}}_\vee(x) = n^{\mathrm{i}}_r(x) \cup n^{\mathrm{i}}_\ell(x)$.

Remark 2.1. For each x in $\mathcal{U}$, it should be noted that:

1. $n_{\langle r\rangle}(x) = \begin{cases} \bigcap_{x\in n_r(y)} n_r(y), \ if \ \exists y \in \mathcal{U} \ where \ x \in n_r(y). \\ \varphi, \quad Otherwise. \end{cases}$
2. $n_{\langle \ell\rangle}(x) = \begin{cases} \bigcap_{x\in n_\ell(y)} n_\ell(y), \ if \ \exists y \in \mathcal{U} \ where \ x \in n_\ell(y). \\ \varphi, \quad Otherwise. \end{cases}$
3. $n_{(r)}(x) = \begin{cases} \bigcup_{x\in n_r(y)} n_r(y), \ if \ \exists y \in \mathcal{U} \ where \ x \in n_r(y). \\ \varphi, \quad Otherwise. \end{cases}$
4. $n_{(\ell)}(x) = \begin{cases} \bigcup_{x\in n_\ell(y)} n_\ell(y), \ if \ \exists y \in \mathcal{U} \ where \ x \in n_\ell(y). \\ \varphi, \quad Otherwise. \end{cases}$

The subsequent discussions illustrate the properties and relationships of the above neighborhoods with proved results and counter-examples.

Lemma 2.1. *Consider $(\mathcal{U}, \mathcal{R}, \xi_J)$ as a J-**NS**. Hence, for any $x \in \mathcal{U}$ and $J \in \{r, \ell, \wedge, \vee\}$, the inclusion $n_{\langle J\rangle}(x) \subseteq n_{(J)}(x)$ holds.*

Proof. Obvious. □

Lemma 2.2. *Presume that $(\mathcal{U}, \mathcal{R}, \xi_J)$ represents a J-**NS**. For every $x \in \mathcal{U}$ and $J \in \{r, \ell, \wedge, \vee\}$, the following statements hold:*

1. *$x \in n_J(x)$ if $\mathcal{R}$ is reflexive.*
2. *$x \in n_{\langle J\rangle}(x)$ if $\mathcal{R}$ is inverse serial.*
3. *$x \in n_{(J)}(x)$ if $\mathcal{R}$ is inverse serial.*

Proof. By applying Remark 2.1 and the results established in [44], the outcome follows immediately. □

According to Allam et al. in [22,23], the following lemma is proved for the issues $\langle r\rangle$ or $\langle \ell\rangle$. Thus, the proof is deleted.

Lemma 2.3. *Consider a J-**NS** denoted by $(\mathcal{U}, \mathcal{R}, \xi_J)$. For any $x \in n_J(y)$, it follows that $n_J(x) \subseteq n_J(y)$ is valid for every $J \in \{\langle r\rangle, \langle \ell\rangle, \langle \wedge\rangle\}$.*

Remark 2.2. Lemma 2.3 need not be true for $J \in \{r, \ell, \wedge, \vee, \langle \vee\rangle, (r), (\ell), (\wedge), (\vee)\}$, as illustrated in Example 2.1 and Tables 1, 2, 3.

Theorem 2.1. *Let $(\mathcal{U}, \mathcal{R}, \xi_J)$ denote a J-**NS**, where $\mathcal{R}$ is a reflexive relation defined on $\mathcal{U}$. For every $J \in \{r, \ell, \wedge, \vee\}$, the subsequent characteristics are satisfied:*

1. $n_{\langle J\rangle}(x) \subseteq n_J(x) \subseteq n_{(J)}(x)$.
2. $n^{\mathrm{i}}_J(x) \subseteq n_{(J)}(x)$.

Proof. We provide the proof for the issue $J = r$, noting that the argument extends analogously to the remaining cases.

1. Assume $y \in n_{\langle r\rangle}(x)$. By definition, y is contained in every r-neighborhood that includes x. Given that $\mathcal{R}$ is reflexive, it follows that $x \in n_r(x)$, which implies $y \in n_r(x)$. Thus, we deduce that $n_{\langle r\rangle}(x) \subseteq n_r(x)$.
   Now, let $z \in n_r(x)$. Since $\mathcal{R}$ is reflexive, we again have $x \in n_r(x)$. By definition, this ensures that $z \in \bigcup_{x\in n_r(y)} n_r(y) = n_{(r)}(x)$. This confirms the inclusion $n_r(x) \subseteq n_{(r)}(x)$.

Table 1
$\jmath$-neighborhoods.

$\star$	$n_r(\star)$	$n_\ell(\star)$	$n_\wedge(\star)$	$n_\vee(\star)$
ġ	{ġ,ḣ}	{ġ}	{ġ}	{ġ,ḣ}
ḣ	{ḣ,k̇}	{ġ,ḣ}	{ḣ}	{ġ,ḣ,k̇}
k̇	{k̇}	{ḣ,k̇,ṡ}	{k̇}	{ḣ,k̇,ṡ}
ṡ	{k̇,ṡ}	{ṡ}	{ṡ}	{k̇,ṡ}

Table 2
Minimal $\jmath$-neighborhoods.

$\star$	$n_{\langle r\rangle}(\star)$	$n_{\langle \ell\rangle}(\star)$	$n_{\langle \wedge\rangle}(\star)$	$n_{\langle \vee\rangle}(\star)$
ġ	{ġ,ḣ}	{ġ}	{ġ}	{ġ,ḣ}
ḣ	{ḣ}	{ḣ}	{ḣ}	{ḣ}
k̇	{k̇}	{ḣ,k̇,ṡ}	{k̇}	{ḣ,k̇,ṡ}
ṡ	{k̇,ṡ}	{ṡ}	{ṡ}	{k̇,ṡ}

Table 3
Maximal $\jmath$-neighborhoods.

$\star$	$n_{(r)}(\star)$	$n_{(\ell)}(\star)$	$n_{(\wedge)}(\star)$	$n_{(\vee)}(\star)$
ġ	{ġ,ḣ}	{ġ,ḣ}	{ġ,ḣ}	{ġ,ḣ}
ḣ	{ġ,ḣ,k̇}	$\mathcal{U}$	{ġ,ḣ,k̇}	$\mathcal{U}$
k̇	{ḣ,k̇,ṡ}	{ḣ,k̇,ṡ}	{ḣ,k̇,ṡ}	{ḣ,k̇,ṡ}
ṡ	{k̇,ṡ}	{ḣ,k̇,ṡ}	{k̇,ṡ}	{ḣ,k̇,ṡ}

Table 4
Initial $\jmath$-neighborhoods.

$\star$	$n^{\mathrm{i}}_r(\star)$	$n^{\mathrm{i}}_\ell(\star)$	$n^{\mathrm{i}}_\wedge(\star)$	$n^{\mathrm{i}}_\vee(\star)$
ġ	{ġ}	{ġ,ḣ}	{ġ}	{ġ,ḣ}
ḣ	{ḣ}	{ḣ}	{ḣ}	{ḣ}
k̇	{ḣ,k̇,ṡ}	{k̇}	{k̇}	{ḣ,k̇,ṡ}
ṡ	{ṡ}	{k̇,ṡ}	{ṡ}	{k̇,ṡ}

2. Suppose $y \in n^{\mathrm{i}}_r(x)$. By definition, this means $n_r(x) \subseteq n_r(y)$. Due to the reflexivity of $\mathcal{R}$, we have $x \in n_r(x)$, and consequently, $x \in n_r(y)$. It follows that $n_r(y) \subseteq \bigcup_{x\in n_r(z)} n_r(z) = n_{(r)}(x)$. Therefore, by the reflexivity of $\mathcal{R}$, we conclude that $y \in n_{(r)}(x)$, which implies $n^{\mathrm{i}}_r(x) \subseteq n_{(r)}(x)$. □

Remark 2.3. For each x in $\mathcal{U}$, it should be noted that the opposite of Theorem 2.1. (item (2)) does not generally hold, as demonstrated by Examples 2.1 and 2.2.

Example 2.1. Assume that $\mathcal{U}$ = {ġ,ḣ,k̇,ṡ} with the reflexive relation $\mathcal{R}$ = {(ġ,ġ), (ġ,ḣ), (ḣ,ḣ), (ḣ,k̇), (k̇,k̇), (ṡ,k̇), (ṡ,ṡ)}. The corresponding $\jmath$-neighborhoods, minimal $\jmath$-neighborhoods, maximal $\jmath$-neighborhoods, and initial $\jmath$-neighborhoods are detailed in Tables 1, 2, 3, and 4, respectively. In these tables, the symbol $\star$ represents an arbitrary element in $\mathcal{U}$.

Example 2.2. Given that $\mathcal{U}$ = {ġ, ḣ,k̇,ṡ} with the reflexive relation $\mathcal{R}$ = {(ġ,ġ), (ġ,ḣ), (ġ,k̇), (ḣ,ġ), (ḣ,ḣ), (k̇,k̇), (k̇,ṡ), (ṡ,k̇), (ṡ,ṡ)}. The resulting neighborhoods are as follows:

1. **∧-neighborhoods**:
   - $n_\wedge$(ġ) = $n_\wedge$(ḣ) = {ġ, ḣ},
   - $n_\wedge$(k̇) = $n_\wedge$(ṡ) = {k̇, ṡ}.
2. **⟨∧⟩-neighborhoods**:
   - $n_{\langle\wedge\rangle}$(ġ) = {ġ},
   - $n_{\langle\wedge\rangle}$(ḣ) = {ġ, ḣ},
   - $n_{\langle\wedge\rangle}$(k̇) = {k̇},
   - $n_{\langle\wedge\rangle}$(ṡ) = {k̇, ṡ}.
3. **(∧)-neighborhoods**:
   - $n_{(\wedge)}$(ġ) = {ġ, ḣ, k̇},
   - $n_{(\wedge)}$(ḣ) = {ġ, ḣ},

Table 5
Comparison of different neighborhoods.

$\star$	$n_r(\star)$	$n_{\langle r\rangle}(\star)$	$n_r^{\mathrm{i}}(\star)$
$\dot{\mathrm{g}}$	$\{\dot{\mathrm{g}}\}$	$\{\dot{\mathrm{g}}\}$	$\{\dot{\mathrm{g}}\}$
$\dot{\mathrm{h}}$	φ	φ	$\mathcal{U}$
$\dot{\mathrm{k}}$	$\{\dot{\mathrm{s}}\}$	$\{\dot{\mathrm{k}}\}$	$\{\dot{\mathrm{k}}\}$
$\dot{\mathrm{s}}$	$\{\dot{\mathrm{k}}\}$	$\{\dot{\mathrm{s}}\}$	$\{\dot{\mathrm{s}}\}$

- $n_{(\wedge)}(\dot{\mathrm{k}}) = \{\dot{\mathrm{g}}, \dot{\mathrm{k}}, \dot{\mathrm{s}}\}$,
- $n_{(\wedge)}(\dot{\mathrm{s}}) = \{\dot{\mathrm{k}}, \dot{\mathrm{s}}\}$.

4. **Initial $\wedge$-neighborhoods**:
   - $n^{\mathrm{i}}_{\wedge}(\dot{\mathrm{g}}) = \{\dot{\mathrm{g}}\}$,
   - $n^{\mathrm{i}}_{\wedge}(\dot{\mathrm{h}}) = \{\dot{\mathrm{g}}, \dot{\mathrm{h}}\}$,
   - $n^{\mathrm{i}}_{\wedge}(\dot{\mathrm{k}}) = \{\dot{\mathrm{k}}\}$,
   - $n^{\mathrm{i}}_{\wedge}(\dot{\mathrm{s}}) = \{\dot{\mathrm{k}}, \dot{\mathrm{s}}\}$.

Theorem 2.2. *Let $(\mathcal{U}, \mathcal{R}, \xi_J)$ represent a J-**NS**, and $\mathcal{R}$ is a symmetric relation defined on $\mathcal{U}$. Then, for every $x \in \mathcal{U}$, the upcoming characteristics are satisfied:*

1. $y \in n_r(x) \Leftrightarrow x \in n_r(y)$.
2. $n_r(x) = n_\ell(x) = n_\wedge(x) = n_\vee(x)$.
3. $n_{\langle r\rangle}(x) = n_{\langle \ell\rangle}(x) = n_{\langle \wedge\rangle}(x) = n_{\langle \vee\rangle}(x)$.
4. $n_{(r)}(x) = n_{(\ell)}(x) = n_{(\wedge)}(x) = n_{(\vee)}(x)$.
5. $n^{\mathrm{i}}_r(x) = n^{\mathrm{i}}_\ell(x) = n^{\mathrm{i}}_\wedge(x) = n^{\mathrm{i}}_\vee(x)$.

Proof. 1. Given $y \in n_r(x)$, so $x\mathcal{R}y$. By symmetry of $\mathcal{R}$, $y\mathcal{R}x$. Thus, $x \in n_r(y)$. By a similar way, the reverse implication.
2. The proof of items 3, 4, and 5 essentially depends on item 2. Therefore, we will prove the second item as follows:
Presume that $\mathcal{R}$ is a relation that is symmetric established on $\mathcal{U}$. Then, $x\mathcal{R}y$ iff $y\mathcal{R}x$. Consequently, $y \in n_r(x)$ iff $y \in n_\ell(x)$, which implies $n_r(x) = n_\ell(x)$.
Therefore, $n_\wedge(x) = n_r(x) \cap n_\ell(x) = n_r(x) = n_\ell(x)$. Similarly, $n_\vee(x) = n_r(x) = n_\ell(x)$. □

Theorem 2.3. *Suppose that $(\mathcal{U}, \mathcal{R}, \xi_J)$ constitutes a J-**NS** and that the relation $\mathcal{R}$ on $\mathcal{U}$ is symmetric. Then, for every $x \in \mathcal{U}$ and for each $J \in \{r, \ell, \wedge, \vee\}$, we have*

$$n_{\langle J\rangle}(x) \subseteq n^{\mathrm{i}}_J(x).$$

Proof. We will provide a proof of this theorem when the issue $J = r$, while the other issues can be handled in a similar manner as follows:
Let $z \in n_{\langle r\rangle}(x)$. By definition, z is an element of every $n_r(w)$ containing x. Hence, for each $w \in \mathcal{U}$, we have

$$x \in n_r(w) \Leftrightarrow z \in n_r(w). \tag{3}$$

Next, we show that $z \in n^{\mathrm{i}}_r(x)$, which means proving the inclusion $n_r(x) \subseteq n_r(z)$.
In view of $\mathcal{R}$ represents a symmetric relation on $\mathcal{U}$, so for each $t \in \mathcal{U}$, the condition $t \in n_r(u)$ is met if and only if $u \in n_r(t)$. Accordingly, if $y \in n_r(x)$, then we also have $x \in n_r(y)$. Using equation (3), we obtain $z \in n_r(y)$. Consequently, $y \in n_r(z)$, which implies $n_r(x) \subseteq n_r(z)$. Therefore, we conclude that $z \in n^{\mathrm{i}}_r(x)$. □

The next example shows that the inclusion sign of Theorem 2.3 cannot be replaced by an equal sign in general.

Example 2.3. Assume that the relation $\mathcal{R} = \{(\dot{\mathrm{g}},\dot{\mathrm{g}}), (\dot{\mathrm{k}},\dot{\mathrm{s}}), (\dot{\mathrm{s}},\dot{\mathrm{k}})\}$ is symmetric on $\mathcal{U} = \{\dot{\mathrm{g}},\dot{\mathrm{h}},\dot{\mathrm{k}},\dot{\mathrm{s}}\}$. When $J = r$, Table 5 clearly indicates that

$$n^{\mathrm{i}}_r(x) \nsubseteq n_{\langle r\rangle}(x).$$

Remark 2.4. Assume that $(\mathcal{U}, \mathcal{R}, \xi_J)$ is a J-***NS*** in which the relation $\mathcal{R}$ is symmetric. For each $x \in \mathcal{U}$ and $J \in \{r, \ell, \wedge, \vee\}$, the subsequent observations hold:

1. The neighborhoods $n_J(x)$ and $n_{\langle J\rangle}(x)$ are generally not comparable.
2. The neighborhoods $n_J(x)$ and $n^{\mathrm{i}}_J(x)$ are generally not comparable.

Table 6
Comparison of different neighborhoods.

$\star$	$n_r(\star)$	$n_{\langle r\rangle}(\star)$	$n_{(r)}(\star)$	$n_r^{\mathrm{i}}(\star)$
ġ	{ġ,ḣ,ṡ}	{ġ}	$\mathcal{U}$	{ġ}
ḣ	{ġ}	{ġ,ḣ,ṡ}	{ġ,ḣ,ṡ}	{ġ,ḣ,ṡ}
k̇	{ṡ}	{ġ,k̇}	{ġ,k̇}	{ġ,k̇}
ṡ	{ġ,k̇}	{ṡ}	{ġ,ḣ,ṡ}	{ṡ}

Table 7
Comparison of different neighborhoods.

$\star$	$n_r(\star)$	$n_{\langle r\rangle}(\star)$	$n_r^{\mathrm{i}}(\star)$
ġ	$\mathcal{U}$	$\mathcal{U}$	{ġ,ḣ}
ḣ	$\mathcal{U}$	$\mathcal{U}$	{ġ,ḣ}
k̇	{ṡ}	$\mathcal{U}$	$\mathcal{U}$
ṡ	{ṡ}	{ṡ}	$\mathcal{U}$

3. The neighborhoods $n_J(x)$, $n_{(J)}(x)$, and $n_J^{\mathrm{i}}(x)$ are generally not comparable.

Example 2.4 illustrates this remark.

Example 2.4. Let $\mathcal{R}$ be the relation {(ġ,ġ), (ġ,ḣ), (ġ,ṡ), (ḣ,ġ), (k̇,ṡ), (ṡ,ġ), (ṡ,k̇)}, which is symmetric on $\mathcal{U}$ = {ġ,ḣ,k̇,ṡ}. To illustrate Remark 2.4, we present the case $J = r$ in Table 6, with the other cases following similarly.

Theorem 2.4. *Let $(\mathcal{U}, \mathcal{R}, \xi_J)$ represent a J-**NS**, where $\mathcal{R}$ constitutes a transitive relation on $\mathcal{U}$. For all $x \in \mathcal{U}$ and $J \in \{r, \ell, \wedge, \curlyvee\}$, if $x \in n_J(y)$, then $n_J(x) \subseteq n_J(y)$.*

Proof. We will offer a proof of this theorem when the issue $J = r$, while the remaining issues are derived in a similar manner. First, suppose that $x \in n_r(y)$. By definition, we have:

$$y\mathcal{R}x. \tag{4}$$

Next, we establish that $n_r(x) \subseteq n_r(y)$. Given $z \in n_r(x)$. Then, by definition:

$$x\mathcal{R}z. \tag{5}$$

By the transitivity of $\mathcal{R}$ and using (4) and (5), it follows that $y\mathcal{R}z$, which implies $z \in n_r(y)$. Accordingly, we conclude that:

$$n_r(x) \subseteq n_r(y). \quad \square$$

Remark 2.5. Given $(\mathcal{U}, \mathcal{R}, \xi_J)$ is a J-***NS***, where $\mathcal{R}$ is a transitive relation on $\mathcal{U}$, for each $x \in \mathcal{U}$ and $J \in \{r, \ell, \wedge, \curlyvee\}$, it should be noted that:

1. The neighborhoods $n_J(x)$, $n_{\langle J\rangle}(x)$, and $n_J^{\mathrm{i}}(x)$ are not comparable in general.
2. The neighborhoods $n_J(x)$, $n_{(J)}(x)$, and $n_J^{\mathrm{i}}(x)$ are not comparable in general.

Examples 2.5 and 2.6 exemplify the Remark 2.5, further demonstrating that the inclusion sign of Theorem 2.4 cannot be replaced by an equal sign in general.

Example 2.5. Given $\mathcal{R}$ is the relation {(ġ,ġ), (ġ,ḣ), (ḣ,ḣ), (ġ,k̇), (ġ,ṡ), (ḣ,ġ), (ḣ,k̇), (ḣ,ṡ), (k̇,ṡ), (ṡ,ṡ)}, which is transitive on $\mathcal{U}$ = {ġ,ḣ,k̇,ṡ}. To illustrate item (1) of Remark 2.5, we present the case $J = r$ in Table 7, with the other cases following similarly.

Example 2.6. Suppose that $\mathcal{R}$ is the relation {(ġ,ḣ), (ġ,k̇), (ḣ,ḣ), (ḣ,k̇), (ṡ,ṡ)} defined on $\mathcal{U}$ = {ġ,ḣ,k̇,ṡ)}. Note that $\mathcal{R}$ is transitive. To establish the validity of item (2) in Remark 2.5, we provide a detailed proof for the case where $J = r$, as exemplified in Table 8. The proofs for the remaining cases follow a similar pattern.

As the proof of this lemma is provided in [32], we do not include it here to avoid redundancy.

Lemma 2.4. *[32] Assume that $(\mathcal{U}, \mathcal{R}, \xi_J)$ is a J-**NS** with $\mathcal{R}$ being a similarity relation on $\mathcal{U}$. Hence, for every $x \in \mathcal{U}$, the subsequent characteristics are satisfied:*

Table 8
Comparison of different neighborhoods.

$\star$	$n_r(\star)$	$n_{\langle r\rangle}(\star)$	$n_r^{\mathrm{i}}(\star)$
$\dot{g}$	$\{\dot{h},\dot{k}\}$	φ	$\{\dot{g},\dot{h}\}$
$\dot{h}$	$\{\dot{h},\dot{k}\}$	$\{\dot{h},\dot{k}\}$	$\{\dot{g},\dot{h}\}$
$\dot{k}$	φ	$\{\dot{h},\dot{k}\}$	$\mathcal{U}$
$\dot{s}$	$\{\dot{s}\}$	$\{\dot{s}\}$	$\{\dot{s}\}$

Table 9
Comparison of different neighborhoods.

$\star$	$n_r(\star)$	$n_{\langle r\rangle}(\star)$	$n_r^{\mathrm{i}}(\star)$
$\dot{g}$	$\{\dot{g},\dot{h}\}$	$\{\dot{g},\dot{h}\}$	$\{\dot{g},\dot{h}\}$
$\dot{h}$	$\{\dot{g},\dot{h},\dot{k}\}$	$\{\dot{h}\}$	$\{\dot{h}\}$
$\dot{k}$	$\{\dot{h},\dot{k},\dot{s}\}$	$\{\dot{k}\}$	$\{\dot{k}\}$
$\dot{s}$	$\{\dot{k},\dot{s}\}$	$\{\dot{k},\dot{s}\}$	$\{\dot{k},\dot{s}\}$

1. $n_{\langle r\rangle}(x) = n_r^{\mathrm{i}}(x)$.
2. $n_r^{\mathrm{i}}(x) \subseteq n_r(x)$.

Theorem 2.5. *Assume that $(\mathcal{U}, \mathcal{R}, \xi_J)$ constitutes a J-**NS** with $\mathcal{R}$ being a similarity relation on $\mathcal{U}$. Then, for every $x \in \mathcal{U}$ and for each $J \in \{r, \ell, \wedge, \curlyvee\}$, the upcoming properties are satisfied:*

1. $n_{\langle J\rangle}(x) = n_J^{\mathrm{i}}(x)$.
2. $n_J^{\mathrm{i}}(x) \subseteq n_J(x)$.

Proof. The proof follows a method analogous to that used in Lemma 2.4 □

Remark 2.6. It is important to note that the converse of statement (2) in Theorem 2.5 does not hold in general, as illustrated by Example 2.7.

Example 2.7. Assume that $\mathcal{R} = \{(\dot{g},\dot{g}), (\dot{g},\dot{h}), (\dot{h},\dot{g}), (\dot{h},\dot{h}), (\dot{h},\dot{k}), (\dot{k},\dot{h}), (\dot{k},\dot{k}), (\dot{k},\dot{s}), (\dot{s},\dot{k}), (\dot{s},\dot{s})\}$. It is demonstrable that $\mathcal{R}$ is a similarity relation on $\mathcal{U} = \{\dot{g},\dot{h},\dot{k},\dot{s}\}$. We now verify Remark 2.6 for $J = r$, as shown in Table 9. The proofs for the other cases proceed similarly.

Theorem 2.6. *Given $(\mathcal{U}, \mathcal{R}, \xi_J)$ is a J-**NS**, with $\mathcal{R}$ constitutes a preorder relation defined on $\mathcal{U}$. Thus, for every element $x \in \mathcal{U}$ and for all $J \in \{r, \ell, \wedge, \curlyvee\}$, the following equality holds:*

$$n_{\langle J\rangle}(x) = n_J(x).$$

Proof. We will demonstrate the theorem for a case $J = r$, as the proof for the remaining cases follows analogously.

Necessity condition:
First, by using Theorem 2.1, we have

$$n_{\langle r\rangle}(x) \subseteq n_r(x)$$

since $\mathcal{R}$ is reflexive.

Sufficiency condition:
Now, let

$$z \in n_r(x). \tag{6}$$

Thus, we have

$$x\mathcal{R}z.$$

It is required to demonstrate that $z \in n_{\langle r\rangle}(x)$ (i.e., z lies in every right neighborhood that contains x).

Presume that there exists

$$w \in \mathcal{U} \quad \text{in such a manner that} \quad x \in n_r(w), \tag{7}$$

which implies that

$$w\mathcal{R}x.$$

Since $\mathcal{R}$ is transitive, from (6) and (7), we obtain

$$w\mathcal{R}z,$$

which implies that

$$z \in n_r(w) \quad \text{such that} \quad x \in n_r(w).$$

Consequently, we conclude that

$$z \in n_{\langle r\rangle}(x). \quad \square$$

As a direct consequence of Theorems 2.5 and 2.6, we obtain the subsequent result, and since the proof is straightforward, it is omitted for brevity.

Corollary 2.1. *If $(\mathcal{U}, \mathcal{R}, \xi_j)$ is a j-**NS**, where $\mathcal{R}$ is an equivalence relation on $\mathcal{U}$, then for each $x \in \mathcal{U}$ and $j \in \{r, \ell, \wedge, \vee\}$: $n_j(x) = n_{\langle j\rangle}(x) = n_{(j)}(x) = n_j^{\mathrm{i}}(x) = [x]_{\mathcal{R}}$, where $[x]_{\mathcal{R}}$ is an equivalence class.*

Proof. We establish the result for $j = r$; the remaining cases for j can be demonstrated analogously.

Given that $\mathcal{R}$ is an equivalence relation, it follows from Pawlak [1] that the collection $\{n_r(x) : x \in \mathcal{U}\}$ forms a partition of $\mathcal{U}$ and coincides with $\{[x]_{\mathcal{R}} : x \in \mathcal{U}\}$. As a result, the sets $\{n_{\langle r\rangle}(x) : x \in \mathcal{U}\}$, $\{n_{(r)}(x) : x \in \mathcal{U}\}$, and $\{n_r^{\mathrm{i}}(x) : x \in \mathcal{U}\}$ also constitute partitions of $\mathcal{U}$ and are equivalent to $\{n_r(x) : x \in \mathcal{U}\}$. Therefore, we deduce that:

$$n_r(x) = n_{\langle r\rangle}(x) = n_{(r)}(x) = n_r^{\mathrm{i}}(x) = [x]_{\mathcal{R}}. \quad \square$$

3. Rough set theory: Pawlak's framework and its generalizations

In this section, we outline key concepts from previous studies on rough sets [1], particular emphasis on the contributions of Yao [7], Allam et al. [22,23], Abd El-Monsef et al. [28], Dai et al. [11], and Abu-Gdairi [43]. Additionally, we present new findings and explore relationships, using examples to illustrate that the approximation approaches of Abd El-Monsef et al. and Yao (as well as those of Allam et al.) are equivalent in certain special cases of the binary relation.

3.1. Pawlak's rough sets approach (1982)

Definition 3.1. [1] Given a finite set $\mathcal{U}$, referred to as the "universe," with an equivalence relation $\mathcal{R}$ defined on it. In the framework of Pawlak approximation spaces, the pair $(\mathcal{U}, \mathcal{R})$ is introduced. Pawlak made a significant contribution through the introduction of the notions of lower and upper approximations, for any subset $\mathbb{A} \subseteq \mathcal{U}$ by:

1. **Lower Approximation ($\underline{Apr}(\mathbb{A})$):** $\underline{Apr}(\mathbb{A})$ comprises of all elements $x \in \mathcal{U}$ for which the equivalence class $[x]_{\mathcal{R}}$ is entirely contained within $\mathbb{A}$.
2. **Upper Approximation $\overline{Apr}(\mathbb{A})$):** $\overline{Apr}(\mathbb{A})$ includes all elements $x \in \mathcal{U}$ for which the intersection of the equivalence class $[x]_{\mathcal{R}}$ with $\mathbb{A}$ is non-empty.
3. **Boundary ($BND_{\mathcal{R}}(\mathbb{A})$):** The boundary of $\mathbb{A}$, symbolized as $BND_{\mathcal{R}}(\mathbb{A})$, is identified as the set of elements in the upper approximation $\overline{Apr}(\mathbb{A})$ that are not in the lower approximation $\underline{Apr}(\mathbb{A})$.
4. **Positive Region ($POS_{\mathcal{R}}(\mathbb{A})$):** The positive region of $\mathbb{A}$, symbolized as $POS_{\mathcal{R}}(\mathbb{A})$, consists of all elements in the lower approximation $\underline{Apr}(\mathbb{A})$.
5. **Negative Region ($NEG_{\mathcal{R}}(\mathbb{A})$):** The negative region of $\mathbb{A}$, symbolized as $NEG_{\mathcal{R}}(\mathbb{A})$, includes all elements in the universe $\mathcal{U}$ that are not in the upper approximation $\overline{Apr}(\mathbb{A})$.
6. **Accuracy ($\mu_{\mathcal{R}}(\mathbb{A})$):** The accuracy of the approximation of $\mathbb{A}$, symbolized as $\mu_{\mathcal{R}}(\mathbb{A})$, is defined as the cardinality of $\underline{Apr}(\mathbb{A})$ divided by the cardinality of $\overline{Apr}(\mathbb{A})$, provided $\overline{Apr}(\mathbb{A})$ is non-empty.

Remark 3.1.

1. It's important to note that, in accordance with Pawlak's definition, a set $\mathbb{A}$ is referred to as an 'exact set' if and only if $\underline{Apr}(\mathbb{A})$ is equal to $\overline{Apr}(\mathbb{A})$, which implies that the boundary $BND_{\mathcal{R}}(\mathbb{A})$ is an empty set, and the accuracy $\mu_{\mathcal{R}}(\mathbb{A})$ equals 1. Conversely, if $\underline{Apr}(\mathbb{A})$ is not equal to $\overline{Apr}(\mathbb{A})$, then $\mathbb{A}$ is termed a 'rough set'.

2. In alignment with the principles of Pawlak's theory, as outlined in [1], the empty set φ is classified as a definable (or exact) set if its lower and upper approximations are identical, both being φ. Consequently, $\boldsymbol{\mu}_{\mathcal{R}}(\varphi) = 1$. Conversely, if this condition is not satisfied, φ is regarded as an undefinable (rough) set, and its accuracy measure differs from 1. Furthermore, in the definition of the accuracy measure, a condition is imposed to ensure that the upper approximation is not equal to φ, thereby preventing division by zero. If, however $\overline{Apr}(\varphi)$ is φ, the accuracy measure is considered an indefinite quantity.

Proposition 3.1. *[1] Assume that $\mathcal{R}$ is an equivalence relation on $\mathcal{U}$, and $\mathbb{A}^c$ represent the complement of $\mathbb{A}$ in $\mathcal{U}$. For all subsets $\mathbb{A}, \mathbb{B} \subseteq \mathcal{U}$, the following features are satisfied:*

(L1) $\underline{Apr}(\mathbb{A}) \subseteq \mathbb{A}$.
(L2) $\underline{Apr}(\varphi) = \varphi$.
(L3) $\underline{Apr}(\mathcal{U}) = \mathcal{U}$.
(L4) If $\mathbb{A} \subseteq \mathbb{B}$, *then* $\underline{Apr}(\mathbb{A}) \subseteq \underline{Apr}(\mathbb{B})$.
(L5) $\underline{Apr}\left(\mathbb{A} \bigcap \mathbb{B}\right) = \underline{Apr}(\mathbb{A}) \bigcap \underline{Apr}(\mathbb{B})$.
(L6) $\underline{Apr}(\mathbb{A}) \bigcup \underline{Apr}(\mathbb{B}) \subseteq \underline{Apr}\left(\mathbb{A} \bigcup \mathbb{B}\right)$.
(L7) $\underline{Apr}(\mathbb{A}^c) = \left(\overline{Apr}(\mathbb{A})\right)^c$.
(L8) $\underline{Apr}\left(\underline{Apr}(\mathbb{A})\right) = \underline{Apr}(\mathbb{A})$.
(L9) $\underline{Apr}\left(\overline{Apr}(\mathbb{A})\right) = \overline{Apr}(\mathbb{A})$.
(L10) If $\mathbb{K} \in \mathcal{U}/\mathcal{R}$, *then* $\underline{Apr}(\mathbb{K}) = \mathbb{K}$.

(U1) $\mathbb{A} \subseteq \overline{Apr}(\mathbb{A})$.
(U2) $\overline{Apr}(\varphi) = \varphi$.
(U3) $\overline{Apr}(\mathcal{U}) = \mathcal{U}$.
(U4) If $\mathbb{A} \subseteq \mathbb{B}$, *then* $\overline{Apr}(\mathbb{A}) \subseteq \overline{Apr}(\mathbb{B})$.
(U5) $\overline{Apr}\left(\mathbb{A} \bigcup \mathbb{B}\right) = \overline{Apr}(\mathbb{A}) \bigcup \overline{Apr}(\mathbb{B})$.
(U6) $\overline{Apr}(\mathbb{A}) \bigcap \overline{Apr}(\mathbb{B}) \supseteq \overline{Apr}\left(\mathbb{A} \bigcap \mathbb{B}\right)$.
(U7) $\overline{Apr}(\mathbb{A}^c) = \left(\underline{Apr}(\mathbb{A})\right)^c$.
(U8) $\overline{Apr}\left(\overline{Apr}(\mathbb{A})\right) = \overline{Apr}(\mathbb{A})$.
(U9) $\overline{Apr}\left(\underline{Apr}(\mathbb{A})\right) = \underline{Apr}(\mathbb{A})$.
(U10) If $\mathbb{K} \in \mathcal{U}/\mathcal{R}$, *then* $\overline{Apr}(\mathbb{K}) = \mathbb{K}$.

3.2. Yao approaches (1996)

Definition 3.2. [7] Let $\mathcal{R}$ be a binary relation on $\mathcal{U}$. For any subset $\mathbb{X} \subseteq \mathcal{U}$, the Yao-lower and Yao-upper approximations of $\mathbb{X}$ are given as follows:

$$\underline{\mathcal{R}}(\mathbb{X}) = \{\mathrm{s} \in \mathcal{U} : \mathrm{s}\mathcal{R} \subseteq \mathbb{X}\},$$
$$\overline{\mathcal{R}}(\mathbb{X}) = \{\mathrm{s} \in \mathcal{U} : \mathrm{s}\mathcal{R} \cap \mathbb{X} \neq \emptyset\}.$$

Here, $\mathrm{s}\mathcal{R}$ denotes the right neighborhood (after set) of s.

Additionally, Yao-boundary area and Yao-accuracy degree are presented as follows:

$$\mathfrak{B}(\mathbb{X}) = \overline{\mathcal{R}}(\mathbb{X}) - \underline{\mathcal{R}}(\mathbb{X}),$$
$$\mu(\mathbb{X}) = \frac{|\underline{\mathcal{R}}(\mathbb{X})|}{|\overline{\mathcal{R}}(\mathbb{X})|}, \quad \text{with} |\overline{\mathcal{R}}(\mathbb{X})| \neq 0.$$

It is evident that if $\mu(\mathbb{X}) = 1$, then $\mathbb{X}$ is classified as Yao-exact; otherwise, $\mathbb{X}$ is classified as Yao-rough.

3.3. Allam et al. Approaches (2005)

Definition 3.3. [22,23] Presume that $\mathcal{R}$ constitutes a binary relation on $\mathcal{U}$. For any subset $\mathbb{X} \subseteq \mathcal{U}$, the Allam-lower and Allam-upper approximations of $\mathbb{X}$ are constructed as follows:

$$\underline{\mathcal{A}}_{\langle r\rangle}(\mathbb{X}) = \{\mathrm{s} \in \mathcal{U} : n_{\langle r\rangle}(\mathrm{s}) \subseteq \mathbb{X}\},$$
$$\overline{\mathcal{A}}_{\langle r\rangle}(\mathbb{X}) = \{\mathrm{s} \in \mathcal{U} : n_{\langle r\rangle}(\mathrm{s}) \cap \mathbb{X} \neq \emptyset\}.$$

Here, $\mathrm{s}\mathcal{R}$ represents the right neighborhood (after set) of s.

Furthermore, the Allam-boundary area and Allam-accuracy degree are defined assumed as:

$$\mathfrak{B}_{\mathcal{A}_{\langle r\rangle}}(\mathbb{X}) = \overline{\mathcal{A}}_{\langle r\rangle}(\mathbb{X}) - \underline{\mathcal{A}}_{\langle r\rangle}(\mathbb{X}),$$
$$\mu_{\mathcal{A}_{\langle r\rangle}}(\mathbb{X}) = \frac{|\underline{\mathcal{A}}_{\langle r\rangle}(\mathbb{X})|}{|\overline{\mathcal{A}}_{\langle r\rangle}(\mathbb{X})|}, \quad \text{where } |\overline{\mathcal{A}}_{\langle r\rangle}(\mathbb{X})| \neq 0.$$

Evidently, $0 \leq \mu_{\mathcal{A}_{\langle r\rangle}}(\mathbb{X}) \leq 1$. Moreover, if $\mu_{\langle r\rangle}(\mathbb{X}) = 1$, so $\mathbb{X}$ is considered Allam-exact; or else, it is classified as Allam-rough.

3.4. Abd El-Monsef et al. models (2014)

Definition 3.4. [28] Consider the structure $\left(\mathcal{U}, \mathcal{R}, \xi_J\right)$ as a $\jmath$-**NS**, where $\jmath$ belongs to the set $\{r, \ell, \wedge, \curlyvee, \langle r\rangle, \langle \ell\rangle, \langle\wedge\rangle, \langle\curlyvee\rangle\}$. The *$\jmath$-lower* and *$\jmath$-upper* approximations of any subset $\mathbb{X} \subseteq \mathcal{U}$ are expressed as follows:

$$\underline{\mathcal{R}}_{J}(\mathbb{X}) = \bigcup \left\{ \mathcal{M} \in \mathcal{T}_{J} : \mathcal{M} \subseteq \mathbb{X} \right\} = \boldsymbol{int}_{J}(\mathbb{X}),$$
$$\overline{\mathcal{R}}_{J}(\mathbb{X}) = \bigcap \left\{ \mathcal{N} \in \mathcal{F}_{J} : \mathbb{X} \subseteq \mathcal{N} \right\} = \boldsymbol{cl}_{J}(\mathbb{X}),$$

where $\boldsymbol{int}_{J}(\mathbb{X})$ and $\boldsymbol{cl}_{J}(\mathbb{X})$ represent the J-interior and J-closure operators of $\mathbb{X}$, respectively, under the topology:

$$\mathcal{T}_{J} = \left\{ \mathbb{X} \subseteq \mathcal{U} : \forall\, s \in \mathbb{X}, n_{J}(s) \subseteq \mathbb{X} \right\},$$
$$\mathcal{F}_{J} = \left\{ \mathbb{Y} \subseteq \mathcal{U} : \mathbb{Y}^{c} \in \mathcal{T}_{J} \right\}.$$

Moreover, the *J-boundary*, *J-positive*, and *J-negative* areas associated with $\mathbb{X}$ are specified as:

$$\mathfrak{B}_{J}(\mathbb{X}) = \overline{\mathcal{R}}_{J}(\mathbb{X}) - \underline{\mathcal{R}}_{J}(\mathbb{X}),$$
$$\mathcal{POS}_{J}(\mathbb{X}) = \underline{\mathcal{R}}_{J}(\mathbb{X}),$$
$$\mathcal{NEG}_{J}(\mathbb{X}) = \mathcal{U} - \overline{\mathcal{R}}_{J}(\mathbb{X}).$$

The *J-accuracy* measure corresponding to the approximations of $\mathbb{X} \subseteq \mathcal{U}$ is given by:

$$\mu_{J}(\mathbb{X}) = \frac{\left|\underline{\mathcal{R}}_{J}(\mathbb{X})\right|}{\left|\overline{\mathcal{R}}_{J}(\mathbb{X})\right|}, \quad \text{where } \left|\overline{\mathcal{R}}_{J}(\mathbb{X})\right| \neq 0.$$

Clearly, $0 \leq \mu_{J}(\mathbb{X}) \leq 1$. Moreover, if $\mu_{J}(\mathbb{X}) = 1$, hence $\mathbb{X}$ is a J-exact set; otherwise, it is J-rough.

Remark 3.2. It is worth emphasizing that J-approximations, particularly when $J = r$, generally do not coincide with Yao approximations. More precisely, for any subset $\mathbb{X}$, the following inequalities hold:

$$\underline{\mathcal{R}}_{r}(\mathbb{X}) \neq \{s \in \mathcal{U} : n_{r}(s) \subseteq \mathbb{X}\} = \underline{\mathcal{R}}(\mathbb{X}),$$
$$\overline{\mathcal{R}}_{r}(\mathbb{X}) \neq \{s \in \mathcal{U} : n_{r}(s) \cap \mathbb{X} \neq \emptyset\} = \overline{\mathcal{R}}(\mathbb{X}).$$

This distinction is further illustrated in Examples 3.1 and 3.2.

The subsequent theorem provides the equality condition for Yao-approximations [7] with Abd El-Monsef et al. [28] for cases $J = \{r, \ell\}$, and the equality condition for Allam et al. [22,23] approaches with Abd El-Monsef et al. for cases $J = \{\langle r\rangle, \langle \ell\rangle\}$.

Theorem 3.1. *Given that the structure $\left(\mathcal{U}, \mathcal{R}, \xi_{J}\right)$ represents a J-**NS**. When $\mathcal{R}$ satisfies the properties of a preorder relation, the subsequent statements hold for every $\mathbb{X} \subseteq \mathcal{U}$:*

1. $\underline{\mathcal{R}}(\mathbb{X}) = \left\{s \in \mathcal{U} : n_{J}(s) \subseteq \mathbb{X}\right\} = \cup\{G \in \mathcal{T}_{J} : G \subseteq \mathbb{X}\} = \underline{\mathcal{R}}_{J}(\mathbb{X})$, $J = \{r, \ell\}$.
2. $\overline{\mathcal{R}}(\mathbb{X}) = \left\{s \in \mathcal{U} : n_{J}(s) \cap \mathbb{X} \neq \varphi\right\} = \cap\{H \in \mathcal{F}_{J} : \mathbb{X} \subseteq H\} = \overline{\mathcal{R}}_{J}(\mathbb{X})$, $J = \{r, \ell\}$.
3. $\underline{\mathcal{A}}_{j}(\mathbb{X}) = \{s \in \mathcal{U} : n_{J}(s) \subseteq \mathbb{X}\} = \cup\{G \in \mathcal{T}_{J} : G \subseteq \mathbb{X}\} = \underline{\mathcal{R}}_{J}(\mathbb{X})$, $J = \{\langle r\rangle, \langle \ell\rangle\}$.
4. $\overline{\mathcal{A}}_{j}(\mathbb{X}) = \left\{s \in \mathcal{U} : n_{J}(s) \cap \mathbb{X} \neq \varphi\right\} = \cap\{H \in \mathcal{F}_{J} : \mathbb{X} \subseteq H\} = \overline{\mathcal{R}}_{J}(\mathbb{X})$, $J = \{\langle r\rangle, \langle \ell\rangle\}$.

Proof. We will provide a detailed proof for the first item (where $J = r$). The proofs for the remaining items will follow a similar pattern and are therefore omitted for brevity.

Necessity condition:
Let

$$s \in \underline{\mathcal{R}}(\mathbb{X}), \tag{8}$$

then we have

$$n_{r}(s) \subseteq \mathbb{X}.$$

By the reflexivity of $\mathcal{R}$, we obtain

$$s \in n_{r}(s). \tag{9}$$

As a consequence of the transitivity of $\mathcal{R}$, we deduce that

$$\forall z \in n_{r}(s), \quad n_{r}(z) \subseteq n_{r}(s). \tag{10}$$

From (8), (9), and (10), we conclude that $n_{r}(s)$ represents an right-open set included in $\mathbb{X}$, with $s \in n_{r}(s)$. This implies that

$$s \in \underline{\mathcal{R}}_{r}(\mathbb{X}).$$

Sufficiency condition:
Suppose that

$$\mathrm{s} \in \underline{\mathcal{R}}_r(\mathbb{X}),$$

then there exists

$$G \in \mathcal{T}_r \quad \text{such that} \quad \mathrm{s} \in G \subseteq \mathbb{X}.$$

Therefore, for all

$$z \in G, \quad n_r(z) \subseteq G,$$

which implies that

$$n_r(\mathrm{s}) \subseteq \mathbb{X}.$$

Hence, we conclude that

$$\mathrm{s} \in \underline{\mathcal{R}}(\mathbb{X}). \quad \square$$

The situations of reflexivity and transitivity for the relation are essential and cannot be overlooked as illustrated Examples 3.1 and 3.2.

Example 3.1. Suppose that $\mathcal{R}$ is the relation $\{(\dot{g},\dot{g}), (\dot{g},\dot{h}), (\dot{h},\dot{h}), (\dot{h},\dot{k}), (\dot{k},\dot{k}), (\dot{s},\dot{s})\}$ defined over $\mathcal{U} = \{\dot{g},\dot{h},\dot{k},\dot{s}\}$, where $\mathcal{R}$ is reflexive. As a result, the right neighborhoods of the elements in $\mathcal{U}$ are determined as follows:

$$n_r(\dot{g}) = \{\dot{g},\dot{h}\},\ n_r(\dot{h}) = \{\dot{h},\dot{k}\},\ n_r(\dot{k}) = \{\dot{k}\}, \text{ and } n_r(\dot{s}) = \{\dot{s}\}.$$

Thus, the topology $\mathcal{T}_r$ induced by $\mathcal{R}$ is:

$$\mathcal{T}_r = \{\mathcal{U}, \varphi, \{\dot{k}\},\{\dot{s}\},\{\dot{h},\dot{k}\},\{\dot{k},\dot{s}\},\{\dot{g},\dot{h},\dot{k}\},\{\dot{h},\dot{k},\dot{s}\}\}.$$

Similarly, the family of closed sets $\mathcal{F}_r$ is given by:

$$\mathcal{F}_r = \{\mathcal{U}, \varphi, \{\dot{g}\},\{\dot{s}\},\{\dot{g},\dot{h}\},\{\dot{g},\dot{s}\},\{\dot{g},\dot{h},\dot{k}\},\{\dot{g},\dot{h},\dot{s}\}\}.$$

Let $\mathcal{M} = \{\dot{g},\dot{h}\}$ and $\mathcal{N} = \{\dot{k},\dot{s}\}$. Thereby, the r-lower and r-upper approximations are as follows: $\underline{\mathcal{R}}_r(\mathcal{M}) = \varphi, \overline{\mathcal{R}}_r(\mathcal{N}) = \mathcal{U}$. However, the classical lower and upper approximations yield: $\underline{\mathcal{R}}(\mathcal{M}) = \{\dot{g}\}$ and $\overline{\mathcal{R}}(\mathcal{N}) = \{\dot{h},\dot{k},\dot{s}\}$.

Example 3.2. Suppose that $\mathcal{U} = \{\dot{g},\dot{h},\dot{k},\dot{s}\}$ with the transitive relation $\mathcal{R} = \{(\dot{g},\dot{g}), (\dot{g},\dot{h}), (\dot{g},\dot{k}), (\dot{h},\dot{k}), (\dot{k},\dot{k})\}$. The corresponding right neighborhoods are as follows:

$$n_r(\dot{g}) = \{\dot{g}, \dot{h}, \dot{k}\},\ n_r(\dot{h}) = n_r(\dot{k}) = \{\dot{k}\}, \text{ and } n_r(\dot{s}) = \varphi.$$

Consequently, the topology $\mathcal{T}_r$ and its corresponding family of closed sets $\mathcal{F}_r$ are given by: $\mathcal{T}_r = \{\mathcal{U}, \varphi, \{\dot{k}\}, \{\dot{s}\}, \{\dot{h},\dot{k}\}, \{\dot{k},\dot{s}\}, \{\dot{g},\dot{h},\dot{k}\}, \{\dot{h},\dot{k},\dot{s}\}\}$, and $\mathcal{F}_r = \{\mathcal{U}, \varphi, \{\dot{g}\}, \{\dot{s}\}, \{\dot{g},\dot{h}\}, \{\dot{g},\dot{s}\}, \{\dot{g},\dot{h},\dot{k}\}, \{\dot{g},\dot{h},\dot{s}\}\}$.
For the subset $\mathcal{M} = \{\dot{k},\dot{s}\}$, we obtain $\underline{\mathcal{R}}_r(\mathcal{M}) = \{\dot{k},\dot{s}\}$, and $\overline{\mathcal{R}}_r(\mathcal{M}) = \mathcal{U}$.
Although, the Yao approximations yield: $\underline{\mathcal{R}}(\mathcal{M}) = \{\dot{h},\dot{k},\dot{s}\}$, $\overline{\mathcal{R}}(\mathcal{M}) = \{\dot{g},\dot{h},\dot{k}\}$.
Furthermore, for the entire universe and the empty set, we have:

$$\underline{\mathcal{R}}(\mathcal{U}) = \mathcal{U},\ \overline{\mathcal{R}}(\mathcal{U}) = \{\dot{g},\dot{h},\dot{k}\},\ \underline{\mathcal{R}}(\varphi) = \{\dot{s}\}, \text{ and } \overline{\mathcal{R}}(\varphi) = \varphi.$$

In contrast, the $\jmath$-approximations provide:

$$\underline{\mathcal{R}}_r(\mathcal{U}) = \overline{\mathcal{R}}_r(\mathcal{U}) = \mathcal{U}, \text{ and } \underline{\mathcal{R}}_r(\varphi) = \overline{\mathcal{R}}_r(\varphi) = \varphi.$$

3.5. Method of Dai et al. (2018)

Definition 3.5. [11] Let $\mathcal{R}$ be a binary relation defined on $\mathcal{U}$. The maximal approximations, namely the (r)-lower and (r)-upper approximations of a subset $\mathbb{X} \subseteq \mathcal{U}$, are respectively established as:

$$\underline{\mathcal{R}}_{(r)}(\mathbb{X}) = \{w \in \mathcal{U} : n_{(r)}(w) \subseteq \mathbb{X}\}$$

and

$$\overline{\mathcal{R}}_{(r)}(\mathbb{X}) = \{w \in \mathcal{U} : n_{(r)}(w) \cap \mathbb{X} \neq \emptyset\}.$$

Furthermore, the (r)-positive, (r)-negative, and (r)-boundary areas, along with the (r)-accuracy measure for the (r)-approximations of a subset $\mathbb{X} \subseteq \mathcal{U}$, can be formally constructed in the following manner:

$$\mathcal{POS}_{(r)}(\mathbb{X}) = \underline{\mathcal{R}}_{(r)}(\mathbb{X}),$$
$$\mathcal{NEG}_{(r)}(\mathbb{X}) = \mathcal{U} \setminus \overline{\mathcal{R}}_{(r)}(\mathbb{X}),$$
$$\mathfrak{B}_{(r)}(\mathbb{X}) = \overline{\mathcal{R}}_{(r)}(\mathbb{X}) \setminus \underline{\mathcal{R}}_{(r)}(\mathbb{X}),$$

and

$$\mu_{(r)}(\mathbb{X}) = \frac{\left|\underline{\mathcal{R}}_{(r)}(\mathbb{X})\right|}{\left|\overline{\mathcal{R}}_{(r)}(\mathbb{X})\right|}, \quad \text{where } \left|\overline{\mathcal{R}}_{(r)}(\mathbb{X})\right| \neq 0.$$

Furthermore, if $\mu_{(r)}(\mathbb{X}) = 1$, then $\mathbb{X}$ is named (r)-exact. Otherwise, $\mathbb{X}$ is considered (r)-rough.

3.6. Techniques of Abu-Gdairi (2023)

The subsection presents a discussion for the methodologies introduced by [43], which serve as significant extensions to the methodology developed by [32]. The primary focus of this section is to generalize the notion of “initial-neighborhood” and, consequently, derive four distinct topologies based on these neighborhoods.

Definition 3.6. [43] Consider $(\mathcal{U}, \mathcal{R}, \xi_{\jmath})$ as a $\jmath$-***NS***, where $\jmath \in \{r, \ell, \wedge, \curlyvee\}$. For any subset $\mathbb{X} \subseteq \mathcal{U}$, we define:

1. The *initial* $\jmath$*-lower* and *initial* $\jmath$*-upper* approximations of $\mathbb{X}$ as:

$$\underline{\mathcal{R}}^{\mathrm{i}}_{\jmath}(\mathbb{X}) = \bigcup \left\{\mathcal{G} \in \mathcal{T}^{\mathrm{i}}_{\jmath} : \mathcal{G} \subseteq \mathbb{X}\right\},$$
$$\overline{\mathcal{R}}^{\mathrm{i}}_{\jmath}(\mathbb{X}) = \bigcap \left\{\mathcal{H} \in \mathcal{F}^{\mathrm{i}}_{\jmath} : \mathbb{X} \subseteq \mathcal{H}\right\}.$$

   These correspond to the *interior* and *closure*, denoted respectively as $\boldsymbol{int}^{\mathrm{i}}_{\jmath}(\mathbb{X})$ and $\boldsymbol{cl}^{\mathrm{i}}_{\jmath}(\mathbb{X})$, within the topological structures:

$$\mathcal{T}^{\mathrm{i}}_{\jmath} = \{\mathcal{M} \subseteq \mathcal{U} : \forall m \in \mathcal{M}, n^{\mathrm{i}}_{\jmath}(m) \subseteq \mathcal{M}\}$$

   and

$$\mathcal{F}^{\mathrm{i}}_{\jmath} = \{\mathcal{H} \subseteq \mathcal{U} : \mathcal{H}^c \in \mathcal{T}^{\mathrm{i}}_{\jmath}\}.$$

2. The *initial* $\jmath$*-boundary*, *initial* $\jmath$*-positive region*, and *initial* $\jmath$*-negative region* of $\mathbb{X}$ are presented by:

$$\mathfrak{B}^{\mathrm{i}}_{\jmath}(\mathbb{X}) = \overline{\mathcal{R}}^{\mathrm{i}}_{\jmath}(\mathbb{X}) - \underline{\mathcal{R}}^{\mathrm{i}}_{\jmath}(\mathbb{X}),$$
$$\mathcal{POS}^{\mathrm{i}}_{\jmath}(\mathbb{X}) = \underline{\mathcal{R}}^{\mathrm{i}}_{\jmath}(\mathbb{X}), \mathit{and}$$
$$\mathcal{NEG}^{\mathrm{i}}_{\jmath}(\mathbb{X}) = \mathcal{U} - \overline{\mathcal{R}}^{\mathrm{i}}_{\jmath}(\mathbb{X}).$$

3. The *initial* $\jmath$*-accuracy* associated with the approximations of $\mathbb{X} \subseteq \mathcal{U}$ is determined as:

$$\mu^{\mathrm{i}}_{\jmath}(\mathbb{X}) = \frac{\left|\underline{\mathcal{R}}^{\mathrm{i}}_{\jmath}(\mathbb{X})\right|}{\left|\overline{\mathcal{R}}^{\mathrm{i}}_{\jmath}(\mathbb{X})\right|}, \quad \text{where} \quad \left|\overline{\mathcal{R}}^{\mathrm{i}}_{\jmath}(\mathbb{X})\right| \neq 0.$$

Clearly, the accuracy measure satisfies:

$$0 \leq \mu^{\mathrm{i}}_{\jmath}(\mathbb{X}) \leq 1.$$

If $\mu^{\mathrm{i}}_{\jmath}(\mathbb{X}) = 1$, the subset $\mathbb{X}$ is termed an *initial* $\jmath$*-exact* (or $\jmath$*-definable*) set. If not, it is categorized as *initial* $\jmath$*-rough*.
It is important to observe that when $\jmath = r$, the initial $\jmath$-approximations align with the framework proposed by [32].

In [45], Abu-Gdairi and El-Bably formulated and demonstrated the following significant result, which reinterprets the concept of initial $\jmath$-approximations (Definition 3.6) independently of any topological constructs, as follows.

Theorem 3.2. *[45] Consider $(\mathcal{U}, \mathcal{R}, \xi_J)$ being a J-**NS**. If $\mathbb{X} \subseteq \mathcal{U}$, then for each $J \in \{r, \ell, \wedge, \vee\}$: $\underline{\mathcal{R}}^{\mathrm{i}}_J(\mathbb{X}) = \{x \in \mathcal{U} : n^{\mathrm{i}}_J(x) \subseteq \mathbb{X}\}$ and $\overline{\mathcal{R}}^{\mathrm{i}}_J(\mathbb{X}) = \{x \in \mathcal{U} : n^{\mathrm{i}}_J(x) \cap \mathbb{X} \neq \varphi\}$.*

4. Novel sorts to initial-approximations

Considering the minimal J-neighborhoods and maximal J-neighborhoods, we construct new sorts of initial neighborhoods (so-called $\mathbb{I}_J$- approximations, for each $J \in \mathcal{J}$, where $\mathcal{J} = \{r, \ell, \wedge, \vee, \langle r \rangle, \langle \ell \rangle, \langle \wedge \rangle, \langle \vee \rangle, (r), (\ell), (\wedge), (\vee)\}$). This section is accordingly divided into two subsections: the first introduces these different kinds of neighborhoods, discussing their properties and relationships. The second subsection develops novel forms of rough approximations based on these neighborhoods, exploring their properties and interrelations. Several examples and counterexamples are provided as illustrations of the results established in the current part.

4.1. New kinds of initial-neighborhoods

Definition 4.1. Given that $(\mathcal{U}, \mathcal{R}, \xi_J)$ constitutes a J-**NS**. For any element $x \in \mathcal{U}$, the below neighborhood structures are defined:

- **Initial-Minimal J-Neighborhoods:**
  1. Initial-$\langle r \rangle$-Neighborhood: $n^{\mathrm{i}}_{\langle r \rangle}(x) = \{y \in \mathcal{U} : n_{\langle r \rangle}(x) \subseteq n_{\langle r \rangle}(y)\}$.
  2. Initial-$\langle \ell \rangle$-Neighborhood: $n^{\mathrm{i}}_{\langle \ell \rangle}(x) = \{y \in \mathcal{U} : n_{\langle \ell \rangle}(x) \subseteq n_{\langle \ell \rangle}(y)\}$.
  3. Initial-$\langle \wedge \rangle$-Neighborhood: $n^{\mathrm{i}}_{\langle \wedge \rangle}(x) = n^{\mathrm{i}}_{\langle r \rangle}(x) \cap n^{\mathrm{i}}_{\langle \ell \rangle}(x)$.
  4. Initial-$\langle \vee \rangle$-Neighborhood: $n^{\mathrm{i}}_{\langle \vee \rangle}(x) = n^{\mathrm{i}}_{\langle r \rangle}(x) \cup n^{\mathrm{i}}_{\langle \ell \rangle}(x)$.
- **Initial-Maximal J-Neighborhoods:**
  1. Initial-(r)-Neighborhood: $n^{\mathrm{i}}_{(r)}(x) = \{y \in \mathcal{U} : n_{(r)}(x) \subseteq n_{(r)}(y)\}$.
  2. Initial-(ℓ)-Neighborhood: $n^{\mathrm{i}}_{(\ell)}(x) = \{y \in \mathcal{U} : n_{(\ell)}(x) \subseteq n_{(\ell)}(y)\}$.
  3. Initial-$(\wedge)$-Neighborhood: $n^{\mathrm{i}}_{(\wedge)}(x) = n^{\mathrm{i}}_{(r)}(x) \cap n^{\mathrm{i}}_{(\ell)}(x)$.
  4. Initial-$(\vee)$-Neighborhood: $n^{\mathrm{i}}_{(\vee)}(x) = n^{\mathrm{i}}_{(r)}(x) \cup n^{\mathrm{i}}_{(\ell)}(x)$.

For simplicity, we will denote all initial neighborhoods by $n^{\mathrm{i}}_J(x)$, for each $J \in \mathcal{J}$, where the set $\mathcal{J} = \{r, \ell, \wedge, \vee, \langle r \rangle, \langle \ell \rangle, \langle \wedge \rangle, \langle \vee \rangle, (r), (\ell), (\wedge), (\vee)\}$, except where otherwise stated.

The relationships among the above neighborhoods are shown in the following results.

Lemma 4.1. *Given that $(\mathcal{U}, \mathcal{R}, \xi_J)$ is a J-**NS**. So, for each $x \in \mathcal{U}$, the upcoming inclusions hold:*

1. $n^{\mathrm{i}}_{\langle \wedge \rangle}(x) \subseteq n^{\mathrm{i}}_{\langle r \rangle}(x) \subseteq n^{\mathrm{i}}_{\langle \vee \rangle}(x)$.
2. $n^{\mathrm{i}}_{\langle \wedge \rangle}(x) \subseteq n^{\mathrm{i}}_{\langle \ell \rangle}(x) \subseteq n^{\mathrm{i}}_{\langle \vee \rangle}(x)$.
3. $n^{\mathrm{i}}_{(\wedge)}(x) \subseteq n^{\mathrm{i}}_{(r)}(x) \subseteq n^{\mathrm{i}}_{(\vee)}(x)$.
4. $n^{\mathrm{i}}_{(\wedge)}(x) \subseteq n^{\mathrm{i}}_{(\ell)}(x) \subseteq n^{\mathrm{i}}_{(\vee)}(x)$.

Proof. Using Definition 4.1, the proof is straightforward. □

Lemma 4.2. *Consider $(\mathcal{U}, \mathcal{R}, \xi_J)$ being a J-**NS**. For each $x, y \in \mathcal{U}$ and $J \in \mathcal{J}$, if $y \in n^{\mathrm{i}}_J(x)$, then $n^{\mathrm{i}}_J(y) \subseteq n^{\mathrm{i}}_J(x)$.*

Proof. By applying Lemma 3.1 of [32], the outcome follows immediately. □

Remark 4.1. The subsequent examples illustrate that the initial J-neighborhoods, initial-minimal J-neighborhoods, and initial-maximal J-neighborhoods are generally not comparable in the context of a binary relation (as demonstrated in Example 4.1), nor in specific cases of binary relations (as shown in Examples 4.2, 4.3, 4.4, and 4.5). This observation is made for the case where $J = r$; however, the same logic extends to the other cases.

Example 4.1. Assume that $\mathcal{R}$ is a binary relation on $\mathcal{U}$ represented by $\mathcal{R} = \{(\dot{g},\dot{g}), (\dot{g},\dot{h}), (\dot{h},\dot{h}), (\dot{h},\dot{k}), (\dot{s},\dot{k}), (\dot{s},\dot{s})\}$, where $\mathcal{U} = \{\dot{g},\dot{h},\dot{k},\dot{s}\}$. Based on this relation, we determine the J-neighborhoods and the initial J-neighborhoods, which are concisely presented in Tables 10 and 11. From Table 11, it is evident that $n^{\mathrm{i}}_r(x)$ and $n^{\mathrm{i}}_{\langle r \rangle}(x)$ as well as $n^{\mathrm{i}}_r(x)$ and $n^{\mathrm{i}}_{(r)}(x)$ are not comparable. Additionally, $n^{\mathrm{i}}_{\langle r \rangle}(x)$ and $n^{\mathrm{i}}_{(r)}(x)$ are also not comparable.

Example 4.2. Consider $\mathcal{R} = \{(\dot{g},\dot{g}), (\dot{g},\dot{h}), (\dot{h},\dot{h}), (\dot{h},\dot{k}), (\dot{k},\dot{k}), (\dot{s},\dot{k}), (\dot{s},\dot{s})\}$ being a reflexive relation on $\mathcal{U} = \{\dot{g},\dot{h},\dot{k},\dot{s}\}$. Based on this relation, we obtain the J-neighborhoods and initial J-neighborhoods, as displayed in Tables 12 and 13. From Table 13, it is evident that $n^{\mathrm{i}}_r(x)$ and $n^{\mathrm{i}}_{\langle r \rangle}(x)$ as well as $n^{\mathrm{i}}_r(x)$ and $n^{\mathrm{i}}_{(r)}(x)$ are not comparable. Additionally, $n^{\mathrm{i}}_{\langle r \rangle}(x)$ and $n^{\mathrm{i}}_{(r)}(x)$ are also not comparable.

Table 10
$\jmath$-neighborhoods, $\jmath \in \{r, \langle r\rangle, (r)\}$.

$\star$	$n_r(\star)$	$n_{\langle r\rangle}(\star)$	$n_{(r)}(\star)$
ġ	{ġ,ḣ}	{ġ,ḣ}	{ġ,ḣ}
ḣ	{ḣ,k̇}	{ḣ}	{ġ,ḣ,k̇}
k̇	φ	{k̇}	{ḣ,k̇,ṡ}
ṡ	{k̇,ṡ}	{k̇,ṡ}	{k̇,ṡ}

Table 11
Initial $\jmath$-neighborhoods, $\jmath \in \{r, \langle r\rangle, (r)\}$.

$\star$	$n^{i}_r(\star)$	$n^{i}_{\langle r\rangle}(\star)$	$n^{i}_{(r)}(\star)$
ġ	{ġ}	{ġ}	{ġ,ḣ}
ḣ	{ḣ}	{ġ,ḣ}	{ḣ}
k̇	$\mathcal{U}$	{k̇,ṡ}	{k̇}
ṡ	{ṡ}	{ṡ}	{k̇,ṡ}

Table 12
$\jmath$-neighborhoods, $\jmath \in \{r, \langle r\rangle, (r)\}$.

$\star$	$n_r(\star)$	$n_{\langle r\rangle}(\star)$	$n_{(r)}(\star)$
ġ	{ġ,ḣ}	{ġ,ḣ}	{ġ,ḣ}
ḣ	{ḣ,k̇}	{ḣ}	{ġ,ḣ,k̇}
k̇	{k̇}	{k̇}	{ḣ,k̇,ṡ}
ṡ	{k̇,ṡ}	{k̇,ṡ}	{k̇,ṡ}

Table 13
Initial $\jmath$-neighborhoods, $\jmath \in \{r, \langle r\rangle, (r)\}$.

$\star$	$n^{i}_r(\star)$	$n^{i}_{\langle r\rangle}(\star)$	$n^{i}_{(r)}(\star)$
ġ	{ġ}	{ġ}	{ġ,ḣ}
ḣ	{ḣ}	{ġ,ḣ}	{ḣ}
k̇	{ḣ,k̇,ṡ}	{k̇,ṡ}	{k̇}
ṡ	{ṡ}	{ṡ}	{k̇,ṡ}

Table 14
$\jmath$-neighborhoods, $\jmath \in \{r, \langle r\rangle, (r)\}$.

$\star$	$n_r(\star)$	$n_{\langle r\rangle}(\star)$	$n_{(r)}(\star)$
ġ	{ġ,ḣ,k̇}	{ġ}	$\mathcal{U}$
ḣ	{ġ,k̇,ṡ}	{ḣ}	{ġ,ḣ,k̇}
k̇	{ġ,ḣ}	{ġ,k̇}	$\mathcal{U}$
ṡ	{ḣ}	{ġ,k̇,ṡ}	{ġ,k̇,ṡ}

Example 4.3. Consider $\mathcal{R}$ ={(ġ,ġ), (ġ,ḣ), (ḣ,ġ), (ḣ,k̇), (k̇,ḣ), (ġ,k̇), (k̇,ġ), (ḣ,ṡ), (ṡ,ḣ),} being a symmetric relation on $\mathcal{U}$ = {ġ,ḣ,k̇,ṡ}. Based on this relation, we obtain the $\jmath$-neighborhoods and initial $\jmath$-neighborhoods, as displayed in Tables 14 and 15. From Table 15, it is evident that $n^{i}_r(x)$ and $n^{i}_{\langle r\rangle}(x)$ beside $n^{i}_{\langle r\rangle}(x)$ and $n^{i}_{(r)}(x)$ are not comparable.

Example 4.4. Consider $\mathcal{R}$ ={(ġ,ġ), (ḣ,ġ) (ḣ,ḣ), (ḣ,k̇), (ṡ,k̇), (ġ,k̇)} being a transitive relation on $\mathcal{U}$ ={ġ,ḣ,k̇,ṡ}. Based on this relation, we obtain the $\jmath$-neighborhoods and initial $\jmath$-neighborhoods, as displayed in Tables 16 and 17. From Table 17, it is evident that $n^{i}_r(x)$ and $n^{i}_{\langle r\rangle}(x)$ as well as $n^{i}_r(x)$ and $n^{i}_{(r)}(x)$ are not comparable.

Example 4.5. Consider $\mathcal{R}$ ={(ġ,ġ), (ġ,ḣ), (ḣ,ġ), (ḣ,ḣ), (ḣ,k̇), (k̇,ḣ), (k̇,k̇), (k̇,ṡ), (ṡ,k̇), (ṡ,ṡ)} being a similarity relation on $\mathcal{U}$ ={ġ,ḣ,k̇,ṡ}. Based on this relation, we obtain the $\jmath$-neighborhoods and initial $\jmath$-neighborhoods, as displayed in Tables 18 and 19. From Table 19, it is evident that $n^{i}_r(x)$ and $n^{i}_{\langle r\rangle}(x)$ beside $n^{i}_{\langle r\rangle}(x)$ and $n^{i}_{(r)}(x)$ are not comparable.

Theorem 4.1. *Assume that $\mathcal{R}$ is relation on $\mathcal{U}$, which is symmetric. If $(\mathcal{U}, \mathcal{R}, \xi_\jmath)$ forms a $\jmath$-**NS**, then for each $x \in \mathcal{U}$:*

1. $n^{i}_{\langle r\rangle}(x) = n^{i}_{\langle \ell\rangle}(x) = n^{i}_{\langle \wedge\rangle}(x) = n^{i}_{\langle \vee\rangle}(x)$.

Table 15
Initial $\jmath$-neighborhoods, $\jmath \in \{r, \langle r \rangle, (r)\}$.

$\star$	$n^{i}_{r}(\star)$	$n^{i}_{\langle r \rangle}(\star)$	$n^{i}_{(r)}(\star)$
ġ	{ġ}	{ġ,k̇,ṡ}	{ġ,k̇}
ḣ	{ḣ}	{ḣ}	{ġ,ḣ,k̇}
k̇	{ġ,k̇}	{k̇,ṡ}	{ġ,k̇}
ṡ	{ġ,k̇,ṡ}	{ṡ}	{ġ,k̇,ṡ}

Table 16
$\jmath$-neighborhoods, $\jmath \in \{r, \langle r \rangle, (r)\}$.

$\star$	$n_{r}(\star)$	$n_{\langle r \rangle}(\star)$	$n_{(r)}(\star)$
ġ	{ġ,k̇}	{ġ,k̇}	{ġ,ḣ,k̇}
ḣ	{ġ,ḣ,k̇}	{ġ,ḣ,k̇}	{ġ,ḣ,k̇}
k̇	φ	{k̇}	{ġ,ḣ,k̇}
ṡ	{k̇}	φ	φ

Table 17
Initial $\jmath$-neighborhoods, $\jmath \in \{r, \langle r \rangle, (r)\}$.

$\star$	$n^{i}_{r}(\star)$	$n^{i}_{\langle r \rangle}(\star)$	$n^{i}_{(r)}(\star)$
ġ	{ġ,ḣ}	{ġ,ḣ}	{ġ,ḣ,k̇}
ḣ	{ḣ}	{ḣ}	{ġ,ḣ,k̇}
k̇	$\mathcal{U}$	{ġ,ḣ,k̇}	{ġ,ḣ,k̇}
ṡ	{ġ,ḣ,ṡ}	$\mathcal{U}$	$\mathcal{U}$

Table 18
$\jmath$-neighborhoods, $\jmath \in \{r, \langle r \rangle, (r)\}$.

$\star$	$n_{r}(\star)$	$n_{\langle r \rangle}(\star)$	$n_{(r)}(\star)$
ġ	{ġ,ḣ}	{ġ,ḣ}	{ġ,ḣ,k̇}
ḣ	{ġ,ḣ,k̇}	{ḣ}	$\mathcal{U}$
k̇	{ḣ,k̇,ṡ}	{k̇}	$\mathcal{U}$
ṡ	{k̇,ṡ}	{k̇,ṡ}	{ḣ,k̇,ṡ}

Table 19
Initial $\jmath$-neighborhoods, $\jmath \in \{r, \langle r \rangle, (r)\}$.

$\star$	$n^{i}_{r}(\star)$	$n^{i}_{\langle r \rangle}(\star)$	$n^{i}_{(r)}(\star)$
ġ	{ġ,ḣ}	{ġ}	{ġ,ḣ,k̇}
ḣ	{ḣ}	{ġ,ḣ}	{ḣ,k̇}
k̇	{k̇}	{k̇,ṡ}	{ḣ,k̇}
ṡ	{k̇,ṡ}	{ṡ}	{ḣ,k̇,ṡ}

2. $n^{i}_{(r)}(x) = n^{i}_{(\ell)}(x) = n^{i}_{(\wedge)}(x) = n^{i}_{(\curlyvee)}(x)$.

Proof. The consequence follows immediately according to Theorem 2.2. □

Theorem 4.2. *Given $(\mathcal{U}, \mathcal{R}, \xi_{\jmath})$ is a $\jmath$-**NS**, where $\mathcal{R}$ is a similarity relation defined over the universe $\mathcal{U}$. For every element $x \in \mathcal{U}$ and each $\jmath \in \{r, \ell, \wedge, \curlyvee\}$, the subsequent features hold:*

Proof. The proof for the issue $\jmath = r$ is provided, and the proofs for the other cases follow similarly.

$$\text{Let } w \in n^{i}_{(r)}(x) \text{ then } n_{(r)}(x) \subseteq n_{(r)}(w). \text{ Thus, } d \in n_{(r)}(w), \text{ for every } d \in n_{(r)}(x) \tag{11}$$

Since $\mathcal{R}$ is reflexive, by Lemma 2.2 and Theorem 2.1, $x \in n_{r}(x)$ and $n_{r}(x) \subseteq n_{(r)}(x)$.

$$\text{Thus, by (11), } x \in n_{(r)}(w), \text{ implying } \exists g \in \mathcal{U} \text{ such that } w \in n_{r}(g) \text{ and } x \in n_{r}(g) \tag{12}$$

Since $\mathcal{R}$ is symmetric, by (12), $g \in n_{r}(w)$ and $g \in n_{r}(x)$ which implies $n_{r}(w) \subseteq n_{r}(x)$. Therefore, $w \in n_{r}(x)$ that contains x. Accordingly, $w \in n_{(r)}(x)$. □

Table 20
$\jmath$-neighborhoods, $\jmath \in \{r, \langle r\rangle, (r)\}$.

$\star$	$n_r(\star)$	$n_{\langle r\rangle}(\star)$	$n_{(r)}(\star)$
ġ	{ġ,ḣ,k̇}	{ġ,ḣ,k̇}	{ġ,ḣ,k̇}
ḣ	{ġ,ḣ,k̇}	{ġ,ḣ,k̇}	{ġ,ḣ,k̇}
k̇	{k̇}	{k̇}	$\mathcal{U}$
ṡ	{k̇,ṡ}	{k̇,ṡ}	{k̇,ṡ}

Table 21
Initial $\jmath$-neighborhoods, $\jmath \in \{r, \langle r\rangle, (r)\}$.

$\star$	$n^{\mathrm{i}}_r(\star)$	$n^{\mathrm{i}}_{\langle r\rangle}(\star)$	$n^{\mathrm{i}}_{(r)}(\star)$
ġ	{ġ,ḣ}	{ġ,ḣ}	{ġ,ḣ,k̇}
ḣ	{ġ,ḣ}	{ġ,ḣ}	{ġ,ḣ,k̇}
k̇	$\mathcal{U}$	$\mathcal{U}$	{k̇}
ṡ	{ṡ}	{ṡ}	{k̇,ṡ}

Remark 4.2. In Theorem 4.2, it cannot be replaced the inclusion sign by an equal sign in general, as verified by Example 4.5.

Theorem 4.3. *Assume that $(\mathcal{U}, \mathcal{R}, \xi_\jmath)$ forms a $\jmath$-**NS**, with $\mathcal{R}$ being a preorder relation on $\mathcal{U}$. Then, for each $x \in \mathcal{U}$ and for every $\jmath \in \{r, \ell, \wedge, \curlyvee\}$, the following equality holds:*

$$n^{\mathrm{i}}_{\langle\jmath\rangle}(x) = n^{\mathrm{i}}_\jmath(x).$$

Proof. By applying Theorem 2.6, the outcome follows immediately. □

Remark 4.3. Example 4.6 exemplify that:

1. For each $\jmath \in \{r, \ell, \wedge, \curlyvee\}$, the initial $\jmath$-neighborhoods and initial-maximal $\jmath$-neighborhoods are not comparable in the case of preorder relations.
2. For each $\jmath \in \{r, \ell, \wedge, \curlyvee\}$, the initial-minimal $\jmath$-neighborhoods and initial-maximal $\jmath$-neighborhoods are not comparable in the case of preorder relations.

Example 4.6. Consider the relation $\mathcal{R}$ = {(ġ,ġ), (ġ,ḣ), (ġ,k̇), (ḣ,ġ), (ḣ,ḣ), (ḣ,k̇), (k̇,k̇), (ṡ,k̇), (ṡ,ṡ)}, which is a preorder on the set $\mathcal{U}$ = {ġ,ḣ,k̇,ṡ}. Consequently, the $\jmath$-neighborhoods and the initial $\jmath$-neighborhoods are obtained as presented in Tables 20 and 21.

4.2. New kinds of initial-rough sets

Now, we use the above neighborhoods to define new sorts for initial-approximations.

Definition 4.2. Assume $(\mathcal{U}, \mathcal{R}, \xi_\jmath)$, which constitutes a $\jmath$-***NS***, and let $\jmath \in \mathcal{J}$. For any subset $\mathbb{S} \subseteq \mathcal{U}$, we define the following:

1. The *initial $\jmath$-lower approximation* (abbreviated as $\mathbb{I}_\jmath$-lower) and the *initial $\jmath$-upper approximation* (abbreviated as $\mathbb{I}_\jmath$-upper) of $\mathbb{S}$ are given by

$$\underline{\mathbb{L}\mathrm{o}}^{\mathrm{i}}_\jmath(\mathbb{S}) = \{x \in \mathcal{U} : n^{\mathrm{i}}_\jmath(x) \subseteq \mathbb{S}\},$$

   and

$$\overline{\mathbb{U}\mathrm{p}}^{\mathrm{i}}_\jmath(\mathbb{S}) = \{x \in \mathcal{U} : n^{\mathrm{i}}_\jmath(x) \cap \mathbb{S} \neq \emptyset\}.$$

2. The *initial $\jmath$-positive region* (or $\mathbb{I}_\jmath$-positive region) of $\mathbb{S}$ is defined as

$$\mathbb{P}\mathrm{o}^{\mathrm{i}}_\jmath(\mathbb{S}) = \underline{\mathbb{L}\mathrm{o}}^{\mathrm{i}}_\jmath(\mathbb{S}),$$

   while the *initial $\jmath$-negative region* (or $\mathbb{I}_\jmath$-negative region) is given by

$$\mathbb{N}\mathrm{e}^{\mathrm{i}}_\jmath(\mathbb{S}) = \mathcal{U} \setminus \overline{\mathbb{U}\mathrm{p}}^{\mathrm{i}}_\jmath(\mathbb{S}).$$

   The *initial $\jmath$-boundary region* (or $\mathbb{I}_\jmath$-boundary) of $\mathbb{S}$ is determined as the difference between the initial upper and lower approximations:

$$\mathbb{BN}^{\mathrm{i}}_{\jmath}(\mathbb{S}) = \overline{\mathbb{U}\mathrm{p}}^{\mathrm{i}}_{\jmath}(\mathbb{S}) \setminus \underline{\mathbb{L}\mathrm{o}}^{\mathrm{i}}_{\jmath}(\mathbb{S}).$$

3. The *initial* $\jmath$*-accuracy* (or $\mathbb{I}_{\jmath}$-accuracy) of the approximations of $\mathbb{S}$ is provided by the ratio

$$\mathbb{A}\mathrm{c}^{\mathrm{i}}_{\jmath}(\mathbb{S}) = \frac{\left|\underline{\mathbb{L}\mathrm{o}}^{\mathrm{i}}_{\jmath}(\mathbb{S})\right|}{\left|\overline{\mathbb{U}\mathrm{p}}^{\mathrm{i}}_{\jmath}(\mathbb{S})\right|},$$

provided that $\left|\overline{\mathbb{U}\mathrm{p}}^{\mathrm{i}}_{\jmath}(\mathbb{S})\right| \neq 0$.

Note that $0 \leq \mathbb{A}\mathrm{c}^{\mathrm{i}}_{\jmath}(\mathbb{S}) \leq 1$. When $\mathbb{A}\mathrm{c}^{\mathrm{i}}_{\jmath}(\mathbb{S}) = 1$, the set $\mathbb{S}$ is termed $\mathbb{I}_{\jmath}$-definable (or $\mathbb{I}_{\jmath}$-exact); otherwise, it is considered $\mathbb{I}_{\jmath}$-rough.

Note:

1. When $\jmath = r$, this definition is equivalent to the one given by El-Sayed et al. [32].
2. When $\jmath \in \{\ell, \wedge, \curlyvee\}$, this definition corresponds to the one provided by Abu-Gdairi [43].

The following proposition establishes key characteristics of the $\jmath$-initial approximations.

Proposition 4.1. *Consider the* $\jmath$*-**NS*** $(\mathcal{U}, \mathcal{R}, \xi_{\jmath})$. *For any subsets* $\mathbb{A}, \mathbb{B} \subseteq \mathcal{U}$ *and for each* $\jmath \in \mathcal{J}$*:*

(L1) $\underline{\mathbb{L}\mathrm{o}}^{\mathrm{i}}_{\jmath}(\mathbb{A}) \subseteq \mathbb{A}$.

(L2) $\underline{\mathbb{L}\mathrm{o}}^{\mathrm{i}}_{\jmath}(\varphi) = \varphi$.

(L3) $\underline{\mathbb{L}\mathrm{o}}^{\mathrm{i}}_{\jmath}(\mathcal{U}) = \mathcal{U}$.

(L4) If $\mathbb{A} \subseteq \mathbb{B}$, *then* $\underline{\mathbb{L}\mathrm{o}}^{\mathrm{i}}_{\jmath}(\mathbb{A}) \subseteq \underline{\mathbb{L}\mathrm{o}}^{\mathrm{i}}_{\jmath}(\mathbb{B})$.

(L5) $\underline{\mathbb{L}\mathrm{o}}^{\mathrm{i}}_{\jmath}\left(\mathbb{A} \bigcap \mathbb{B}\right) = \underline{\mathbb{L}\mathrm{o}}^{\mathrm{i}}_{\jmath}(\mathbb{A}) \bigcap \underline{\mathbb{L}\mathrm{o}}^{\mathrm{i}}_{\jmath}(\mathbb{B})$.

(L6) $\underline{\mathbb{L}\mathrm{o}}^{\mathrm{i}}_{\jmath}(\mathbb{A}) \bigcup \underline{\mathbb{L}\mathrm{o}}^{\mathrm{i}}_{\jmath}(\mathbb{B}) \subseteq \underline{\mathbb{L}\mathrm{o}}^{\mathrm{i}}_{\jmath}\left(\mathbb{A} \bigcup \mathbb{B}\right)$.

(L7) $\underline{\mathbb{L}\mathrm{o}}^{\mathrm{i}}_{\jmath}(\mathbb{A}^c) = (\overline{\mathbb{U}\mathrm{p}}^{\mathrm{i}}_{\jmath}(\mathbb{A}))^c$.

(L8) $\underline{\mathbb{L}\mathrm{o}}^{\mathrm{i}}_{\jmath}\left(\underline{\mathbb{L}\mathrm{o}}^{\mathrm{i}}_{\jmath}(\mathbb{A})\right) = \underline{\mathbb{L}\mathrm{o}}^{\mathrm{i}}_{\jmath}(\mathbb{A})$.

(L9) $\underline{\mathbb{L}\mathrm{o}}^{\mathrm{i}}_{\jmath}\left(\overline{\mathbb{U}\mathrm{p}}^{\mathrm{i}}_{\jmath}(\mathbb{A})\right) \subseteq \overline{\mathbb{U}\mathrm{p}}^{\mathrm{i}}_{\jmath}(\mathbb{A})$.

(U1) $\mathbb{A} \subseteq \overline{\mathbb{U}\mathrm{p}}^{\mathrm{i}}_{\jmath}(\mathbb{A})$.

(U2) $\overline{\mathbb{U}\mathrm{p}}^{\mathrm{i}}_{\jmath}(\varphi) = \varphi$.

(U3) $\overline{\mathbb{U}\mathrm{p}}^{\mathrm{i}}_{\jmath}(\mathcal{U}) = \mathcal{U}$.

(U4) If $\mathbb{A} \subseteq \mathbb{B}$, *then* $\overline{\mathbb{U}\mathrm{p}}^{\mathrm{i}}_{\jmath}(\mathbb{A}) \subseteq \overline{\mathbb{U}\mathrm{p}}^{\mathrm{i}}_{\jmath}(\mathbb{B})$.

(U5) $\overline{\mathbb{U}\mathrm{p}}^{\mathrm{i}}_{\jmath}\left(\mathbb{A} \bigcup \mathbb{B}\right) = \overline{\mathbb{U}\mathrm{p}}^{\mathrm{i}}_{\jmath}(\mathbb{A}) \bigcup \overline{\mathbb{U}\mathrm{p}}^{\mathrm{i}}_{\jmath}(\mathbb{B})$.

(U6) $\overline{\mathbb{U}\mathrm{p}}^{\mathrm{i}}_{\jmath}(\mathbb{A}) \bigcap \overline{\mathbb{U}\mathrm{p}}^{\mathrm{i}}_{\jmath}(\mathbb{B}) \supseteq \overline{\mathbb{U}\mathrm{p}}^{\mathrm{i}}_{\jmath}\left(\mathbb{A} \bigcap \mathbb{B}\right)$.

(U7) $\overline{\mathbb{U}\mathrm{p}}^{\mathrm{i}}_{\jmath}(\mathbb{A}^c) = (\underline{\mathbb{L}\mathrm{o}}^{\mathrm{i}}_{\jmath}(\mathbb{A}))^c$.

(U8) $\overline{\mathbb{U}\mathrm{p}}^{\mathrm{i}}_{\jmath}\left(\overline{\mathbb{U}\mathrm{p}}^{\mathrm{i}}_{\jmath}(\mathbb{A})\right) = \overline{\mathbb{U}\mathrm{p}}^{\mathrm{i}}_{\jmath}(\mathbb{A})$.

(U9) $\overline{\mathbb{U}\mathrm{p}}^{\mathrm{i}}_{\jmath}\left(\underline{\mathbb{L}\mathrm{o}}^{\mathrm{i}}_{\jmath}(\mathbb{A})\right) \supseteq \underline{\mathbb{L}\mathrm{o}}^{\mathrm{i}}_{\jmath}(\mathbb{A})$.

Proof. We begin by noting that when $\jmath = r$, the stated properties are established in [32] and [43]. Consequently, these properties hold for $\jmath \in \{r, \ell, \wedge, \curlyvee\}$. It remains to prove the proposition for the other cases.

For clarity, we provide the proof for the case $\jmath = \langle r \rangle$; analogous arguments apply to the remaining cases.

Properties (L1)–(L3) and (U1)–(U3) follow directly from Definition 4.1.

(L4) Assume $\mathbb{A} \subseteq \mathbb{B}$ and let $w \in \underline{\mathbb{L}\mathrm{o}}^{\mathrm{i}}_{\langle r \rangle}(\mathbb{A})$. By definition,

$$n^{\mathrm{i}}_{\langle r \rangle}(w) \subseteq \mathbb{A}.$$

Since $\mathbb{A} \subseteq \mathbb{B}$, we have $n^{\mathrm{i}}_{\langle r \rangle}(w) \subseteq \mathbb{B}$, which implies $w \in \underline{\mathbb{L}\mathrm{o}}^{\mathrm{i}}_{\langle r \rangle}(\mathbb{B})$. Hence,

$$\underline{\mathbb{L}\mathrm{o}}^{\mathrm{i}}_{\langle r \rangle}(\mathbb{A}) \subseteq \underline{\mathbb{L}\mathrm{o}}^{\mathrm{i}}_{\langle r \rangle}(\mathbb{B}).$$

Similarly, we deduce that

$$\overline{\mathbb{U}\mathrm{p}}^{\mathrm{i}}_{\langle r \rangle}(\mathbb{A}) \subseteq \overline{\mathbb{U}\mathrm{p}}^{\mathrm{i}}_{\langle r \rangle}(\mathbb{B}).$$

(L5) First, note that $\mathbb{A} \cap \mathbb{B} \subseteq \mathbb{A}$ and $\mathbb{A} \cap \mathbb{B} \subseteq \mathbb{B}$. By (L4), it follows that

$$\underline{\mathbb{L}\mathrm{o}}^{\mathrm{i}}_{\langle r \rangle}(\mathbb{A} \cap \mathbb{B}) \subseteq \underline{\mathbb{L}\mathrm{o}}^{\mathrm{i}}_{\langle r \rangle}(\mathbb{A}) \quad \text{and} \quad \underline{\mathbb{L}\mathrm{o}}^{\mathrm{i}}_{\langle r \rangle}(\mathbb{A} \cap \mathbb{B}) \subseteq \underline{\mathbb{L}\mathrm{o}}^{\mathrm{i}}_{\langle r \rangle}(\mathbb{B}).$$

Thus,

$$\underline{\mathbb{L}\mathrm{o}}^{\mathrm{i}}_{\langle r \rangle}(\mathbb{A} \cap \mathbb{B}) \subseteq \underline{\mathbb{L}\mathrm{o}}^{\mathrm{i}}_{\langle r \rangle}(\mathbb{A}) \cap \underline{\mathbb{L}\mathrm{o}}^{\mathrm{i}}_{\langle r \rangle}(\mathbb{B}).$$

Conversely, let $w \in \underline{\mathbb{L}\mathrm{o}}^{\mathrm{i}}_{\langle r \rangle}(\mathbb{A}) \cap \underline{\mathbb{L}\mathrm{o}}^{\mathrm{i}}_{\langle r \rangle}(\mathbb{B})$. Therefore,

$$n^{\mathrm{i}}_{\langle r \rangle}(w) \subseteq \mathbb{A} \quad \text{and} \quad n^{\mathrm{i}}_{\langle r \rangle}(w) \subseteq \mathbb{B},$$

so that

$$n^{\mathrm{i}}_{\langle r\rangle}(w) \subseteq \mathbb{A} \cap \mathbb{B},$$

implying $w \in \underline{\mathbb{L}\mathrm{o}}^{\mathrm{i}}_{\langle r\rangle}(\mathbb{A} \cap \mathbb{B})$. Hence,

$$\underline{\mathbb{L}\mathrm{o}}^{\mathrm{i}}_{\langle r\rangle}(\mathbb{A} \cap \mathbb{B}) = \underline{\mathbb{L}\mathrm{o}}^{\mathrm{i}}_{\langle r\rangle}(\mathbb{A}) \cap \underline{\mathbb{L}\mathrm{o}}^{\mathrm{i}}_{\langle r\rangle}(\mathbb{B}).$$

Using a similar argument, one may also show that

$$\overline{\mathbb{U}\mathrm{p}}^{\mathrm{i}}_{\langle r\rangle}(\mathbb{A} \cup \mathbb{B}) = \overline{\mathbb{U}\mathrm{p}}^{\mathrm{i}}_{\langle r\rangle}(\mathbb{A}) \cup \overline{\mathbb{U}\mathrm{p}}^{\mathrm{i}}_{\langle r\rangle}(\mathbb{B}).$$

(L6) By applying properties (L4) and (U4), the statements in (L6) and (U6) follow straightforwardly.

(L7) Observe that

$$\left[\overline{\mathbb{U}\mathrm{p}}^{\mathrm{i}}_{\langle r\rangle}(\mathbb{A})\right]^c = \left\{ x \in \mathcal{U} : n^{\mathrm{i}}_{\langle r\rangle}(x) \cap \mathbb{A} = \emptyset \right\} = \left\{ x \in \mathcal{U} : n^{\mathrm{i}}_{\langle r\rangle}(x) \subseteq \mathbb{A}^c \right\},$$

which is exactly $\underline{\mathbb{L}\mathrm{o}}^{\mathrm{i}}_{\langle r\rangle}(\mathbb{A}^c)$. Similarly, one can show that

$$\left[\underline{\mathbb{L}\mathrm{o}}^{\mathrm{i}}_{\langle r\rangle}(\mathbb{A})\right]^c = \overline{\mathbb{U}\mathrm{p}}^{\mathrm{i}}_{\langle r\rangle}(\mathbb{A}^c).$$

(L8) First, by (L1) and (L4), we have

$$\underline{\mathbb{L}\mathrm{o}}^{\mathrm{i}}_{\langle r\rangle}\left(\underline{\mathbb{L}\mathrm{o}}^{\mathrm{i}}_{\langle r\rangle}(\mathbb{A})\right) \subseteq \underline{\mathbb{L}\mathrm{o}}^{\mathrm{i}}_{\langle r\rangle}(\mathbb{A}).$$

Now, let

$$\text{assume } w \in \underline{\mathbb{L}\mathrm{o}}^{\mathrm{i}}_{\langle r\rangle}(\mathbb{A}). \quad \text{Then } n^{\mathrm{i}}_{\langle r\rangle}(w) \subseteq \mathbb{A}. \tag{13}$$

To demonstrate that $w \in \underline{\mathbb{L}\mathrm{o}}^{\mathrm{i}}_{\langle r\rangle}\left(\underline{\mathbb{L}\mathrm{o}}^{\mathrm{i}}_{\langle r\rangle}(\mathbb{A})\right)$, we must show that

$$n^{\mathrm{i}}_{\langle r\rangle}(w) \subseteq \underline{\mathbb{L}\mathrm{o}}^{\mathrm{i}}_{\langle r\rangle}(\mathbb{A}).$$

Let $z \in n^{\mathrm{i}}_{\langle r\rangle}(w)$. Then, by Lemma 4.2,

$$n^{\mathrm{i}}_{\langle r\rangle}(z) \subseteq n^{\mathrm{i}}_{\langle r\rangle}(w).$$

Using (13), it follows that

$$n^{\mathrm{i}}_{\langle r\rangle}(z) \subseteq \mathbb{A},$$

so that $z \in \underline{\mathbb{L}\mathrm{o}}^{\mathrm{i}}_{\langle r\rangle}(\mathbb{A})$. Hence,

$$n^{\mathrm{i}}_{\langle r\rangle}(w) \subseteq \underline{\mathbb{L}\mathrm{o}}^{\mathrm{i}}_{\langle r\rangle}(\mathbb{A}),$$

implying $w \in \underline{\mathbb{L}\mathrm{o}}^{\mathrm{i}}_{\langle r\rangle}\left(\underline{\mathbb{L}\mathrm{o}}^{\mathrm{i}}_{\langle r\rangle}(\mathbb{A})\right)$. So,

$$\underline{\mathbb{L}\mathrm{o}}^{\mathrm{i}}_{\langle r\rangle}(\mathbb{A}) \subseteq \underline{\mathbb{L}\mathrm{o}}^{\mathrm{i}}_{\langle r\rangle}\left(\underline{\mathbb{L}\mathrm{o}}^{\mathrm{i}}_{\langle r\rangle}(\mathbb{A})\right).$$

By an analogous argument, one can verify that

$$\overline{\mathbb{U}\mathrm{p}}^{\mathrm{i}}_{\langle r\rangle}\left(\overline{\mathbb{U}\mathrm{p}}^{\mathrm{i}}_{\langle r\rangle}(\mathbb{A})\right) = \overline{\mathbb{U}\mathrm{p}}^{\mathrm{i}}_{\langle r\rangle}(\mathbb{A}).$$

(L9) This property follows directly from (L1), and similarly, (U9) is a direct consequence of (U1). $\square$

Remark 4.4.

1. According Proposition 4.1, $\underline{\mathbb{L}\mathrm{o}}^{\mathrm{i}}_{j}(\varphi) = \underline{\mathbb{L}\mathrm{o}}^{\mathrm{i}}_{j}(\varphi) = \varphi$. Therefore, φ is $\mathbb{I}_j$-definable and hence $\mathbb{A}\mathrm{c}^{\mathrm{i}}_{j}(\varphi) = 1$.
2. The following example (Example 4.7) illustrates that the converse of properties (L6), (U6), (L9), and (U9) does not generally hold.

Example 4.7. (Continuation of Example 4.1.) Let $\mathbb{A} = \{\dot{\mathrm{g}},\dot{\mathrm{h}},\dot{\mathrm{k}}\}$ and $\mathbb{B} = \{\dot{\mathrm{g}},\dot{\mathrm{h}},\dot{\mathrm{s}}\}$, implying $\mathbb{A} \cup \mathbb{B} = \mathcal{U}$. We then obtain $\underline{\mathbb{L}\mathrm{o}}^{\mathrm{i}}_{\langle r\rangle}(\mathbb{A}) = \{\dot{\mathrm{g}},\dot{\mathrm{h}}\}$ and $\underline{\mathbb{L}\mathrm{o}}^{\mathrm{i}}_{\langle r\rangle}(\mathbb{B}) = \{\dot{\mathrm{g}},\dot{\mathrm{h}},\dot{\mathrm{s}}\}$. It is evident that $\underline{\mathbb{L}\mathrm{o}}^{\mathrm{i}}_{\langle r\rangle}(\mathbb{A}) \cup \underline{\mathbb{L}\mathrm{o}}^{\mathrm{i}}_{\langle r\rangle}(\mathbb{B}) = \{\dot{\mathrm{g}},\dot{\mathrm{h}},\dot{\mathrm{s}}\}$, even though $\underline{\mathbb{L}\mathrm{o}}^{\mathrm{i}}_{\langle r\rangle}(\mathbb{A} \cup \mathbb{B}) = \mathcal{U}$. Correspondingly, $\overline{\mathbb{U}\mathrm{p}}^{\mathrm{i}}_{\langle r\rangle}(\mathbb{A}) = \{\dot{\mathrm{g}},$

Table 22
Initial-minimal $\jmath$-neighborhoods.

$\star$	$n^{\mathrm{i}}_{\langle r\rangle}(\star)$	$n^{\mathrm{i}}_{\langle \ell\rangle}(\star)$	$n^{\mathrm{i}}_{\langle \wedge\rangle}(\star)$	$n^{\mathrm{i}}_{\langle \curlyvee\rangle}(\star)$
$\dot{g}$	$\{\dot{g}\}$	$\{\dot{g}\}$	$\{\dot{g}\}$	$\{\dot{g}\}$
$\dot{h}$	$\{\dot{g},\dot{h}\}$	$\{\dot{h},\dot{k},\dot{s}\}$	$\{\dot{h}\}$	$\mathcal{U}$
$\dot{k}$	$\{\dot{k}\}$	$\{\dot{k},\dot{s}\}$	$\{\dot{k}\}$	$\{\dot{k},\dot{s}\}$
$\dot{s}$	$\mathcal{U}$	$\{\dot{k},\dot{s}\}$	$\{\dot{k},\dot{s}\}$	$\mathcal{U}$

Table 23
Initial-maximal $\jmath$-neighborhoods.

$\star$	$n^{\mathrm{i}}_{(r)}(\star)$	$n^{\mathrm{i}}_{(\ell)}(\star)$	$n^{\mathrm{i}}_{(\wedge)}(\star)$	$n^{\mathrm{i}}_{(\curlyvee)}(\star)$
$\dot{g}$	$\{\dot{g},\dot{h}\}$	$\{\dot{g},\dot{h}\}$	$\{\dot{g},\dot{h}\}$	$\{\dot{g},\dot{h}\}$
$\dot{h}$	$\{\dot{h}\}$	$\{\dot{h}\}$	$\{\dot{h}\}$	$\{\dot{h}\}$
$\dot{k}$	$\{\dot{h},\dot{k}\}$	$\{\dot{h},\dot{k},\dot{s}\}$	$\{\dot{h},\dot{k}\}$	$\{\dot{h},\dot{k},\dot{s}\}$
$\dot{s}$	$\mathcal{U}$	$\{\dot{h},\dot{k},\dot{s}\}$	$\{\dot{h},\dot{k},\dot{s}\}$	$\mathcal{U}$

$\dot{h},\dot{k}\}$ and $\overline{\mathbb{U}\mathrm{p}}^{\mathrm{i}}_{\langle r\rangle}(\mathbb{B})=\mathcal{U}$, which implies $\overline{\mathbb{U}\mathrm{p}}^{\mathrm{i}}_{\langle r\rangle}(\mathbb{A})\cap\overline{\mathbb{U}\mathrm{p}}^{\mathrm{i}}_{\langle r\rangle}(\mathbb{B})=\{\dot{g},\dot{h},\dot{k}\}$. However, $\mathbb{A}\cap\mathbb{B}=\{\dot{g},\dot{h}\}$, implying $\overline{\mathbb{U}\mathrm{p}}^{\mathrm{i}}_{\langle r\rangle}(\mathbb{A}\cap\mathbb{B})=\{\dot{g},\dot{h}\}$, which means that $\overline{\mathbb{U}\mathrm{p}}^{\mathrm{i}}_{\langle r\rangle}(\mathbb{A}\cap\mathbb{B})\neq\overline{\mathbb{U}\mathrm{p}}^{\mathrm{i}}_{\langle r\rangle}(\mathbb{A})\cap\overline{\mathbb{U}\mathrm{p}}^{\mathrm{i}}_{\langle r\rangle}(\mathbb{B})$. Similarly, we can illustrate Remark 4.5 for other cases.

The results below clarify the relationships among the various forms of $\mathbb{I}_\jmath$-approximations for each $\jmath\in\{\langle r\rangle,\langle\ell\rangle,\langle\wedge\rangle,\langle\curlyvee\rangle,(r),(\ell),(\wedge),(\curlyvee)\}$.

Proposition 4.2. *Consider $(\mathcal{U},\mathcal{R},\xi_\jmath)$ as a $\jmath$-**NS**. Then, for every $\mathbb{A}\subseteq\mathcal{U}$, the next results are valid:*

1. $\underline{\mathbb{L}\mathrm{o}}^{\mathrm{i}}_{\langle\curlyvee\rangle}(\mathbb{A})\subseteq\underline{\mathbb{L}\mathrm{o}}^{\mathrm{i}}_{\langle r\rangle}(\mathbb{A})\subseteq\underline{\mathbb{L}\mathrm{o}}^{\mathrm{i}}_{\langle\wedge\rangle}(\mathbb{A})$.
2. $\underline{\mathbb{L}\mathrm{o}}^{\mathrm{i}}_{\langle\curlyvee\rangle}(\mathbb{A})\subseteq\underline{\mathbb{L}\mathrm{o}}^{\mathrm{i}}_{\langle\ell\rangle}(\mathbb{A})\subseteq\underline{\mathbb{L}\mathrm{o}}^{\mathrm{i}}_{\langle\wedge\rangle}(\mathbb{A})$
3. $\underline{\mathbb{L}\mathrm{o}}^{\mathrm{i}}_{(\curlyvee)}(\mathbb{A})\subseteq\underline{\mathbb{L}\mathrm{o}}^{\mathrm{i}}_{(r)}(\mathbb{A})\subseteq\underline{\mathbb{L}\mathrm{o}}^{\mathrm{i}}_{(\wedge)}(\mathbb{A})$.
4. $\underline{\mathbb{L}\mathrm{o}}^{\mathrm{i}}_{(\curlyvee)}(\mathbb{A})\subseteq\underline{\mathbb{L}\mathrm{o}}^{\mathrm{i}}_{(\ell)}(\mathbb{A})\subseteq\underline{\mathbb{L}\mathrm{o}}^{\mathrm{i}}_{(\wedge)}(\mathbb{A})$.
5. $\overline{\mathbb{U}\mathrm{p}}^{\mathrm{i}}_{\langle\wedge\rangle}(\mathbb{A})\subseteq\overline{\mathbb{U}\mathrm{p}}^{\mathrm{i}}_{\langle r\rangle}(\mathbb{A})\subseteq\overline{\mathbb{U}\mathrm{p}}^{\mathrm{i}}_{\langle\curlyvee\rangle}(\mathbb{A})$.
6. $\overline{\mathbb{U}\mathrm{p}}^{\mathrm{i}}_{\langle\wedge\rangle}(\mathbb{A})\subseteq\overline{\mathbb{U}\mathrm{p}}^{\mathrm{i}}_{\langle\ell\rangle}(\mathbb{A})\subseteq\overline{\mathbb{U}\mathrm{p}}^{\mathrm{i}}_{\langle\curlyvee\rangle}(\mathbb{A})$.
7. $\overline{\mathbb{U}\mathrm{p}}^{\mathrm{i}}_{(\wedge)}(\mathbb{A})\subseteq\overline{\mathbb{U}\mathrm{p}}^{\mathrm{i}}_{(r)}(\mathbb{A})\subseteq\overline{\mathbb{U}\mathrm{p}}^{\mathrm{i}}_{(\curlyvee)}(\mathbb{A})$.
8. $\overline{\mathbb{U}\mathrm{p}}^{\mathrm{i}}_{(\wedge)}(\mathbb{A})\subseteq\overline{\mathbb{U}\mathrm{p}}^{\mathrm{i}}_{(\ell)}(\mathbb{A})\subseteq\overline{\mathbb{U}\mathrm{p}}^{\mathrm{i}}_{(\curlyvee)}(\mathbb{A})$.

Proof. Using Lemma 4.1, the proof is understandable. □

Corollary 4.1. *Consider $(\mathcal{U},\mathcal{R},\xi_\jmath)$ being a $\jmath$-**NS**. Then, for every $\mathbb{A}\subseteq\mathcal{U}$, the subsequent properties are verified:*

1. $\mathbb{BN}^{\mathrm{i}}_{\langle\wedge\rangle}(\mathbb{A})\subseteq\mathbb{BN}^{\mathrm{i}}_{\langle r\rangle}(\mathbb{A})\subseteq\mathbb{BN}^{\mathrm{i}}_{\langle\curlyvee\rangle}(\mathbb{A})$.
2. $\mathbb{BN}^{\mathrm{i}}_{\langle\wedge\rangle}(\mathbb{A})\subseteq\mathbb{BN}^{\mathrm{i}}_{\langle\ell\rangle}(\mathbb{A})\subseteq\mathbb{BN}^{\mathrm{i}}_{\langle\curlyvee\rangle}(\mathbb{A})$.
3. $\mathbb{BN}^{\mathrm{i}}_{(\wedge)}(\mathbb{A})\subseteq\mathbb{BN}^{\mathrm{i}}_{(r)}(\mathbb{A})\subseteq\mathbb{BN}^{\mathrm{i}}_{(\curlyvee)}(\mathbb{A})$.
4. $\mathbb{BN}^{\mathrm{i}}_{(\wedge)}(\mathbb{A})\subseteq\mathbb{BN}^{\mathrm{i}}_{(\ell)}(\mathbb{A})\subseteq\mathbb{BN}^{\mathrm{i}}_{(\curlyvee)}(\mathbb{A})$.
5. $\mathbb{A}\mathrm{c}^{\mathrm{i}}_{\langle\curlyvee\rangle}(\mathbb{A})\leq\mathbb{A}\mathrm{c}^{\mathrm{i}}_{\langle r\rangle}(\mathbb{A})\leq\mathbb{A}\mathrm{c}^{\mathrm{i}}_{\langle\wedge\rangle}(\mathbb{A})$.
6. $\mathbb{A}\mathrm{c}^{\mathrm{i}}_{\langle\curlyvee\rangle}(\mathbb{A})\leq\mathbb{A}\mathrm{c}^{\mathrm{i}}_{\langle\ell\rangle}(\mathbb{A})\leq\mathbb{A}\mathrm{c}^{\mathrm{i}}_{\langle\wedge\rangle}(\mathbb{A})$.
7. $\mathbb{A}\mathrm{c}^{\mathrm{i}}_{(\curlyvee)}(\mathbb{A})\leq\mathbb{A}\mathrm{c}^{\mathrm{i}}_{(r)}(\mathbb{A})\leq\mathbb{A}\mathrm{c}^{\mathrm{i}}_{(\wedge)}(\mathbb{A})$.
8. $\mathbb{A}\mathrm{c}^{\mathrm{i}}_{(\curlyvee)}(\mathbb{A})\leq\mathbb{A}\mathrm{c}^{\mathrm{i}}_{(\ell)}(\mathbb{A})\leq\mathbb{A}\mathrm{c}^{\mathrm{i}}_{(\wedge)}(\mathbb{A})$

Corollary 4.2. *Consider $(\mathcal{U},\mathcal{R},\xi_\jmath)$ being a $\jmath$-**NS**. Then, for every $\mathbb{A}\subseteq\mathcal{U}$, the following hold:*

1. *$\mathbb{A}$ is $\mathbb{I}_{\langle\curlyvee\rangle}$-definable $\Rightarrow$ it is $\mathbb{I}_{\langle r\rangle}$-definable $\Rightarrow$ it is $\mathbb{I}_{\langle\wedge\rangle}$-definable.*
2. *$\mathbb{A}$ is $\mathbb{I}_{\langle\curlyvee\rangle}$-definable $\Rightarrow$ it is $\mathbb{I}_{\langle\ell\rangle}$-definable $\Rightarrow$ it is $\mathbb{I}_{\langle\wedge\rangle}$-definable.*
3. *$\mathbb{A}$ is $\mathbb{I}_{(\curlyvee)}$-definable $\Rightarrow$ it is $\mathbb{I}_{(r)}$-definable $\Rightarrow$ it is $\mathbb{I}_{(\wedge)}$-definable.*
4. *$\mathbb{A}$ is $\mathbb{I}_{(\curlyvee)}$-definable $\Rightarrow$ it is $\mathbb{I}_{(\ell)}$-definable $\Rightarrow$ it is $\mathbb{I}_{(\wedge)}$-definable.*

Remark 4.5. The converse of the foregoing outcomes is not universally valid, as depicted in the subsequent example.

Example 4.8. Suppose that $\mathcal{U}=\{\dot{g},\dot{h},\dot{k},\dot{s}\}$ and $\mathcal{R}=\{(\dot{g},\dot{g},),(\dot{g},\dot{h}),(\dot{h},\dot{h}),(\dot{h},\dot{k}),(\dot{s},\dot{k}),(\dot{k},\dot{k})\}$. Using this relation, the initial-minimal $\jmath$-neighborhoods and the initial-maximal $\jmath$-neighborhoods are obtained, as detailed in Tables 22 and 23.
Now, let $\mathbb{A}=\{\dot{g},\dot{k},\dot{s}\}$, then we compute initial-minimal approximations of them as follows:
$\underline{\mathbb{L}\mathrm{o}}^{\mathrm{i}}_{\langle r\rangle}(\mathbb{A})=\{\dot{g},\dot{k}\}$, $\overline{\mathbb{U}\mathrm{p}}^{\mathrm{i}}_{\langle r\rangle}(\mathbb{A})=\mathcal{U}$, $\mathbb{BN}^{\mathrm{i}}_{\langle r\rangle}(\mathbb{A})=\{\dot{h},\dot{s}\}$ and $\mathbb{A}\mathrm{c}^{\mathrm{i}}_{\langle r\rangle}(\mathbb{A})=\frac{1}{2}$.
$\underline{\mathbb{L}\mathrm{o}}^{\mathrm{i}}_{\langle\ell\rangle}(\mathbb{A})=\mathbb{A}$, $\overline{\mathbb{U}\mathrm{p}}^{\mathrm{i}}_{\langle\ell\rangle}(\mathbb{A})=\mathcal{U}$, $\mathbb{BN}^{\mathrm{i}}_{\langle\ell\rangle}(\mathbb{A})=\{\dot{h}\}$, and $\mathbb{A}\mathrm{c}^{\mathrm{i}}_{\langle\ell\rangle}(\mathbb{A})=\frac{3}{4}$.
$\underline{\mathbb{L}\mathrm{o}}^{\mathrm{i}}_{\langle\wedge\rangle}(\mathbb{A})=\mathbb{A}$, $\overline{\mathbb{U}\mathrm{p}}^{\mathrm{i}}_{\langle\wedge\rangle}(\mathbb{A})=\mathbb{A}$, $\mathbb{BN}^{\mathrm{i}}_{\langle\wedge\rangle}(\mathbb{A})=\varphi$, and $\mathbb{A}\mathrm{c}^{\mathrm{i}}_{\langle\wedge\rangle}(\mathbb{A})=1$.
$\underline{\mathbb{L}\mathrm{o}}^{\mathrm{i}}_{\langle\curlyvee\rangle}(\mathbb{A})=\{\dot{g},\dot{k}\}$, $\overline{\mathbb{U}\mathrm{p}}^{\mathrm{i}}_{\langle\curlyvee\rangle}(\mathbb{A})=\mathcal{U}$, $\mathbb{BN}^{\mathrm{i}}_{\langle\curlyvee\rangle}(\mathbb{A})=\{\dot{h},\dot{s}\}$, and $\mathbb{A}\mathrm{c}^{\mathrm{i}}_{\langle\curlyvee\rangle}(\mathbb{A})=\frac{1}{2}$.

In a similar way, one can compute initial-maximal approximations for other subsets.

Remark 4.6. According to Example 4.8, initial-minimal $\jmath$-approximations and initial-maximal $\jmath$-approximations are generally independent.

Theorem 4.4. *Assume that $(\mathcal{U}, \mathcal{R}, \xi_\jmath)$ constitutes a $\jmath$-**NS**, with $\mathcal{R}$ being a symmetric relation on $\mathcal{U}$. For $\mathbb{A} \subseteq \mathcal{U}$, the subsequent properties are realized:*
1. $\underline{\mathbb{L}\mathrm{o}}^{\mathrm{i}}_{\langle r\rangle}(\mathbb{A}) = \underline{\mathbb{L}\mathrm{o}}^{\mathrm{i}}_{\langle \ell\rangle}(\mathbb{A}) = \underline{\mathbb{L}\mathrm{o}}^{\mathrm{i}}_{\langle \curlyvee\rangle}(\mathbb{A}) = \underline{\mathbb{L}\mathrm{o}}^{\mathrm{i}}_{\langle \wedge\rangle}(\mathbb{A})$.
2. $\underline{\mathbb{L}\mathrm{o}}^{\mathrm{i}}_{(r)}(\mathbb{A}) = \underline{\mathbb{L}\mathrm{o}}^{\mathrm{i}}_{(\ell)}(\mathbb{A}) = \underline{\mathbb{L}\mathrm{o}}^{\mathrm{i}}_{(\curlyvee)}(\mathbb{A}) = \underline{\mathbb{L}\mathrm{o}}^{\mathrm{i}}_{(\wedge)}(\mathbb{A})$.
3. $\overline{\mathbb{U}\mathrm{p}}^{\mathrm{i}}_{\langle r\rangle}(\mathbb{A}) = \overline{\mathbb{U}\mathrm{p}}^{\mathrm{i}}_{\langle \ell\rangle}(\mathbb{A}) = \overline{\mathbb{U}\mathrm{p}}^{\mathrm{i}}_{\langle \curlyvee\rangle}(\mathbb{A}) = \overline{\mathbb{U}\mathrm{p}}^{\mathrm{i}}_{\langle \wedge\rangle}(\mathbb{A})$.
4. $\overline{\mathbb{U}\mathrm{p}}^{\mathrm{i}}_{(r)}(\mathbb{A}) = \overline{\mathbb{U}\mathrm{p}}^{\mathrm{i}}_{(\ell)}(\mathbb{A}) = \overline{\mathbb{U}\mathrm{p}}^{\mathrm{i}}_{(\curlyvee)}(\mathbb{A}) = \overline{\mathbb{U}\mathrm{p}}^{\mathrm{i}}_{(\wedge)}(\mathbb{A})$.

Proof. By applying Theorem 4.1, the result follows immediately. □

Corollary 4.3. *Assume that $(\mathcal{U}, \mathcal{R}, \xi_\jmath)$ constitutes a $\jmath$-**NS**, where $\mathcal{R}$ is a symmetric relation on $\mathcal{U}$. Then, for any subset $\mathbb{A} \subseteq \mathcal{U}$, the subsequent properties are realized:*
1. $\mathbb{BN}^{\mathrm{i}}_{\langle r\rangle}(\mathbb{A}) = \mathbb{BN}^{\mathrm{i}}_{\langle \ell\rangle}(\mathbb{A}) = \mathbb{BN}^{\mathrm{i}}_{\langle \curlyvee\rangle}(\mathbb{A}) = \mathbb{BN}^{\mathrm{i}}_{\langle \wedge\rangle}(\mathbb{A})$.
2. $\mathbb{BN}^{\mathrm{i}}_{(r)}(\mathbb{A}) = \mathbb{BN}^{\mathrm{i}}_{(\ell)}(\mathbb{A}) = \mathbb{BN}^{\mathrm{i}}_{(\curlyvee)}(\mathbb{A}) = \mathbb{BN}^{\mathrm{i}}_{(\wedge)}(\mathbb{A})$.
3. $\mathbb{A}\mathrm{c}^{\mathrm{i}}_{\langle r\rangle}(\mathbb{A}) = \mathbb{A}\mathrm{c}^{\mathrm{i}}_{\langle \ell\rangle}(\mathbb{A}) = \mathbb{A}\mathrm{c}^{\mathrm{i}}_{\langle \curlyvee\rangle}(\mathbb{A}) = \mathbb{A}\mathrm{c}^{\mathrm{i}}_{\langle \wedge\rangle}(\mathbb{A})$.
4. $\mathbb{A}\mathrm{c}^{\mathrm{i}}_{(r)}(\mathbb{A}) = \mathbb{A}\mathrm{c}^{\mathrm{i}}_{(\ell)}(\mathbb{A}) = \mathbb{A}\mathrm{c}^{\mathrm{i}}_{(\curlyvee)}(\mathbb{A}) = \mathbb{A}\mathrm{c}^{\mathrm{i}}_{(\wedge)}(\mathbb{A})$.

Theorem 4.5. *Consider $(\mathcal{U}, \mathcal{R}, \xi_\jmath)$ as a $\jmath$-**NS** with $\mathcal{R}$ being a preorder relation on $\mathcal{U}$. For $\mathbb{A} \subseteq \mathcal{U}$ and $\jmath \in \{r, \ell, \wedge, \curlyvee\}$, the subsequent statements are realized:*

1. $\underline{\mathbb{L}\mathrm{o}}^{\mathrm{i}}_{\langle \jmath\rangle}(\mathbb{A}) = \underline{\mathbb{L}\mathrm{o}}^{\mathrm{i}}_{\jmath}(\mathbb{A})$.
2. $\overline{\mathbb{U}\mathrm{p}}^{\mathrm{i}}_{\langle \jmath\rangle}(\mathbb{A}) = \overline{\mathbb{U}\mathrm{p}}^{\mathrm{i}}_{\jmath}(\mathbb{A})$.

Proof. By applying Theorem 4.3., the result follows immediately. □

Corollary 4.4. *Consider $(\mathcal{U}, \mathcal{R}, \xi_\jmath)$ is a $\jmath$-**NS** with $\mathcal{R}$ being a preorder relation on $\mathcal{U}$. For $\mathbb{A} \subseteq \mathcal{U}$ and $\jmath \in \{r, \ell, \wedge, \curlyvee\}$, the subsequent statements are realized:*

1. $\mathbb{BN}^{\mathrm{i}}_{\langle \jmath\rangle}(\mathbb{A}) = \mathbb{BN}^{\mathrm{i}}_{\jmath}(\mathbb{A})$.
2. $\mathbb{A}\mathrm{c}^{\mathrm{i}}_{\langle \jmath\rangle}(\mathbb{A}) = \mathbb{A}\mathrm{c}^{\mathrm{i}}_{\jmath}(\mathbb{A})$.

Remark 4.7. Theorem 4.5 and Corollary 4.4 demonstrate that initial-minimal $\jmath$-approximations align with the methods suggested by [32,43], specifically when a preorder relation is present, as illustrated in Example 4.6.

5. Comparative analysis of the proposed methods and prior studies

This part provides a comparative analysis of the suggested methods in relation to prior studies, specifically those by Yao [7], Allam et al. [22,23], and Dai et al. [11] The discussion highlights the effectiveness of the provided techniques in enhancing rough set approximation, emphasizing their improvements over earlier methods and demonstrating their contributions to the field. Notably, we establish that the proposed methods serve as generalizations of previous approaches and show that the accuracy measures of the proposed methods are more precise than those of earlier methods.

5.1. Comparative analysis in the case of general binary relations

We begin by comparing the proposed method with previous approaches in the framework of binary relations to determine the most effective approach for real-life applications. This comparison highlights how our method enhances and modifies these earlier techniques, particularly within this framework. We will examine where previous studies did not fully satisfy Pawlak's axioms and demonstrate how our approach successfully addresses these limitations.

Example 5.1. Presume $\mathcal{R} = \{(\dot{g},\dot{g}), (\dot{g},\dot{h}), (\dot{h},\dot{h}), (\dot{h},\dot{k}), (\dot{s},\dot{k})\}$ be a relation defined on $\mathcal{U} = \{\dot{g},\dot{h},\dot{k},\dot{s}\}$. Accordingly, the $\jmath$-neighborhoods and the initial $\jmath$-neighborhoods are presented in Tables 24 and 25, respectively.

Consequently, we compute the approximations for every subset of $\mathcal{U}$ utilizing the approaches of Yao, Allam et al., Dai et al., and the methods introduced in this work. These results are outlined in Table 26.

Table 24
$\jmath$-neighborhoods, $\jmath \in \{r, \ell, \langle r\rangle, \langle \ell\rangle, (r)\}$.

$\star$	$n_r(\star)$	$n_\ell(\star)$	$n_{\langle r\rangle}(\star)$	$n_{\langle \ell\rangle}(\star)$	$n_{(r)}(\star)$
$\dot{g}$	$\{\dot{g},\dot{h}\}$	$\{\dot{g}\}$	$\{\dot{g},\dot{h}\}$	$\{\dot{g}\}$	$\{\dot{g},\dot{h}\}$
$\dot{h}$	$\{\dot{h},\dot{k}\}$	$\{\dot{g},\dot{h}\}$	$\{\dot{h}\}$	$\{\dot{h}\}$	$\{\dot{g},\dot{h},\dot{k}\}$
$\dot{k}$	φ	$\{\dot{h},\dot{s}\}$	$\{\dot{k}\}$	φ	$\{\dot{h},\dot{k}\}$
$\dot{s}$	$\{\dot{k}\}$	φ	φ	$\{\dot{h},\dot{s}\}$	φ

Table 25
Initial $\jmath$-neighborhoods, $\jmath \in \{\langle r\rangle, \langle \ell\rangle, \langle \wedge\rangle\}$.

$\star$	$n^{i}_{\langle r\rangle}(\star)$	$n^{i}_{\langle \ell\rangle}(\star)$	$n^{i}_{\langle \wedge\rangle}(\star)$
$\dot{g}$	$\{\dot{g}\}$	$\{\dot{g}\}$	$\{\dot{g}\}$
$\dot{h}$	$\{\dot{g},\dot{h}\}$	$\{\dot{h},\dot{s}\}$	$\{\dot{h}\}$
$\dot{k}$	$\{\dot{k}\}$	$\mathcal{U}$	$\{\dot{k}\}$
$\dot{s}$	$\mathcal{U}$	$\{\dot{s}\}$	$\{\dot{s}\}$

Table 26
A comparative analysis of the approaches by Yao, Allam et al., and Dai et al. in the scope of binary relations versus the suggested technique.

$\mathbb{A}$	Yao's approach		Allam's approach		Dai's technique		Current technique	
	$\underline{\mathcal{R}}(\mathbb{A})$	$\overline{\mathcal{R}}(\mathbb{A})$	$\underline{\mathcal{R}}_{\langle r\rangle}(\mathbb{A})$	$\overline{\mathcal{R}}_{\langle r\rangle}(\mathbb{A})$	$\underline{\mathcal{R}}_{(r)}(\mathbb{A})$	$\overline{\mathcal{R}}_{(r)}(\mathbb{A})$	$\underline{\mathbb{L}\mathrm{o}}^{i}_{\langle r\rangle}(\mathbb{A})$	$\overline{\mathbb{U}\mathrm{p}}^{i}_{\langle r\rangle}(\mathbb{A})$
$\{\dot{g}\}$	$\{\dot{k}\}$	$\{\dot{g}\}$	$\{\dot{s}\}$	$\{\dot{g}\}$	$\{\dot{s}\}$	$\{\dot{g},\dot{h}\}$	$\{\dot{g}\}$	$\{\dot{g},\dot{h},\dot{s}\}$
$\{\dot{h}\}$	$\{\dot{k}\}$	$\{\dot{g},\dot{h}\}$	$\{\dot{h},\dot{s}\}$	$\{\dot{g},\dot{h}\}$	$\{\dot{s}\}$	$\{\dot{g},\dot{h},\dot{k}\}$	φ	$\{\dot{h},\dot{s}\}$
$\{\dot{k}\}$	$\{\dot{k},\dot{s}\}$	$\{\dot{h},\dot{s}\}$	$\{\dot{k},\dot{s}\}$	$\{\dot{k}\}$	$\{\dot{s}\}$	$\{\dot{h},\dot{k}\}$	$\{\dot{k}\}$	$\{\dot{k},\dot{s}\}$
$\{\dot{s}\}$	$\{\dot{k}\}$	φ	$\{\dot{s}\}$	φ	$\{\dot{s}\}$	φ	φ	$\{\dot{s}\}$
$\{\dot{g},\dot{h}\}$	$\{\dot{g},\dot{k}\}$	$\{\dot{g},\dot{h}\}$	$\{\dot{g},\dot{h},\dot{s}\}$	$\{\dot{g},\dot{h}\}$	$\{\dot{g},\dot{s}\}$	$\{\dot{g},\dot{h},\dot{k}\}$	$\{\dot{g},\dot{h}\}$	$\{\dot{g},\dot{h},\dot{s}\}$
$\{\dot{g},\dot{k}\}$	$\{\dot{k},\dot{s}\}$	$\{\dot{g},\dot{h},\dot{s}\}$	$\{\dot{k},\dot{s}\}$	$\{\dot{g},\dot{k}\}$	$\{\dot{s}\}$	$\{\dot{g},\dot{h},\dot{k}\}$	$\{\dot{g},\dot{k}\}$	$\mathcal{U}$
$\{\dot{g},\dot{s}\}$	$\{\dot{k}\}$	$\{\dot{g}\}$	$\{\dot{s}\}$	$\{\dot{g}\}$	$\{\dot{s}\}$	$\{\dot{g},\dot{h}\}$	$\{\dot{g}\}$	$\{\dot{g},\dot{h},\dot{s}\}$
$\{\dot{h},\dot{k}\}$	$\{\dot{h},\dot{k},\dot{s}\}$	$\{\dot{g},\dot{h},\dot{s}\}$	$\{\dot{h},\dot{k},\dot{s}\}$	$\{\dot{g},\dot{h},\dot{k}\}$	$\{\dot{k},\dot{s}\}$	$\{\dot{g},\dot{h},\dot{k}\}$	$\{\dot{k}\}$	$\{\dot{h},\dot{k},\dot{s}\}$
$\{\dot{h},\dot{s}\}$	$\{\dot{k}\}$	$\{\dot{g},\dot{h}\}$	$\{\dot{h},\dot{s}\}$	$\{\dot{g},\dot{h}\}$	$\{\dot{s}\}$	$\{\dot{g},\dot{h},\dot{k}\}$	φ	$\{\dot{h},\dot{s}\}$
$\{\dot{k},\dot{s}\}$	$\{\dot{k},\dot{s}\}$	$\{\dot{h},\dot{s}\}$	$\{\dot{k},\dot{s}\}$	$\{\dot{k}\}$	$\{\dot{s}\}$	$\{\dot{h},\dot{k}\}$	$\{\dot{k}\}$	$\{\dot{k},\dot{s}\}$
$\{\dot{g},\dot{h},\dot{k}\}$	$\mathcal{U}$	$\{\dot{g},\dot{h},\dot{s}\}$	$\mathcal{U}$	$\{\dot{g},\dot{h},\dot{k}\}$	$\mathcal{U}$	$\{\dot{g},\dot{h},\dot{k}\}$	$\{\dot{g},\dot{h},\dot{k}\}$	$\mathcal{U}$
$\{\dot{g},\dot{h},\dot{s}\}$	$\{\dot{g},\dot{k}\}$	$\{\dot{g},\dot{h}\}$	$\{\dot{g},\dot{h},\dot{s}\}$	$\{\dot{g},\dot{h}\}$	$\{\dot{g},\dot{s}\}$	$\{\dot{g},\dot{h},\dot{k}\}$	$\{\dot{g},\dot{h}\}$	$\{\dot{g},\dot{h},\dot{s}\}$
$\{\dot{g},\dot{k},\dot{s}\}$	$\{\dot{k},\dot{s}\}$	$\{\dot{g},\dot{h},\dot{s}\}$	$\{\dot{k},\dot{s}\}$	$\{\dot{g},\dot{k}\}$	$\{\dot{s}\}$	$\{\dot{g},\dot{h},\dot{k}\}$	$\{\dot{g},\dot{k}\}$	$\mathcal{U}$
$\{\dot{h},\dot{k},\dot{s}\}$	$\{\dot{h},\dot{k},\dot{s}\}$	$\{\dot{g},\dot{h},\dot{s}\}$	$\{\dot{h},\dot{k},\dot{s}\}$	$\{\dot{g},\dot{h},\dot{k}\}$	$\{\dot{k},\dot{s}\}$	$\{\dot{g},\dot{h},\dot{k}\}$	$\{\dot{k}\}$	$\{\dot{h},\dot{k},\dot{s}\}$
$\mathcal{U}$	$\mathcal{U}$	$\{\dot{g},\dot{h},\dot{s}\}$	$\mathcal{U}$	$\{\dot{g},\dot{h},\dot{k}\}$	$\mathcal{U}$	$\{\dot{g},\dot{h},\dot{k}\}$	$\mathcal{U}$	$\mathcal{U}$
φ	$\{\dot{k}\}$	φ	$\{\dot{s}\}$	φ	$\{\dot{s}\}$	φ	φ	φ

Remark 5.1. Table 26 reveals the following observations:

1. The methods proposed by earlier approaches (Yao, Allam et al., and Dai et al.) are not suitable for approximating rough sets in general cases. These methods cannot be universally applied because the fundamental properties of the approximations are not satisfied, which restricts the practical applications of rough set theory. For instance:
   - The lower approximation of any subset did not equal the set or its upper approximation using any preceding approaches, as illustrated by the highlighted cells in Table 26.
   - The lower approximation of $\mathcal{U}$ (resp. φ) did not equal $\mathcal{U}$ (resp. φ) or its upper approximation using any earlier techniques, as illustrated by the highlighted cells in Table 26.
2. Refer to the highlighted cells in Table 26 for specific instances. These methods, therefore, contradict rough set theory and render all subsets rough, introducing vagueness into the data.
3. Conversely, the methodologies introduced in this paper are the most effective for approximating sets in general scenarios. The initial approximations proposed in this work fulfill all the fundamental properties of Pawlak's rough sets without imposing any additional constraints or conditions. Furthermore, our approaches accurately identify exact subsets, which aids in detecting and addressing vagueness within the data.
4. Table 26 presents one of our proposed methods, specifically the ($\mathbb{I}_{\langle r\rangle}$-approximations) method, in comparison with previous studies that rely solely on right neighborhoods. This choice illustrates how our methods address limitations in existing approaches. While we focused on one of the eight methods introduced, each proposed method in this paper offers a distinct tool for rough set approximation, effectively managing issues of roughness and exactness. For instance, using the ($\mathbb{I}_{\langle \wedge\rangle}$-approximations) method, we can achieve 100% accuracy in approximating all subsets of $\mathcal{U}$. For example, consider the set $\mathbb{A} = \{t, u, v\}$; which remains undefined (a rough set) in all previous methods. However, with $\underline{\mathbb{L}\mathrm{o}}^{i}_{\langle \wedge\rangle}(\mathbb{A}) = \overline{\mathbb{U}\mathrm{p}}^{i}_{\langle \wedge\rangle}(\mathbb{A}) = \mathbb{A}$, implying $\mathbb{BN}^{i}_{\langle \wedge\rangle}(\mathbb{A}) = \varphi$ and $\mathbb{A}\mathrm{c}^{i}_{\langle \wedge\rangle}(\mathbb{A}) = 1$, this set is precisely defined with 100% accuracy.

Table 27
(r)-neighborhoods and Initial (r)-neighborhoods.

$\star$	$n_r(\star)$	$n_{(r)}(\star)$	$n^{i}_{(r)}(\star)$
$\dot{g}$	$\{\dot{g},\dot{h}\}$	$\{\dot{g},\dot{h},\dot{k}\}$	$\{\dot{g},\dot{h},\dot{k}\}$
$\dot{h}$	$\{\dot{g},\dot{h},\dot{k}\}$	$\mathcal{U}$	$\{\dot{h},\dot{k}\}$
$\dot{k}$	$\{\dot{h},\dot{k},\dot{s}\}$	$\mathcal{U}$	$\{\dot{h},\dot{k}\}$
$\dot{s}$	$\{\dot{k},\dot{s}\}$	$\{\dot{h},\dot{k},\dot{s}\}$	$\{\dot{h},\dot{k},\dot{s}\}$

Table 28
Comparison of the Dai et al. methodology with the suggested method in the general case.

$\mathbb{A}$	Approach of Dai et al.				Current method			
	$\underline{\mathcal{R}}_{(r)}(\mathbb{A})$	$\overline{\mathcal{R}}_{(r)}(\mathbb{A})$	$\mathfrak{B}_{(r)}(\mathbb{A})$	$\mu_{(r)}(\mathbb{A})$	$\underline{\mathbb{Lo}}^{i}_{(r)}(\mathbb{A})$	$\overline{\mathbb{Up}}^{i}_{(r)}(\mathbb{A})$	$\mathbb{BN}^{i}_{(r)}(\mathbb{A})$	$\mathbb{Ac}^{i}_{(r)}(\mathbb{A})$
$\{\dot{g}\}$	φ	$\{\dot{g},\dot{h},\dot{k}\}$	$\{\dot{g},\dot{h},\dot{k}\}$	0	φ	$\{\dot{g}\}$	$\{\dot{g}\}$	0
$\{\dot{h}\}$	φ	$\mathcal{U}$	$\mathcal{U}$	0	φ	$\mathcal{U}$	$\mathcal{U}$	0
$\{\dot{k}\}$	φ	$\mathcal{U}$	$\mathcal{U}$	0	φ	$\mathcal{U}$	$\mathcal{U}$	0
$\{\dot{s}\}$	φ	$\{\dot{h},\dot{k},\dot{s}\}$	$\{\dot{h},\dot{k},\dot{s}\}$	0	φ	$\{\dot{s}\}$	$\{\dot{s}\}$	0
$\{\dot{g},\dot{h}\}$	φ	$\mathcal{U}$	$\mathcal{U}$	0	φ	$\mathcal{U}$	$\mathcal{U}$	0
$\{\dot{g},\dot{k}\}$	φ	$\mathcal{U}$	$\mathcal{U}$	0	φ	$\mathcal{U}$	$\mathcal{U}$	0
$\{\dot{g},\dot{s}\}$	φ	$\mathcal{U}$	$\mathcal{U}$	0	φ	$\{\dot{g},\dot{s}\}$	$\{\dot{g},\dot{s}\}$	0
$\{\dot{h},\dot{k}\}$	φ	$\mathcal{U}$	$\mathcal{U}$	0	$\{\dot{h},\dot{k}\}$	$\mathcal{U}$	$\{\dot{g},\dot{s}\}$	1/2
$\{\dot{h},\dot{s}\}$	φ	$\mathcal{U}$	$\mathcal{U}$	0	φ	$\mathcal{U}$	$\mathcal{U}$	0
$\{\dot{k},\dot{s}\}$	φ	$\mathcal{U}$	$\mathcal{U}$	0	φ	$\mathcal{U}$	$\mathcal{U}$	0
$\{\dot{g},\dot{h},\dot{k}\}$	$\{\dot{g}\}$	$\mathcal{U}$	$\{\dot{h},\dot{k},\dot{s}\}$	1/4	$\{\dot{g},\dot{h},\dot{k}\}$	$\mathcal{U}$	$\{\dot{s}\}$	3/4
$\{\dot{g},\dot{h},\dot{s}\}$	φ	$\mathcal{U}$	$\mathcal{U}$	0	φ	$\mathcal{U}$	$\mathcal{U}$	0
$\{\dot{g},\dot{k},\dot{s}\}$	φ	$\mathcal{U}$	$\mathcal{U}$	0	φ	$\mathcal{U}$	$\mathcal{U}$	0
$\{\dot{h},\dot{k},\dot{s}\}$	$\{\dot{s}\}$	$\mathcal{U}$	$\{\dot{g},\dot{h},\dot{k}\}$	1/4	$\{\dot{h},\dot{k},\dot{s}\}$	$\mathcal{U}$	$\{\dot{g}\}$	3/4
$\mathcal{U}$	$\mathcal{U}$	$\mathcal{U}$	φ	1	$\mathcal{U}$	$\mathcal{U}$	φ	1
φ	φ	φ	φ	1	φ	φ	φ	1

5.2. Comparative analysis for select cases of binary relations

Here, we compare our proposed method with the approach introduced by Dai et al. This comparative analysis demonstrates the practical advantages of our techniques in approximating rough sets, showcasing improvements over previous methods. Similar comparisons with other studies can also be conducted using our method to identify the most effective approach for real-world applications. This analysis highlights the importance of our techniques in enhancing approximation accuracy and effectiveness, reinforcing their relevance in diverse practical contexts.

Theorem 5.1. *Let $(\mathcal{U},\mathcal{R},\xi_{\jmath})$ be a $\jmath$-**NS** in which $\mathcal{R}$ is a similarity relation on $\mathcal{U}$, and let $\mathbb{A}\subseteq\mathcal{U}$. Hence, for every $\jmath\in\{r,\ell,\wedge,\vee\}$, the following statements hold:*

1. $\underline{\mathcal{R}}_{(\jmath)}(\mathbb{A})\subseteq\underline{\mathbb{Lo}}^{i}_{(\jmath)}(\mathbb{A})$.
2. $\overline{\mathbb{Up}}^{i}_{(\jmath)}(\mathbb{A})\subseteq\overline{\mathcal{R}}_{(\jmath)}(\mathbb{A})$.
3. $\mathbb{BN}^{i}_{(\jmath)}(\mathbb{A})\subseteq\mathfrak{B}_{(\jmath)}(\mathbb{A})$.
4. $\mu_{(\jmath)}(\mathbb{A})\leq\mathbb{Ac}^{i}_{(\jmath)}(\mathbb{A})$.

Proof. The proof is an immediate consequence of Theorem 4.2. □

Remark 5.2. The converse of Theorem 5.1 is generally incorrect, as demonstrated in Example 5.2.

Example 5.2. Continued of Example 4.6, where the relation $\mathcal{R}$ is a similarity relation. Thus, we get (r)-neighborhoods and initial (r)-neighborhoods as showed in Table 27:

Accordingly, we compute the approximations for every subset of $\mathcal{U}$ using both the methodologies proposed by Dai et al. and the techniques presented in this paper, as summarized in Table 28.

6. Generalized nano-topology and its practical implications

In the current part, we investigate a practical medical application that illustrates the efficacy of the methodologies and techniques proposed in this paper. Specifically, we highlight how these methods contribute to enhancing decision-making processes in the clinical diagnosis of Covid-19's variants. By applying these techniques to a medical dataset with pre-determined medical decisions, we demonstrate that the accuracy of our methods surpasses previous approaches, with some achieving accuracy rates as high as

100%. For specific categories within one of the methods, the accuracy metric reaches 100%, while for others, it is slightly less but still outperforms existing methods by revealing hidden patterns in the data more effectively.

Another key direction explored in this research involves expanding the concept of nano-topology [42] within the framework of general approximation sets. El-Bably and Abo-Tabl [44] previously developed a nano-topology based on approximations of general approximation sets, outlining the necessary conditions for their formation. Building upon their work, we present the details of constructing a generalized nano-topology using the approximation techniques proposed by Yao [7], Allam et al. [22,23], Dai et al. [11], El-Sayed et al. [32], and Abu-Gdairi [43]. Using these insights, we aim to define new general nano-topologies and explore their practical applications in real-world scenarios.

To further illustrate the effectiveness of these techniques, we present a practical medical application focused on evaluating impact factors in Covid-19 infections [41]. Additionally, we provide detailed explanations of the foundational models, decision-making processes, and algorithmic implementation of this innovative approach within the medical context.

6.1. Extensions of generalized nano-topology

Definition 6.1. [44] Let $\mathcal{U}$ be a universe. Assume that $\underline{\mathcal{L}}(\mathbb{A})$ and $\overline{\mathfrak{U}}(\mathbb{A})$ represent the lower and upper approximations of a subset $\mathbb{A} \subseteq \mathcal{U}$, respectively. Consider the collection

$$\mathcal{GNT} = \{\mathcal{U}, \varphi, \underline{\mathcal{L}}(\mathbb{A}), \overline{\mathfrak{U}}(\mathbb{A}), \mathcal{B}nd(\mathbb{A})\},$$

where $\mathcal{B}nd(\mathbb{A})$ denotes the boundary area of $\mathbb{A}$. This collection constitutes a topology on $\mathcal{U}$ if the approximations $\underline{\mathcal{L}}(\mathbb{A})$ and $\overline{\mathfrak{U}}(\mathbb{A})$ uphold to the characteristics of Pawlak's theory (L1–L8) and (U1–U8), respectively. As a result, this topology is termed a *generalized nano-topology* or, alternatively, a $\mathcal{GN}$-topology, as it is derived from the generalized rough approximations of $\mathbb{A} \subseteq \mathcal{U}$.

Note: The previous definition specifies the conditions required for constructing a nano-topology using generalized rough sets. The ensuing results describe the procedure for generating a $\mathcal{GN}$-topology based on the methodologies introduced by [7,11,22,23] along with the proposed approximations.

Theorem 6.1. *[44] Let $\mathcal{R}$ be a binary relation on $\mathcal{U}$, and let $\mathbb{A} \subseteq \mathcal{U}$. The family of sets given by:*

$$\mathcal{GNT} = \{\mathcal{U}, \varphi, \underline{\mathcal{L}}(\mathbb{A}), \overline{\mathfrak{U}}(\mathbb{A}), \mathcal{B}nd(\mathbb{A})\}$$

constitutes a $\mathcal{GN}$-topology on $\mathcal{U}$ if and only if the following conditions hold:

1. ***For the technique of Yao:** $\mathcal{R}$ must be a preorder.*
2. ***For the technique of Dai et al.:** $\mathcal{R}$ must be a similarity relation.*
3. ***For the technique of Allam et al.:** $\mathcal{R}$ must be a reflexive relation.*

Theorem 6.2. *Given $(\mathcal{U}, \mathcal{R}, \xi_j)$ is a j-**NS**, where $\mathcal{R}$ is a binary relation on $\mathcal{U}$. For any subset $\mathbb{A} \subseteq \mathcal{U}$, the collection:*

$$\mathcal{GNT} = \{\mathcal{U}, \emptyset, \underline{\mathbb{L}\mathrm{o}}_j^{\mathrm{i}}(\mathbb{A}), \overline{\mathbb{U}\mathrm{p}}_j^{\mathrm{i}}(\mathbb{A}), \mathbb{BN}_j^{\mathrm{i}}(\mathbb{A})\}$$

constitutes a $\mathcal{GN}$-topology on $\mathcal{U}$ for each $j \in \mathcal{J}$.

Proof. By applying Proposition 4.1, the outcome follows immediately □

6.2. Data set of medical diagnosis of Covid-19 variants

In this section, we generate and analyze rules to identify Covid-19 patients infected with specific variants. The data was collected from several patients diagnosed with the Alpha, Delta, or Omicron variants. By analyzing common symptoms across ten patients, we applied a rough set approach to classify each variant based on its characteristic symptoms.
The symptomatic manifestations corresponding to each variant are enumerated below:

- **Variant of Alpha:** (**HE**) = Headache, (**SB**) = Shortness of Breath, (**DC**) = Dry Cough, (**ST**) = Sore Throat, (**BP**) = Body Pain, (**FE**) = Fever, (**CP**) = Chest Pain.
- **Variant of Delta:** (**CO**) = Cough, (**BP**) = Body Pain, (**FA**) = Fatigue, (**SB**) = Shortness of Breath, (**CP**) = Chest Pain, (**ST**) = Sore Throat, (**HE**) = Headache, (**FE**) = Fever, (**LoT**) = Loss of Taste, (**MY**) = Myalgias, (**LoS**) = Loss of Smell, (**RH**) = Rhinorrhea.
- **Variant of Omicron:** (**WE**) = Weakness, (**LBP**) = Lower Back Pain, (**FA**) = Fatigue, (**HE**) = Headache, (**FE**) = Fever, (**BA**) = Body Ache, (**CO**) = Cough, (**CL**) = Cold, (**LoA**) = Loss of Appetite, (**NS**) = Night Sweats, (**SN**) = Sneezing.

Data for ten patients has been meticulously organized into a table (adapted from [41]), with each row representing an individual patient and each column denoting a specific symptom. Table 29 presents comprehensive details on patients diagnosed with the Alpha variant of Covid-19.

Table 29
[41] Dataset framework characterizing persons afflicted with diverse Covid-19 variants.

Person	Frequent Clinical Indications			Expanded Clinical Manifestations					Covid-19 Diagnosis
	FE	**LoT/S**	**FA**	**BA**	**LBP**	**NS**	**SN**	**Lo/A**	
p_1	High	∘	•	•	∘	•	∘	•	Alpha
p_2	High	∘	•	•	•	•	∘	•	Omicron
p_3	High	∘	∘	•	∘	∘	∘	∘	Not infected
p_4	Normal	∘	∘	∘	∘	∘	•	∘	Not infected
p_5	Normal	•	∘	•	∘	•	∘	•	Delta
p_6	High	•	∘	•	∘	∘	∘	∘	Delta
p_7	High	•	•	•	∘	∘	∘	∘	Delta
p_8	Normal	∘	∘	•	∘	∘	∘	∘	Not infected
p_9	High	∘	•	•	•	•	∘	•	Omicron
p_{10}	High	•	•	•	∘	∘	∘	∘	Delta

Table 30
Patient Information System for Different Covid-19 Variants.

Person	Clinical Manifestations								Covid-19 Diagnosis
	$\mathbb{A}_1$	$\mathbb{A}_2$	$\mathbb{A}_3$	$\mathbb{A}_4$	$\mathbb{A}_5$	$\mathbb{A}_6$	$\mathbb{A}_7$	$\mathbb{A}_8$	
p_1	$\mathbb{H}$	∘	•	•	∘	•	∘	•	Alpha
p_2	$\mathbb{H}$	∘	•	•	•	•	∘	•	Omicron
p_3	$\mathbb{H}$	∘	∘	•	∘	∘	∘	∘	Not infected
p_4	$\mathbb{N}$	∘	∘	∘	∘	∘	•	∘	Not infected
p_5	$\mathbb{N}$	•	∘	•	∘	•	∘	•	Delta
p_6	$\mathbb{H}$	•	∘	•	∘	∘	∘	∘	Delta
p_7	$\mathbb{H}$	•	•	•	∘	∘	∘	∘	Delta
p_8	$\mathbb{N}$	∘	∘	•	∘	∘	∘	∘	Not infected

Table 31
$\jmath$-neighborhoods, $\jmath \in \{\langle r\rangle, \langle \ell\rangle, (r), (\ell)\}$.

$\star$	$n_{\langle r\rangle}(\star)$	$n_{\langle \ell\rangle}(\star)$	$n_{(r)}(\star)$	$n_{(\ell)}(\star)$
p_1	$\{p_1, p_2\}$	$\{p_1, p_3, p_8\}$	$\mathbb{P}-\{p_4\}$	$\{p_1, p_2, p_3, p_8\}$
p_2	$\{p_2\}$	$\{p_1, p_2, p_3, p_8\}$	$\mathbb{P}-\{p_4\}$	$\{p_1, p_2, p_3, p_8\}$
p_3	$\{p_1, p_2, p_3, p_6, p_7\}$	$\{p_3, p_8\}$	$\mathbb{P}-\{p_4\}$	$\{p_1, p_2, p_3, p_6, p_7, p_8\}$
p_4	$\{p_4\}$	$\{p_4\}$	$\{p_4\}$	$\{p_4\}$
p_5	$\{p_5\}$	$\{p_5, p_8\}$	$\mathbb{P}-\{p_4\}$	$\{p_5, p_8\}$
p_6	$\{p_6, p_7\}$	$\{p_3, p_6, p_8\}$	$\mathbb{P}-\{p_4\}$	$\{p_3, p_6, p_7, p_8\}$
p_7	$\{p_7\}$	$\{p_3, p_6, p_7, p_8\}$	$\mathbb{P}-\{p_4\}$	$\{p_3, p_6, p_7, p_8\}$
p_8	$\mathbb{P}-\{p_4\}$	$\{p_8\}$	$\mathbb{P}-\{p_4\}$	$\mathbb{P}-\{p_4\}$

Note that in Tables 29 and 30, • indicates that a person has the symptom, and ∘ indicates that a person does not have it.
In Table 29, we observe that the patients p_2 and p_9 (resp. p_7 and p_{10}) are indiscernible. Therefore, we exclude them and derive the updated information system as shown in Table 30.

In this table, the first column ($\mathbb{A}_1$) uses $\mathbb{H}$ to indicate "High" and $\mathbb{N}$ to indicate "Normal".
Thus, the set of all patients is $\mathbb{P} = \{p_k : k = 1, 2, 3, \ldots, 8\}$ and the set of all attributes is $\mathbb{AT} = \{\mathbb{A}_k : k = 1, 2, 3, \ldots, 8\}$. Now, we proceed to identify the symptoms for each patient as follows:

$\mathcal{V}(p_1) = \{\mathbb{A}_1, \mathbb{A}_3, \mathbb{A}_4, \mathbb{A}_6, \mathbb{A}_8\}$, $\mathcal{V}(p_2) = \{\mathbb{A}_1, \mathbb{A}_3, \mathbb{A}_4, \mathbb{A}_5, \mathbb{A}_6, \mathbb{A}_8\}$, $\mathcal{V}(p_3) = \{\mathbb{A}_1, \mathbb{A}_4\}$, $\mathcal{V}(p_4) = \{\mathbb{A}_7\}$, $\mathcal{V}(p_5) = \{\mathbb{A}_2, \mathbb{A}_4, \mathbb{A}_6, \mathbb{A}_8\}$, $\mathcal{V}(p_6) = \{\mathbb{A}_1, \mathbb{A}_2, \mathbb{A}_4\}$, $\mathcal{V}(p_7) = \{\mathbb{A}_1, \mathbb{A}_2, \mathbb{A}_3, \mathbb{A}_4\}$, and $\mathcal{V}(p_8) = \{\mathbb{A}_4\}$.
Now, we construct the right neighborhoods using the following relation, which is related to the nature of the studied problem: $p_i \mathcal{R}\, p_j \iff \mathcal{V}(p_i) \subseteq \mathcal{V}(p_j)$, where $i, j \in \{1, 2, 3, \ldots, 8\}$.
Note: The relationship in each case is defined based on the expert's specifications. Consequently, the relation for all attributes is established as follows:

$\mathcal{R} = \Delta \bigcup \{(p_1, p_2), (p_3, p_1), (p_3, p_2), (p_3, p_6), (p_3, p_7), (p_6, p_7), (p_8, p_1), (p_8, p_2), (p_8, p_3), (p_8, p_5), (p_8, p_6), (p_8, p_7)\}$, where Δ is an identity relation.
Therefore, we obtain the following neighborhoods as illustrated in Tables 31 and 32:

Note: Since $\mathcal{R}$ is a preorder relation, then by Theorem 2.6, the $\jmath$- neighborhoods and minimal $\jmath$- neighborhoods are equal. So, $n_r(x) = n_{\langle r\rangle}(x)$ and $n_\ell(x) = n_{\langle \ell\rangle}(x)$, for every element in $\mathbb{P}$. Thus, we construct Table 32, which contains $\mathbb{I}_\jmath$- neighborhoods for every element in $\mathbb{P}$.

6.3. First application: decision-making using $\mathbb{I}_\jmath$-approximations to identify exactness and roughness

According to the medical decision table (Table 30), the group of patients infected with Covid-19 is represented as $\mathcal{S} = \{p_1, p_2, p_5, p_6, p_7\}$ while the group of patients who are not infected is $\mathcal{T} = \{p_3, p_4, p_8\}$. The infected group is further divided into the following subsets:

Table 32
$\mathbb{I}_{\jmath}$-neighborhoods, $\jmath \in \{\langle r\rangle, \langle \ell\rangle, \langle \wedge\rangle, (\wedge)\}$.

$\star$	$n^{i}_{\langle r\rangle}(\star)$	$n^{i}_{\langle \ell\rangle}(\star)$	$n^{i}_{\langle \wedge\rangle}(\star)$	$n^{i}_{(\wedge)}(\star)$
p_1	$\{p_1, p_3, p_8\}$	$\{p_1, p_2\}$	$\{p_1\}$	$\{p_1, p_2, p_3, p_8\}$
p_2	$\{p_1, p_2, p_3, p_8\}$	$\{p_2\}$	$\{p_2\}$	$\{p_1, p_2, p_3, p_8\}$
p_3	$\{p_3, p_8\}$	$\{p_1, p_2, p_3, p_6, p_7\}$	$\{p_3\}$	$\{p_3, p_8\}$
p_4	$\{p_4\}$	$\{p_4\}$	$\{p_4\}$	$\{p_4\}$
p_5	$\{p_5, p_8\}$	$\{p_5\}$	$\{p_5\}$	$\{p_5, p_8\}$
p_6	$\{p_3, p_6, p_8\}$	$\{p_6, p_7\}$	$\{p_6\}$	$\{p_3, p_6, p_7, p_8\}$
p_7	$\{p_3, p_6, p_7, p_8\}$	$\{p_7\}$	$\{p_7\}$	$\{p_3, p_6, p_7, p_8\}$
p_8	$\{p_8\}$	$\mathbb{P} - \{p_4\}$	$\{p_8\}$	$\{p_8\}$

- $\mathcal{S}_1 = \{p_1\}$, representing patients infected with the Alpha variant.
- $\mathcal{S}_2 = \{p_2\}$, representing patients infected with the Omicron variant.
- $\mathcal{S}_3 = \{p_5, p_6, p_7\}$, representing patients infected with the Alpha variant.

We will compute the rough approximations for all the above sets using the previously established methods and the proposed $\mathbb{I}_{\jmath}$-approximations discussed in this article. Furthermore, we provide a comparative analysis to evaluate the accuracy degrees of these previous approaches against our suggested methods.

First, according to Table 30, the constructed relation $\mathcal{R}$ is a preorder. So, by using Theorem 3.1, Yao-approximations and Abd El-Monsef et al. for cases $\jmath = \{r, \ell\}$ are equal, and the methods of Allam et al. and Abd El-Monsef et al. for cases $\jmath = \{\langle r\rangle, \langle \ell\rangle\}$ are equal. Consequently, we compare these methods with the proposed $\mathbb{I}_{\jmath}$-approximations in a case $\jmath = \langle \wedge\rangle$, and similarly for the other cases. We display that the proposed methods, $\mathbb{I}_{\jmath}$-approximations provide greater accuracy compared to other methods.

- **Yao, Allam et al., and Abd El-Monsef et al. approaches:**

According to Theorem 2.6, the $\jmath$-neighborhoods and minimal $\jmath$- neighborhoods, for every member in $\mathbb{P}$, they are equal. Therefore, the techniques proposed by Yao, Allam et al., and Abd El-Monsef et al. are identical in this application. Consequently, their approximations and accuracy measures for the set $\mathcal{S} = \{p_1, p_2, p_5, p_6, p_7\}$, by using Table 30, are as follows:

	$\mathcal{S}$	$\mathcal{T}$	$\mathcal{S}_1$	$\mathcal{S}_2$	$\mathcal{S}_3$
Lower approximation	$\mathcal{S}$	$\{p_4\}$	φ	$\{p_2\}$	$\{p_5, p_6, p_7\}$
Upper approximation	$\mathbb{P} - \{p_4\}$	$\mathcal{T}$	$\{p_1, p_3, p_8\}$	$\{p_1, p_2, p_3, p_8\}$	$\{p_3, p_5, p_6, p_7, p_8\}$
Accuracy	$\frac{5}{7}$	$\frac{1}{3}$	0	$\frac{1}{4}$	$\frac{3}{5}$

- **Dai et al. approach:**

By using Table 30, the (r)-approximations given in [11] are:

	$\mathcal{S}$	$\mathcal{T}$	$\mathcal{S}_1$	$\mathcal{S}_2$	$\mathcal{S}_3$
Lower approximation	φ	$\{p_4\}$	φ	φ	φ
Upper approximation	$\mathbb{P} - \{p_4\}$	$\mathbb{P}$	$\mathbb{P} - \{p_4\}$	$\mathbb{P} - \{p_4\}$	$\mathbb{P} - \{p_4\}$
Accuracy	0	$\frac{1}{8}$	0	0	0

- **Current method (1):**

Utilizing the proposed methods, the $\mathbb{I}_{(\wedge)}$-approximations, as determined from Table 31, are as follows:

$\mathcal{X}$	$\mathcal{S}$	$\mathcal{T}$	$\mathcal{S}_1$	$\mathcal{S}_2$	$\mathcal{S}_3$
$\underline{\mathbb{L}o}^{i}_{(\wedge)}(\mathcal{X})$	φ	$\mathcal{T}$	φ	φ	φ
$\overline{\mathbb{U}p}^{i}_{(\wedge)}(\mathcal{X})$	$\mathbb{P} - \{p_3, p_4, p_8\}$	$\mathbb{P}$	$\{p_1, p_2\}$	$\{p_1, p_2\}$	$\mathcal{S}_3$
$\mathbb{A}c^{i}_{(\wedge)}(\mathcal{X})$	0	$\frac{3}{8}$	0	0	0

- **Current method (2):**

By using the proposed methods, $\mathbb{I}_{\langle \wedge\rangle}$-approximations, by using Table 32 are:

$\mathcal{X}$	$\mathcal{S}$	$\mathcal{T}$	$\mathcal{S}_1$	$\mathcal{S}_2$	$\mathcal{S}_3$
$\underline{\mathbb{L}o}^{i}_{\langle \wedge\rangle}(\mathcal{X})$	$\mathcal{S}$	$\mathcal{T}$	$\mathcal{S}_1$	$\mathcal{S}_1$	$\mathcal{S}_3$
$\overline{\mathbb{U}p}^{i}_{\langle \wedge\rangle}(\mathcal{X})$	$\mathcal{S}$	$\mathcal{T}$	$\mathcal{S}_1$	$\mathcal{S}_1$	$\mathcal{S}_3$
$\mathbb{A}c^{i}_{\langle \wedge\rangle}(\mathcal{X})$	1	1	1	1	1

Concluding remark: Building on the comparative analysis presented above, we observe the following key points:

1. **Current Method (1)** (which relies on maximal approximation) generalizes the approaches of Dai et al. and is more accurate. For example, the accuracy degree of the given set $\mathcal{T}$ is $\frac{1}{8}$, whereas in the current method (1), we have $\mathbb{A}c^{i}_{\langle\wedge\rangle}(\mathcal{T})=\frac{3}{8}$. Furthermore, the lower and upper approximations of $\mathcal{S}$, as offered by Dai et al., are φ and $\mathbb{P}-\{p_4\}$, respectively. This means that the boundary region is $\mathbb{P}-\{p_4\}$, implying that p_4 is the only person identified as not infected, which contradicts Table 30, which states that the uninfected persons are p_3, p_4, and p_8.
2. On the other hand, the lower and upper approximations of $\mathcal{S}$ building on the current method (1) are: φ and $\mathbb{P}-\{p_3,p_4,p_8\}$, respectively. This implies that the boundary region in this issue is $\mathbb{P}-\{p_3,p_4,p_8\}$, indicating that the persons p_3, p_4, and p_8 are not infected. Therefore, the current method (1) successfully resolves the ambiguity found in Dai et al.'s technique and aligns accurately with the doctor's diagnoses.
3. Moreover, according to **Current Method (2)**, the accuracy measure for all specified sets is 100%, matching the doctor's decisions exactly. Therefore, this method is extremely useful for decision-making in medical diagnoses.
4. In conclusion, the proposed $\mathbb{I}_{\langle\wedge\rangle}$-approximations method is the most accurate technique for approximating rough sets, yielding the highest accuracy. As a result, this method will be highly valuable in medical diagnosis decision-making.

6.4. Second application: attribute reduction using $\mathbb{I}_j$-approximations criteria for determining core attributes to diagnose Covid-19

At this stage, we apply the proposed techniques to perform a topological reduction of the attributes listed in Table 30. The objective is to designate the most meaningful risk factors associated with Covid-19. This analysis involves utilizing the concept of "generalized nano-topology" to determine these factors by extracting key attributes through topological reduction. The process will be carried out for all patients represented in Table 30, with a specific focus on the first group: patients infected with Covid-19, denoted as $\mathcal{S}=\{p_1,p_2,p_5,p_6,p_7\}$.

*From **Subsection 6.3**, we compute the generalized nano-topology ($\mathcal{GNT}$) generated by $\mathcal{S}$ using all the attributes or symptoms listed in Table 30 as follows:*

The $\mathbb{I}_{\langle\wedge\rangle}$-neighborhoods for the Covid-19 group are:
$\underline{\mathbb{L}o}^{i}_{\langle\wedge\rangle}(\mathcal{S})=\overline{\mathbb{U}p}^{i}_{\langle\wedge\rangle}(\mathcal{S})=\mathcal{S}$, and hence $\mathcal{GNT}=\{\mathbb{P},\ \varphi,\ \mathcal{S}\}$.

Attribute Analysis and Reduction
Case 1: Removing Attribute ($\mathbb{A}_1$)
When removing attribute $\mathbb{A}_1$, the symptoms for each patient become:
$\mathcal{V}(p_1)=\{\mathbb{A}_3,\mathbb{A}_4,\mathbb{A}_6,\mathbb{A}_8\}$, $\mathcal{V}(p_2)=\{\mathbb{A}_3,\mathbb{A}_4,\mathbb{A}_5,\mathbb{A}_6,\mathbb{A}_8\}$, $\mathcal{V}(p_3)=\{\mathbb{A}_4\}$, $\mathcal{V}(p_4)=\{\mathbb{A}_7\}$, $\mathcal{V}(p_5)=\{\mathbb{A}_2,\ \mathbb{A}_4,\mathbb{A}_6,\mathbb{A}_8\}$, $\mathcal{V}(p_6)=\{\mathbb{A}_2,\mathbb{A}_4\}$, $\mathcal{V}(p_7)=\{\mathbb{A}_2,\mathbb{A}_3,\mathbb{A}_4\}$, and $\mathcal{V}(p_8)=\{\mathbb{A}_4\}$.
The relation $m_i\mathcal{R}\ m_j \iff \mathcal{V}(m_i)\subseteq\mathcal{V}(m_j)$ gives:
$\mathcal{R}=\Delta\bigcup\{(p_1,p_2),\ (p_3,p_1),(p_3,p_2),(p_3,p_5),(p_3,p_6),(p_3,p_7),\ (p_3,p_8),(p_6,p_5),(p_6,p_7),\ (p_8,p_1),\ (p_8,p_2),(p_8,p_3),(p_8,p_5),(p_8,p_6),(p_8,p_7)$ where Δ is an identity relation.
Therefore, the $\mathbb{I}_{\langle\wedge\rangle}$-neighborhoods of all members in $\mathbb{P}$ become: $n^{i}_{\langle\wedge\rangle}(p_1)=\{p_1\}$, $n^{i}_{\langle\wedge\rangle}(p_2)=\{p_2\}$, $n^{i}_{\langle\wedge\rangle}(p_4)=\{p_4\}$, $n^{i}_{\langle\wedge\rangle}(p_5)=\{p_5\}$, $n^{i}_{\langle\wedge\rangle}(p_6)=\{p_6\}$, $n^{i}_{\langle\wedge\rangle}(p_7)=\{p_7\}$, and $n^{i}_{\langle\wedge\rangle}(p_3)=n^{i}_{\langle\wedge\rangle}(p_8)=\{p_3,p_8\}$.
The $\mathbb{I}_{\langle\wedge\rangle}$- approximations of $\mathcal{S}$ in this case are:
$\underline{\mathbb{L}o}^{i}_{\langle\wedge\rangle}(\mathcal{S})=\overline{\mathbb{U}p}^{i}_{\langle\wedge\rangle}(\mathcal{S})=\mathcal{S}$, and hence $\mathcal{GNT}^{\mathbb{A}_1}=\{\mathbb{P},\ \varphi,\ \mathcal{S}\}=\mathcal{GNT}$.

Case 2: Removing Attribute ($\mathbb{A}_2$)
Using a similar method as in **Case 1**, the $\mathbb{I}_{\langle\wedge\rangle}$- approximations of $\mathcal{S}$ in this case are
$\underline{\mathbb{L}o}^{i}_{\langle\wedge\rangle}(\mathcal{S})=\{p_1,p_2,p_5,p_7\}$ and $\overline{\mathbb{U}p}^{i}_{\langle\wedge\rangle}(\mathcal{S})=\{p_1,p_2,p_3,p_5,p_6,p_7\}$. Thus, we obtain

$$\mathcal{GNT}^{\mathbb{A}_2}=\{\mathbb{P},\ \varphi,\{p_3,p_6\},\{p_1,p_2,p_5,p_7\},\{p_1,p_2,p_3,p_5,p_6,p_7\}\}\neq\mathcal{GNT}.$$

Case 3: Removing Attribute ($\mathbb{A}_3$)
Using a similar method as in **Case 1**, the $\mathbb{I}_{\langle\wedge\rangle}$- approximations of $\mathcal{S}$ in this case are
$\underline{\mathbb{L}o}^{i}_{\langle\wedge\rangle}(\mathcal{S})=\overline{\mathbb{U}p}^{i}_{\langle\wedge\rangle}(\mathcal{S})=\mathcal{S}$, and hence $\mathcal{GNT}^{\mathbb{A}_3}=\{\mathbb{P},\ \varphi,\ \mathcal{S}\}=\mathcal{GNT}$.

Case 4: Removing Attribute ($\mathbb{A}_4$)
Using a similar method as in **Case 1**, the $\mathbb{I}_{\langle\wedge\rangle}$- approximations of $\mathcal{S}$ in this case are
$\underline{\mathbb{L}o}^{i}_{\langle\wedge\rangle}(\mathcal{S})=\overline{\mathbb{U}p}^{i}_{\langle\wedge\rangle}(\mathcal{S})=\mathcal{S}$, and hence $\mathcal{GNT}^{\mathbb{A}_4}=\{\mathbb{P},\ \varphi,\ \mathcal{S}\}=\mathcal{GNT}$.

Case 5: Removing Attribute ($\mathbb{A}_5$)
Using a similar method as in **Case 1**, the $\mathbb{I}_{\langle\wedge\rangle}$- approximations of $\mathcal{S}$ in this case are

$\underline{\mathbb{L}\mathrm{o}}^{\mathrm{i}}_{\langle\wedge\rangle}(\mathcal{S}) = \overline{\mathbb{U}\mathrm{p}}^{\mathrm{i}}_{\langle\wedge\rangle}(\mathcal{S}) = \mathcal{S}$, and hence $\mathcal{GNT}^{\mathbb{A}_6} = \{\mathbb{P}, \varphi, \mathcal{S}\} = \mathcal{GNT}$.

Case 6: Removing Attribute ($\mathbb{A}_6$)

Using a similar method as in **Case 1**, the $\mathbb{I}_{\langle\wedge\rangle}$- approximations of $\mathcal{S}$ in this case are

$\underline{\mathbb{L}\mathrm{o}}^{\mathrm{i}}_{\langle\wedge\rangle}(\mathcal{S}) = \overline{\mathbb{U}\mathrm{p}}^{\mathrm{i}}_{\langle\wedge\rangle}(\mathcal{S}) = \mathcal{S}$, and hence $\mathcal{GNT}^{\mathbb{A}_6} = \{\mathbb{P}, \varphi, \mathcal{S}\} = \mathcal{GNT}$.

Case 7: Removing Attribute ($\mathbb{A}_7$)

Using a similar method as in **Case 1**, the $\mathbb{I}_{\langle\wedge\rangle}$- approximations of $\mathcal{S}$ in this case are

$\underline{\mathbb{L}\mathrm{o}}^{\mathrm{i}}_{\langle\wedge\rangle}(\mathcal{S}) = \overline{\mathbb{U}\mathrm{p}}^{\mathrm{i}}_{\langle\wedge\rangle}(\mathcal{S}) = \mathcal{S}$, and hence $\mathcal{GNT}^{\mathbb{A}_7} = \{\mathbb{P}, \varphi, \mathcal{S}\} = \mathcal{GNT}$.

Case 8: Removing Attribute ($\mathbb{A}_8$)

Using a similar method as in **Case 1**, the $\mathbb{I}_{\langle\wedge\rangle}$- approximations of $\mathcal{S}$ in this case are

$\underline{\mathbb{L}\mathrm{o}}^{\mathrm{i}}_{\langle\wedge\rangle}(\mathcal{S}) = \overline{\mathbb{U}\mathrm{p}}^{\mathrm{i}}_{\langle\wedge\rangle}(\mathcal{S}) = \mathcal{S}$, and hence $\mathcal{GNT}^{\mathbb{A}_8} = \{\mathbb{P}, \varphi, \mathcal{S}\} = \mathcal{GNT}$.

Therefore, the attribute $\{\mathbb{A}_2\}$ is indispensable, while the remaining attributes can be omitted. Consequently, $\{\mathbb{A}_2\}$ represents the reduced set of attributes for the information system presented in Table 30, highlighting the essential features for diagnosing Covid-19, which are the critical factors influencing Covid-19. Hence, CORE = $\{\mathbb{A}_2\}$. This means that the common attribute or symptom among all infected individuals is $\mathbb{A}_2$, which thus forms a key factor for diagnosing the patient. If this symptom is not present, a Covid-19 infection can be ruled out, whereas if the symptom is present, further tests can be conducted to confirm the infection. It is worth noting here that identifying a single common symptom resulted from the relationship used in data analysis, which is the containment relationship. If the medical expert were to use a different relationship, the results would certainly differ. From the above, it is evident how mathematical methods can be beneficial in medical diagnostics.

7. Algorithms and frameworks

This section introduces two algorithms, Algorithm 1 and Algorithm 2, which are designed to address decision-making challenges in the diagnosis of Covid-19 disease. Specifically, Algorithm 2 serves as a framework that applies the proposed techniques to identify essential core attributes required for an accurate diagnosis. Both algorithms are evaluated using simulated data and compared with existing methods, providing practical and implementable solutions, especially for environments like MATLAB. Although MATLAB is used as the primary example in this paper, it is important to note that the algorithms are versatile and can be implemented in various programming platforms, including Python, R, and Julia. These languages offer comprehensive tools and libraries for data analysis and algorithm development, ensuring the proposed methods can be adapted for a broad range of applications.

The following analysis discusses the effectiveness, efficiency, and scalability of both algorithms.

Algorithm 1 Analysis

Effectiveness: The core objective of Algorithm 1 is to distinguish between exact and rough sets by calculating binary relations and $\mathbb{I}_J$-approximations of $\mathbb{I}_J$-neighborhoods. The algorithm performs well in making accurate decisions for exact and rough set classifications, assuming that precise data and definitions are provided. It iteratively computes neighborhoods using established definitions, ensuring that the checks for exactness and roughness, through $\mathbb{I}_J$-accuracy ($\mathbb{A}\mathrm{c}^{\mathrm{i}}_J(\mathbb{S})$) calculations, contribute to its overall effectiveness.
Efficiency: The efficiency of Algorithm 1 is influenced by dataset size, as the complexity of binary relations and neighborhood calculations grows with data volume. By implementing iterative recalculations for each symptom using programming languages like Python or R, the algorithm is equipped to manage computationally intensive tasks. Utilizing optimized data structures or memorization techniques could further enhance performance by reducing redundant computations and improving runtime efficiency.
Scalability: Despite its iterative structure, Algorithm 1 may face scalability challenges with very large datasets. However, programming languages like Python or R can help address these challenges, facilitating the processing of larger and more complex datasets. Additionally, techniques such as parallel processing or data reduction can further improve scalability, enabling the algorithm to handle extensive data effectively and supporting its application in real-time or large-scale contexts.

Algorithm 2 Analysis

Effectiveness: Algorithm 2 is specifically designed to analyze patient datasets and identify core symptoms using rough set and generalized nano-topology techniques. The algorithm constructs binary relations for each patient and recalculates neighborhoods, effectively distinguishing between significant and non-significant symptoms. By refining the symptom set—removing irrelevant symptoms and recalculating relationships—it ensures the accurate identification of core symptoms, provided that the underlying data and symptom definitions are reliable.
Efficiency: The algorithm demonstrates strong efficiency for large datasets, as it can be adapted to handle extensive data effectively. By implementing iterative recalculations for each symptom using programming languages like Python or R, the algorithm is equipped to manage computationally intensive tasks. Further optimization through caching or reducing redundant calculations can enhance runtime performance.

Algorithm 1 Decision-Making Using $\mathbb{I}_{j}$-Approximation to Identify Exactness and Roughness.

Input: Insert the information table generated from the given data such that the first column contains the set of objects $\mathcal{U}$, and the set of attributes $\mathbb{AT}$ as a first row.

Output: An accurate decision for exact and rough sets.

1. Define the binary relations $q_m \mathcal{R}_{a_k} q_n \Leftrightarrow \mathcal{V}_{a_k}(q_m) \subseteq \mathcal{V}_{a_k}(q_n)$, for each $a_k \in \mathbb{AT}$, where $m, n, k \in \{1,2,3,\dots,8\}$.

2. Choose $j \in \mathcal{J}$ which you want to use.

3. **for** each $q_m \in \mathcal{U}$, **do**

  Compute all j-neighborhoods $n_j(q_m)$, using **Definition 2.3** and the first part of **Definition 2.5**.

  Compute all $\mathbb{I}_j$-neighborhoods $n_j^{\mathrm{i}}(q_m)$, using the second part of **Definition 2.5** and **Definition 4.2**.

end

4. **for** each $\mathbb{S} \subseteq \mathcal{U}$, **do**

5. Compute the $\mathbb{I}_j$-lower approximation $\underline{\mathbb{Lo}}_j^{\mathrm{i}}(\mathbb{S}) = \{q_m \in \mathcal{U}: n_j^{\mathrm{i}}(q_m) \subseteq \mathbb{S}\}$.

6. **if** $\underline{\mathbb{Lo}}_j^{\mathrm{i}}(\mathbb{S}) = \varphi$, **then**

7. **return** $\mathbb{S}$ is a rough set.

8. **else**

9. Compute the $\mathbb{I}_j$-upper approximation $\overline{\mathbb{Up}}_j^{\mathrm{i}}(\mathbb{S}) = \{q_m \in \mathcal{U}: n_j^{\mathrm{i}}(q_m) \cap \mathbb{S} \neq \varphi\}$.

10. Compute the $\mathbb{I}_j$-accuracy $\mathbb{Ac}_j^{\mathrm{i}}(\mathbb{S}) = \frac{\left|\underline{\mathbb{Lo}}_j^{\mathrm{i}}(\mathbb{S})\right|}{\left|\overline{\mathbb{Up}}_j^{\mathrm{i}}(\mathbb{S})\right|}$.

11. **if** $\mathbb{Ac}_j^{\mathrm{i}}(\mathbb{S}) = 1$, **then**

12. **return** $\mathbb{S}$ is an $\mathbb{I}_j$-exact set.

13. **else**

14. **return** $\mathbb{S}$ is an $\mathbb{I}_j$-rough set.

15. **end**

16. **end**

17. end

Scalability: Despite its iterative structure, the algorithm may face scalability challenges with very large datasets. However, programming languages like Python or R can improve scalability, facilitating the processing of larger and more complex datasets. Techniques such as parallel processing or data reduction can further enhance scalability, making the algorithm suitable for real-time or large-scale applications.

In a conclusion, both algorithms offer effective solutions for Covid-19 diagnostic decision-making but may require optimization for large datasets. Further development, including parallelization and computational improvements, could enhance their scalability and efficiency for broader use.

8. Conclusion and discussion

In this article, we presented eight novel types of initial-neighborhoods and examined twelve distinct types of neighborhoods derived from binary relations to enhance rough set theory. By defining initial-minimal and initial-maximal neighborhoods, we developed eight types of rough approximations ($\mathbb{I}_j$-approximations for each $j \in \mathcal{J}$, where $\mathcal{J} = \{r, \ell, \wedge, \vee, \langle r \rangle, \langle \ell \rangle, \langle \wedge \rangle, \langle \vee \rangle, (r), (\ell), (\wedge), (\vee)\}$) that generalize Pawlak's theory. Our methods represent a significant improvement over previous techniques, achieving accuracy rates of up to 100% for specific patient subsets, with results that align precisely with physicians' diagnoses in the dataset. We demonstrated the effectiveness of our approach in medical applications, specifically focusing on Covid-19, and introduced two algorithms for decision-making in information systems. These advancements underscore the potential of our methods across various fields, particularly in medical decision-making, where traditional approaches may fall short.

In this article, we also presented a novel approach to medical diagnostics by developing an accurate framework for diagnosing Covid-19 using rough set theory. Our methodology was tested on a dataset of 10 patients, achieving a diagnostic accuracy of up to 100%, matching physician diagnoses for specific patient subsets. This high level of accuracy, which surpasses those achieved by traditional methods, marks a significant breakthrough in mathematical modeling for medical applications. Unlike existing approaches, our framework accurately differentiates between Covid-19 patients and healthy individuals, highlighting its potential in streamlining diagnostic processes and conserving critical resources for both patients and healthcare providers.

Algorithm 2 Attribute Analysis and Reduction Using $\mathbb{I}_j$-Approximation Criteria for Determining Core Attributes to Diagnose Covid-19 Disease.

Input: Insert an information table generated from the given data such that the first column contains the set of patients $\mathcal{U}$, and the set of attributes (symptoms) $\mathbb{AT}$ as a first row.
Output: An accurate decision for exact and rough sets.
1. Define the binary relation $q_m \mathcal{R}_{a_k} q_n \Leftrightarrow \boldsymbol{\mathcal{V}}_{a_k}(q_m) \subseteq \boldsymbol{\mathcal{V}}_{a_k}(q_n)$, for each $a_k \in \mathbb{AT}$, where $m, n, k \in \{1,2,3,\ldots,8\}$. Another relation can be constructed based on the doctor's report.
2. Choose $j \in \mathcal{J}$ which you want to use.
3. **for** each $q_m \in \mathcal{U}$, **do**
   Compute all j-neighborhoods $n_j(q_m)$, using **Definition 2.3** and the first part of **Definition 2.5**.
   Compute all $\mathbb{I}_j$-neighborhoods $n_j^{\mathrm{i}}(q_m)$, using the second part of **Definition 2.5** and **Definition 4.2**.
   end
4. Construct the generalized nano-topology $\mathcal{GNT}$ using the technique outlined in **Theorem 6.2**.
5. **for** each attribute a_k in $\mathbb{AT}$, **do**
6. Remove the attribute a_k from the information system.
7. Extract elements of the relation $\mathcal{R}_{a_k}$ from the data in the information system.
8. Compute all j-neighborhoods $n_j(q_m)$, using **Step 3**.
9. Compute all $\mathbb{I}_j$-neighborhoods $n_j^{\mathrm{i}}(q_m)$, using **Step 3**.
   Construct the generalized nano-topology $\mathcal{GNT}^{a_k}$ using the technique outlined in **Theorem 6.2** according to **Step 4**.
10. **if** $\mathcal{GNT}^{a_k} = \mathcal{GNT}$, **then**
11. Attribute a_k is identified as a core symptom.
12. Add a_k to set $\mathbb{AT}$.
13. **else**
14. Attribute a_k is not an important symptom and can be removed.
15. **end**
16. Display the core set ($CORE$) of symptoms $\mathbb{AT}$.
17. **end**

Furthermore, this paper extended the concept of *Nano-Topology* within generalized rough sets, building on previous research by El-Bably et al. [44]. By employing approximation techniques from Yao [7], Allam et al. [22,23], Dai et al. [11], El-Sayed et al. [32], and Abu-Gdairi [43], we developed new generalized nano-topologies suited to practical applications of initial-rough sets. To validate this approach, we applied it in a medical context to analyze factors affecting Covid-19 infections [41], achieving accuracy rates comparable to exact diagnoses. Our method supports decision-making by identifying key Covid-19 risk factors using binary relations, thereby enabling healthcare providers to make more informed and effective diagnostic decisions.

This paper also introduced two novel algorithms specifically designed for Covid-19 diagnostics. Algorithm 1 focuses on distinguishing between exact and rough sets by computing binary relations and $\mathbb{I}_j$-approximations, demonstrating robust accuracy in classifying sets. Although computationally intensive, performance can be enhanced through optimization strategies, such as iterative recalculations and data structure enhancements. Algorithm 2, which identifies core Covid-19 symptoms through rough set and nano-topology techniques, was also tested with high accuracy. While both algorithms are implemented in MATLAB, they are adaptable to other programming environments, such as Python, R, and Julia, for wider practical use.

Advantages

1. **High Accuracy:** The proposed $\mathbb{I}_j$-approximations achieve high accuracy rates, with some methods reaching up to 100%, thereby enhancing the reliability of diagnostic processes.
2. **Generalization of Pawlak's Theory:** Our approach extends traditional rough set theory by generalizing it to accommodate a broader range of applications.
3. **Effective Decision-Making:** The introduced algorithms and approximations improve decision-making by providing clearer insights and reducing data ambiguity.

4. **Versatility:** The methods are applicable across various fields, as demonstrated by their successful application to Covid-19 diagnosis and decision-making.
5. **Robust Methodology:** Using initial neighborhoods and binary relations provides a robust framework for addressing uncertainty and ambiguity in data.

Disadvantages

1. **Similarity to Previous Studies:** One of the proposed techniques, specifically the $\mathbb{I}_{\langle \jmath \rangle}$-approximations for each $\jmath \in \{r, \ell, \wedge, \curlyvee\}$, is identical to Abu-Gdairi's [43] methods (namely, $\mathbb{I}_{\jmath}$-approximations, $\jmath \in \{r, \ell, \wedge, \curlyvee\}$) when the relation is a preorder. This overlap may limit the perceived novelty of our contribution.
2. **Specialized Knowledge Requirement:** The application of these methods necessitates a thorough understanding of rough set theory and the generalizations introduced in this article. As a result, their accessibility may be limited for practitioners who lack specialized expertise in this field.

Future work

Future research can build on the findings of this study in several ways:

1. **Optimization of Algorithms:** Additional research could focus on optimizing the proposed algorithms to reduce computational demands and improve efficiency, making them more suitable for real-time applications.
2. **Broader Medical Applications:** Expanding these approaches to other medical environments beyond Covid-19 can validate their flexibility and effectiveness across various domains.
3. **User-Friendly Implementations:** Developing user-friendly software tools and frameworks that incorporate these advanced rough set methods could simplify adoption for practitioners without specialized knowledge.
4. **Integration with AI Techniques:** Integrating the proposed rough set approaches with other artificial intelligence and machine learning techniques may further increase the accuracy and applicability of diagnostic tools.
5. **Longitudinal Studies:** Conducting longitudinal studies to evaluate the effectiveness of these methods over time across different fields could provide deeper insights into their practical benefits and limitations. This includes exploring related concepts such as new types of neighborhoods based on topological structures, as cited in references [5,6,46,47].

In summary, the advancements presented in this study offer significant improvements to rough set theory and its practical applications, particularly in the medical field. Despite the challenges that remain, the potential benefits of these methods underscore the importance of continued research and development in this area.

CRediT authorship contribution statement

Mostafa K. El-Bably: Writing – review & editing, Writing – original draft, Supervision, Project administration, Methodology, Formal analysis, Data curation, Conceptualization. **Rodyna A. Hosny:** Writing – review & editing, Visualization, Validation, Resources, Formal analysis. **Mostafa A. El-Gayar:** Visualization, Validation, Resources, Investigation, Formal analysis, Data curation.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Data availability

No data was used for the research described in the article.

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