mathematics

MDPI

Article

Differential Equations of Fourth-Order with p-Laplacian-Like Operator: Oscillation Theorems

Omar Bazighifan [1,2,*], Nawa Alshammari [3], Khalil S. Al-Ghafri [4] and Loredana Florentina Iambor [5,*]

[1] Department of Mathematics, Faculty of Education, Seiyun University, Hadhramout 50512, Yemen
[2] Jadara Research Center, Jadara University, Irbid 21110, Jordan
[3] Department of Basic Sciences, College of Science and Theoretical Studies, Saudi Electronic University, Riyadh 11673, Saudi Arabia; n.alshammari@seu.edu.sa
[4] College of Applied Sciences, University of Technology and Applied Sciences, P.O. Box 14, Ibri 516, Oman; khalil.alghafri@utas.edu.om
[5] Department of Mathematics and Computer Science, University of Oradea, University Street, 410087 Oradea, Romania
* Correspondence: o.bazighifan@gmail.com (O.B.); iambor.loredana@gmail.com (L.F.I.)

Abstract: In this work, we find new oscillation criteria for fourth-order advanced differential equations with a p-Laplace-type operator. We established our results through a comparison method with integral averaging and Riccati techniques to obtain new oscillatory properties for the considered equation. Our criteria substantially simplify and complement a number of existing ones. We give some examples to illustrate the significance of the obtained results.

Keywords: oscillation; differential equations; p-Laplace

MSC: 34C10; 34K11

check for updates

Citation: Bazighifan, O.; Alshammari, N.; Al-Ghafri, K.S.; Iambor, L.F. Differential Equations of Fourth-Order with p-Laplacian-Like Operator: Oscillation Theorems. *Mathematics* **2024**, *12*, 3558. https://doi.org/10.3390/math12223558

Academic Editors: Quanxin Zhu and Yu Tian

Received: 15 October 2024
Revised: 5 November 2024
Accepted: 13 November 2024
Published: 14 November 2024

1. Introduction

In this manuscript, we investigate the oscillatory properties of solutions for fourth-order differential equations in the form

$$\left(m(t)\left|(\varpi'''(t))\right|^{p-2}\varpi'''(t)\right)' + b(t)\Phi(\varpi(\varphi(t))) = 0, \quad t \geq t_0, \tag{1}$$

under the following assumptions:

(H_1) $1 < p < \infty$, where p is a p-Laplace-type operator, introducing a degree of nonlinearity and complexity to the equation;

(H_2) $m \in C^1([t_0,\infty),\mathbb{R})$, with $m(t) > 0$ and $m'(t) \geq 0$ ensuring that m is continuous, positive, and non-decreasing;

(H_3) $b, \varphi \in C([t_0,\infty),\mathbb{R})$, where $b(t) \geq 0$ and $\varphi(t) \geq t$, with $\lim_{t\to\infty} \varphi(t) = \infty$, ensuring that b is non-negative, and φ represents an unbounded delay function;

(H_4) $\Phi \in C(\mathbb{R},\mathbb{R})$ such that $\Phi(s)/s^{p-1} \geq \varsigma > 0$, where ς is a constant for $s \neq 0$ in the canonical case, meaning

$$\int_{t_0}^{\infty} \frac{1}{m^{1/p-1}(s)} \mathrm{d}s = \infty. \tag{2}$$

Definition 1. *A solution to (1) is called oscillatory if it is neither eventually positive nor eventually negative. Otherwise, it is said to be nonoscillatory.*

Definition 2. *Equation (1) is said to be oscillatory if all of its solutions are oscillatory.*

Advanced equations and population models are essential tools for understanding the dynamics of species populations over time, allowing for the study of the effects of environmental factors and the development of effective conservation strategies (see [1,2]). These models rely on nonlinear differential equations to accurately represent complex interactions, and the introduction of fractional differentials has helped them to capture memory properties and long-term dependence more accurately, making them more effective than traditional models [3–5].

Oscillation theory represents one basic research field in both physics and mathematics that deals with the behavior of some systems that exhibit periodic motion. Various natural phenomena exhibit oscillatory behavior, such as the swinging of a pendulum, or even the rhythmic beating of the human heart, population dynamics, and electrical circuits; see [6–8]. The study of oscillations is important not only in such natural phenomena as the tides and planetary motion but also in technological systems such as mechanical engineering, control systems, and signal processing [9–11].
Fourth-order differential equations involving p-Laplace operators are important mathematical models for studying the oscillatory behavior of complex dynamic systems. These equations represent a powerful tool for analyzing nonlinear phenomena that arise in many fields, such as mechanics, fluid mechanics, and elasticity theory. The p-Laplace operator has mathematical properties that allow it to be applied to a wide range of problems characterized by complex interactions and unconventional growth conditions, making it particularly suitable for studying oscillations in systems affected by multiple nonlinear factors. These properties enable researchers to formulate new oscillation parameters and understand the responses of systems under higher-order nonlinear effects, which contribute to expanding knowledge about how complex systems interact with these operators and their role in the dynamics of oscillatory behavior (see [12,13]).

Investigations by some authors in [14] have yielded techniques and methodologies aimed at enhancing the oscillatory attributes of these equations. Furthermore, the work carried out in [15–17] has expanded this inquiry to encompass differential equations of the neutral variety. In recent years, there has also been significant exploration of oscillation behaviors in higher-order DDEs with p-Laplace-type operators, as evidenced by studies such as [18–21]. The main motivation for work is to contribute to the development of the oscillation theory for fourth-order neutral equations by finding sufficient conditions that guarantee that the solutions of this type of equation are oscillatory.

The authors in [20,21] established new conditions to improve and extend some of the oscillation results for the equation

$$\left(m(t)\left(\varpi^{(n-1)}(t)\right)^{p-1}\right)' + b(t)z^{p-1}(\varphi(t)) = 0,$$

where n is even, and $\int_{s_0}^{\infty-} m^{1/(p-1)}(v)\mathrm{d}v = \infty$.
Liu et al. [19] introduced good conditions concerning solutions to even-order differential equations featuring a mixed term under the canonical case

$$\left(m(t)\left(\varpi^{(n)}(t)\right)^{p-1}\right)' + r(t)\left(\varpi^{(n)}(s)\right)^{p-1} + b(t)z^{p-1}(\varphi(t)) = 0,$$

where

$$\varpi(t) := z(t) + a(t)z(\varnothing(t)).$$

Chatzarakis et al. [22] explored the oscillation and asymptotic behavior of all solutions of second-order half-linear differential equations with an advanced argument of the form

$$\left(m(t)(\varpi'(t))^{\gamma}\right)' + b(t)\varpi^{\gamma}(\varphi(t)) = 0, \tag{3}$$

and in the noncanonical case,

$$\int_{t_0}^{\infty} \frac{1}{m^{1/\gamma}(s)} \mathrm{d}s < \infty. \tag{4}$$

Hassan [23] investigated (3) in its noncanonical form as a particular case of a more general second-order advanced dynamic equation under the condition

$$\int_{t_0}^{\infty} \left(\frac{1}{m(t)} \int_{t_0}^{t} \pi^{\gamma}(\varphi(s)) b(s) \mathrm{d}s \right)^{1/\gamma} \mathrm{d}t = \infty, \tag{5}$$

which allowed him to eliminate possible positive decreasing solutions.
Later on, Agarwal et al. [24] improved the results of Hassan [23] in the sense that they established their results (of the Kamenev type) without requiring (5). Both the results of Hassan [23] and Agarwal et al. [24] use an approach that leads to two independent conditions, eliminating increasing and decreasing positive solutions, respectively.
Agarwal et al. [25] established some new criteria for the oscillation and asymptotic behavior of solutions of even-order advanced differential equations:

$$\left(m(t) \left(\varpi^{(n-1)}(t) \right)^{\gamma} \right)' + b(t) \varpi^{\gamma}(\varphi(t)) = 0, \tag{6}$$

and under the canonical case,

$$\int_{t_0}^{\infty} \frac{1}{m^{1/\gamma}(s)} \mathrm{d}s = \infty, \tag{7}$$

with the noncanonical case given by (4).

Our aim in this paper is to complement and simplify the results obtained in [22–24]. This manuscript aims to broaden the scope of inquiry and complement the results given in [22–24] by obtaining a new theorem of (1) under condition (7) by using a comparison method with second-order equations, integral averaging, and Riccati techniques. We discuss some examples to illustrate the effectiveness of our main criteria.

2. Main Results

We first introduce some important lemmas, and then we obtain oscillation conditions for (1).

Lemma 1 ([9])**.** *Suppose that $\varpi \in C^a([t_0,\infty),(0,\infty))$ and $\varpi^{(a-1)}(t)\varpi^{(a)}(t) \le 0$, and $\varpi^{(a)}$ is of a fixed sign and not identically zero on $[t_0,\infty)$. If $\lim_{t\to\infty} \varpi(t) \neq 0$; then,*

$$\varpi(t) \geq \frac{\varrho}{(a-1)!} t^{a-1} \left| \varpi^{(a-1)}(t) \right|,$$

for every $t \geq t_{\varrho}$ and $\varrho \in (0,1)$.

Lemma 2 ([26])**.** *If $\varpi^{(a+1)}(t) < 0$, then*

$$\frac{a!\varpi(t)}{t^a} \geq \frac{(a-1)!\varpi'(t)}{t^{a-1}},$$

where $\varpi^{(\imath)}(t) > 0$, $\imath = 0,1,...,a$.

Lemma 3 ([27])**.** *Assume that the function ϖ eventually represents a positive solution to (1). This assumption leads to two cases:*

$$\begin{array}{ll} (\mathbf{S}_1) & \varpi^{(n)}(t) > 0,\ n = 0,1,2,3 \text{ and } \varpi^{(4)}(t) < 0, \\ (\mathbf{S}_2) & \varpi^{(n)}(t) > 0,\ n = 0,1,3 \text{ and } \varpi''(t) < 0, \varpi^{(4)}(t) < 0, \end{array}$$

for $t \geq t_1$, where $t_1 \geq t_0$ is sufficiently large.

Lemma 4. *If ϖ eventually represents a positive solution to (1) and case $(\mathbf{S}_1)$ holds, let*

$$z(t) := \varkappa(t)\left(\frac{m(t)(\varpi'''(t))^{p-1}}{\varpi^{p-1}(t)}\right), \tag{8}$$

where $\varkappa \in C^1([t_0,\infty),(0,\infty))$; then,

$$z'(t) \leq -\frac{(p-1)\varrho t^2}{2(\varkappa(t)m(t))^{\frac{1}{p-1}}} z^{\frac{p}{p-1}}(t). \tag{9}$$

Proof. The function ϖ is identified as the ultimate positive solution to (1) and case $(\mathbf{S}_1)$ holds. Using Lemma 1, we find

$$\varpi'(t) \geq \frac{\varrho}{2} t^2 \varpi'''(t), \tag{10}$$

for every $\varrho \in (0,1)$ and for all large t.
From (8), we see that

$$\begin{aligned} z'(t) &= \varkappa'(t)\frac{m(t)(\varpi'''(t))^{p-1}}{\varpi^{p-1}(t)} + \varkappa(t)\frac{\left(m(\varpi''')^{p-1}\right)'(t)}{\varpi^{p-1}(t)} \\ &\quad -(p-1)\varkappa(t)\frac{\varpi^{p-2}(t)\varpi'(t)m(t)(\varpi'''(t))^{p-1}}{\varpi^{2(p-1)}(t)}. \end{aligned}$$

Combining (8) and (10), we obtain

$$\begin{aligned} z'(t) &\leq \frac{\varkappa'(t)}{\varkappa(t)}z(t) + \varkappa(t)\frac{\left(m(t)(\varpi'''(t))^{p-1}\right)'}{\varpi^{p-1}(t)} \\ &\quad -(p-1)\varkappa(t)\frac{\varrho}{2}t^2\frac{m(t)(\varpi'''(t))^{p}}{\varpi^{p}(t)} \\ &\leq \frac{\varkappa'(t)}{\varkappa(t)}z(t) + \varkappa(t)\frac{\left(m(t)(\varpi'''(t))^{p-1}\right)'}{\varpi^{p-1}(t)} \\ &\quad -\frac{(p-1)\varrho t^2}{2(\varkappa(t)m(t))^{\frac{1}{p-1}}} z^{\frac{p}{p-1}}(t). \end{aligned}$$

Therefore, the proof is finished. □

Lemma 5. *If ϖ eventually represents a positive solution to (1) and case $(\mathbf{S}_2)$ holds, let*

$$\emptyset(t) := \vartheta(t)\frac{\varpi'(t)}{\varpi(t)},$$

where $\vartheta \in C^1([t_0,\infty),(0,\infty))$; then,

$$\varpi''(t) + \varpi(t)\int_t^\infty \left(\frac{\varsigma}{m(\zeta)}\int_\zeta^\infty b(s)\mathrm{d}s\right)^{1/p-1} \mathrm{d}\zeta \leq 0. \tag{11}$$

Proof. The function ϖ is identified as the ultimate positive solution to (1) and case $(\mathbf{S}_2)$ holds. By differentiating $\emptyset(t)$, we find

$$\emptyset'(t) = \frac{\vartheta'(t)}{\vartheta(t)}\emptyset(t) + \vartheta(t)\frac{\varpi''(t)}{\varpi(t)} - \frac{1}{\vartheta(t)}\emptyset^2(t). \tag{12}$$

Now, by integrating (1) from t to h and using $\varpi'(t) > 0$, we have

$$m(h)\left(\varpi'''(h)\right)^{p-1} - m(t)\left(\varpi'''(t)\right)^{p-1} = -\int_t^h b(s)\Phi(\varpi(\varphi(s)))ds.$$

By virtue of $\varpi'(t) > 0$ and $\varphi(t) \geq t$, we obtain

$$m(h)\left(\varpi'''(h)\right)^{p-1} - m(t)\left(\varpi'''(t)\right)^{p-1} \leq -\varsigma\varpi^{p-1}(t)\int_t^u b(s)ds.$$

Letting $h \to \infty$, we see that

$$m(t)\left(\varpi'''(t)\right)^{p-1} \geq \varsigma\varpi^{(p-1)}(t)\int_t^\infty b(s)\mathrm{d}s$$

and so

$$\varpi'''(t) \geq \varpi(t)\left(\frac{\varsigma}{m(t)}\int_t^\infty b(s)\mathrm{d}s\right)^{1/p-1}.$$

Integrating again from t to ∞, we obtain

$$\varpi''(t) + \varpi(t)\int_t^\infty \left(\frac{\varsigma}{m(\zeta)}\int_\zeta^\infty b(s)\mathrm{d}s\right)^{1/p-1}\mathrm{d}\zeta \leq 0.$$

The proof is finished. □

Remark 1 ([28]). *The following differential equation is well known:*

$$\left[m(t)\left(\varpi'(t)\right)^r\right]' + b(t)\varpi^r(\Phi(t)) = 0, \quad t \geq t_0, \tag{13}$$

where $r > 0$ is the ratio of odd positive integers, and $m,\ b \in C([t_0,\infty),\mathbb{R}^+)$. This equation is nonoscillatory if and only if there exists a number $t \geq t_0$ and a function $\zeta \in C^1([t,\infty),\mathbb{R})$ that satisfy the inequality

$$\zeta'(t) + \gamma m^{-1/r}(t)(\zeta(t))^{(1+r)/r} + b(t) \leq 0, \quad on\ [t,\infty).$$

Theorem 1. *Assume that (2) holds. If the equations*

$$\left(\frac{2m^{\frac{1}{p-1}}(t)}{(\varrho t^2)^{p-1}}\left(\varpi'(t)\right)^{p-1}\right)' + \varsigma b(t)\varpi^{p-1}(t) = 0 \tag{14}$$

and

$$\varpi''(t) + \varpi(t)\int_t^\infty \left(\frac{1}{m(\zeta)}\int_\zeta^\infty b(s)\mathrm{d}s\right)^{1/p-1}\mathrm{d}\zeta = 0 \tag{15}$$

are oscillatory, then (1) is oscillatory.

Proof. The function ϖ is identified as the ultimate positive solution to (1). Using Lemma 3, we see cases $(\mathbf{S}_1)$ and $(\mathbf{S}_2)$. Let case $(\mathbf{S}_1)$ hold. Using Lemma 4, along with (1) and (9), we obtain

$$z'(t) \leq \frac{\varkappa'(t)}{\varkappa(t)}z(t) - \varsigma\varkappa(t)\frac{b(t)\varpi^{p-1}(\varphi(t))}{\varpi^{p-1}(t)} - \frac{(p-1)\varrho t^2}{2(\varkappa(t)m(t))^{\frac{1}{p-1}}}z^{\frac{p}{p-1}}(t).$$

Note that $\varpi'(t) > 0$ and $\varphi(t) \geq t$. Thus,

$$z'(t) \leq \frac{\varkappa'(t)}{\varkappa(t)}z(t) - \varsigma\varkappa(t)b(t) - \frac{(p-1)\varrho t^2}{2(\varkappa(t)m(t))^{\frac{1}{p-1}}}z(t)^{\frac{p}{p-1}}. \tag{16}$$

If we set $\varkappa(t) = \varsigma = 1$ in (16), we obtain

$$z'(t) + \frac{(p-1)\varrho t^2}{2m^{\frac{1}{p-1}}(t)} z^{\frac{p}{p-1}}(t) + b(t) \leq 0.$$

Using Remark 1, we can see that Equation (14) is nonoscillatory, which is a contradiction. Let case $(\mathbf{S}_2)$ hold. From Lemma 5, by combining (11) and (12), we obtain

$$\varnothing'(t) \leq \frac{\vartheta'(t)}{\vartheta(t)}\varnothing(t) - \vartheta(t)\int_t^\infty \left(\frac{\varsigma}{m(\zeta)}\int_\zeta^\infty b(s)\mathrm{d}s\right)^{1/p-1}\mathrm{d}\zeta - \frac{1}{\vartheta(t)}\varnothing^2(t). \tag{17}$$

If $\vartheta(t) = \varsigma = 1$ in (17), we obtain

$$\varnothing'(t) + \varnothing^2(t) + \int_t^\infty \left(\frac{1}{m(\zeta)}\int_\zeta^\infty b(s)\mathrm{d}s\right)^{1/p-1}\mathrm{d}\zeta \leq 0.$$

Hence, we see that Equation (15) is nonoscillatory, which is a contradiction. The proof of the theorem is complete. □

Now, we obtain Hille- and Nehari-type oscillation conditions for (1) with $p = 2$ in Theorem 1.

Theorem 2. *Let $p = 2$ and $\varsigma = 1$. Assume that*

$$\int_{t_0}^\infty \frac{\varrho t^2}{2m(t)}\mathrm{d}t = \infty$$

and

$$\liminf_{t\to\infty}\left(\int_{t_0}^t \frac{\varrho s^2}{2m(s)}\mathrm{d}s\right)\int_t^\infty b(s)\mathrm{d}s > \frac{1}{4}, \tag{18}$$

for some constant $\varrho \in (0,1)$,

$$\liminf_{t\to\infty} t\int_{t_0}^t \int_v^\infty \left(\frac{1}{m(\zeta)}\int_\zeta^\infty b(s)\mathrm{d}s\right)\mathrm{d}\zeta\mathrm{d}v > \frac{1}{4}; \tag{19}$$

then, all solutions of (1) are oscillatory.

Definition 3. *Let the functions $\varkappa, \vartheta \in C^1([t_0,\infty),(0,\infty))$ and $F_\iota \in C(D,\mathbb{R})$ with $\iota \in \{1,2\}$; then,*

$$\frac{\partial}{\partial s}F_1(t,s) + \frac{\varkappa'(s)}{\varkappa(s)}F_1(t,s) = f_1(t,s)F_1^{p-1/p}(t,s) \tag{20}$$

and

$$\frac{\partial}{\partial s}F_2(t,s) + \frac{\vartheta'(s)}{\vartheta(s)}F_2(t,s) = f_2(t,s)\sqrt{F_2(t,s)}. \tag{21}$$

where $F_\iota(t,s)$ has a nonpositive partial derivative, $\partial F_\iota/\partial s$.

In this theorem, we obtain an oscillation criterion for (1) by using the integral averaging technique:

Theorem 3. *Let (2) hold. If $\varkappa, \vartheta \in C^1([t_0,\infty),\mathbb{R})$ such that*

$$\limsup_{t\to\infty}\frac{1}{F_1(t,t_1)}\int_{t_1}^t (F_1(t,s)\varsigma\varkappa(s)b(s) - \pi(s))\mathrm{d}s = \infty \tag{22}$$

and

$$\limsup_{t\to\infty} \frac{1}{F_2(t,t_1)} \int_{t_1}^{t} \left(F_2(t,s)\vartheta(s)\chi(s) - \frac{\vartheta(s) f_2^2(t,s)}{4} \right) \mathrm{d}s = \infty, \tag{23}$$

where

$$\pi(s) = \frac{f_1^p(t,s) F_1^{p-1}(t,s)}{p^p} \frac{2^{p-1}\varkappa(s) m(s)}{(\varrho s^2)^{p-1}},$$

for all $\varrho \in (0,1)$, *and*

$$\chi(s) = \int_t^\infty \left(\frac{\varsigma}{m(\zeta)} \int_\zeta^\infty b(s)\mathrm{d}s \right)^{1/p-1} \mathrm{d}\zeta,$$

then (1) is oscillatory.

Proof. The function ϖ is identified as the ultimate positive solution to (1). Using Lemma 3, we see cases $(\mathbf{S}_1)$ and $(\mathbf{S}_2)$.
Assume that $(\mathbf{S}_1)$ holds. From Theorem 1, we find that (16) holds.
Multiplying (16) by $F_1(t,s)$ and integrating the resulting inequality from t_1 to t, we find that

$$\begin{aligned} \int_{t_1}^{t} F_1(t,s)\varsigma\varkappa(s) b(s)\mathrm{d}s \quad &\leq \quad z(t_1)F_1(t,t_1) + \int_{t_1}^{t} \left(\frac{\partial}{\partial s} F_1(t,s) + \frac{\varkappa'(s)}{\varkappa(s)} F_1(t,s) \right) z(s)\mathrm{d}s \\ &\quad - \int_{t_1}^{t} \frac{(p-1)\varrho s^2}{2(\varkappa(s) m(s))^{\frac{1}{p-1}}} F_1(t,s) z^{\frac{p}{p-1}}(s)\mathrm{d}s. \end{aligned}$$

From (20), we see that

$$\begin{aligned} \int_{t_1}^{t} F_1(t,s)\varsigma\varkappa(s) b(s)\mathrm{d}s \quad &\leq \quad z(t_1)F_1(t,t_1) + \int_{t_1}^{t} f_1(t,s) F_1^{p-1/P}(t,s) z(s)\mathrm{d}s \\ &\quad - \int_{t_1}^{t} \frac{(p-1)\varrho s^2}{2(\varkappa(s) m(s))^{\frac{1}{p-1}}} F_1(t,s) z^{\frac{p}{p-1}}(s)\mathrm{d}s. \end{aligned} \tag{24}$$

Let $H = (p-1)\varrho s^2 / \left(2(\varkappa(s)m(s))^{\frac{1}{p-1}} \right) F_1(t,s)$, $G = f_1(t,s) F_1^{p-1/P}(t,s)$
and $\varpi = z(s)$. Using the inequality

$$\frac{(p-1)^{p-1}}{p^p} \frac{G^p}{H^{(p-1)}} \geq Gs - Hs^{p/p-1}, \; H > 0,$$

we see that

$$\begin{aligned} & f_1(t,s) F_1^{p-1/P}(t,s) z(s) - \frac{(p-1)\varrho s^2}{2(\varkappa(s)m(s))^{\frac{1}{p-1}}} F_1(t,s) z^{\frac{p}{p-1}}(s) \\ \leq \quad & \frac{f_1^P(t,s) F_1^{p-1}(t,s)}{p^P} \frac{2^{p-1}\varkappa(s) m(s)}{(\varrho s^2)^{p-1}}, \end{aligned}$$

which, with (24), gives

$$\frac{1}{F_1(t,t_1)} \int_{t_1}^{t} (F_1(t,s)\varsigma\varkappa(s) b(s) - \pi(s))\mathrm{d}s \leq z(t_1).$$

This contradicts (22).
Assume that $(\mathbf{S}_2)$ holds. From Theorem 1, (17) holds.
Multiplying (17) by $F_2(t,s)$ and integrating the resulting inequality from t_1 to t, we obtain

$$\begin{aligned}\int_{t_1}^{t} F_2(t,s)\vartheta(s)\chi(s)\mathrm{d}s \leq\ & \varnothing(t_1)F_2(t,t_1) \\ & + \int_{t_1}^{t}\left(\frac{\partial}{\partial s}F_2(t,s) + \frac{\vartheta'(s)}{\vartheta(s)}F_2(t,s)\right)\varnothing(s)\mathrm{d}s \\ & - \int_{t_1}^{t}\frac{1}{\vartheta(s)}F_2(t,s)\varnothing^2(s)\mathrm{d}s.\end{aligned}$$

Thus, from (21), we obtain

$$\begin{aligned}\int_{t_1}^{t} F_2(t,s)\vartheta(s)\chi(s)\mathrm{d}s &\leq \varnothing(t_1)F_2(t,t_1) + \int_{t_1}^{t} f_2(t,s)\sqrt{F_2(t,s)}\varnothing(s)\mathrm{d}s \\ &\quad - \int_{t_1}^{t}\frac{1}{\vartheta(s)}F_2(t,s)\varnothing^2(s)\mathrm{d}s \\ &\leq \varnothing(t_1)F_2(t,t_1) + \int_{t_1}^{t}\frac{\vartheta(s)f_2^2(t,s)}{4}\mathrm{d}s,\end{aligned}$$

and so

$$\frac{1}{F_2(t,t_1)}\int_{t_1}^{t}\left(F_2(t,s)\vartheta(s)\chi(s) - \frac{\vartheta(s)f_2^2(t,s)}{4}\right)\mathrm{d}s \leq \varnothing(t_1),$$

which contradicts (23). The proof of the theorem is complete. □

3. Applications and Discussion

In this part, we discuss some applications and numerical examples to highlight the significance of the conditions we obtained in Theorem 2.

Example 1. *Consider the equation*

$$\varpi^{(4)}(t) + \frac{b_0}{t^4}\varpi(2t) = 0,\ t \geq 1, b_0 > 0. \tag{25}$$

Let $p = 2$, $m(t) = 1$, $b(t) = b_0/t^4$, and $\varphi(t) = 2t$. If we set $\varsigma = 1$, then conditions (18) and (19) become

$$\begin{aligned}\liminf_{t\to\infty}\left(\int_{t_0}^{t}\frac{\varrho s^2}{2m(s)}\mathrm{d}s\right)\int_{t}^{\infty} b(s)\mathrm{d}s &= \liminf_{t\to\infty}\left(\frac{t^3}{3}\right)\int_{t}^{\infty}\frac{b_0}{s^4}ds \\ &= \frac{b_0}{9} > \frac{1}{4},\end{aligned}$$

and

$$\begin{aligned}\liminf_{t\to\infty} t\int_{t_0}^{t}\int_{v}^{\infty}\left(\frac{1}{m(\zeta)}\int_{\zeta}^{\infty} b(s)\mathrm{d}s\right)^{1/p-1}\mathrm{d}\zeta\mathrm{d}v &= \liminf_{t\to\infty} t\left(\frac{b_0}{6t}\right), \\ &= \frac{b_0}{6} > \frac{1}{4},\end{aligned}$$

respectively. From Theorem 2, we see that (25) is oscillatory if $b_0 > 2.25$.

Example 2. *Consider the equation*

$$\left(t\left(\varpi'''(t)\right)\right)' + \frac{b_0}{t}\varpi(3t) = 0, \tag{26}$$

where $t \geq 1, b_0 > 0$. *Let* $p = 2$, $m(t) = t$, $b(t) = b_0/t$, *and* $\varphi(t) = 3t$. *If we set* $\varsigma = 1$, $f_1(t,s) = \varkappa(s) = 1$, *and* $F_1(t,s) = t$, *then*

$$\begin{aligned} &\int_{t_0}^{\infty} \frac{1}{m^{1/p-1}(s)} \mathrm{d}s \\ = &\int_{t_0}^{\infty} \frac{1}{s} \mathrm{d}s = \infty, \end{aligned}$$

and

$$\begin{aligned} \pi(s) &= \frac{f_1^p(t,s) F_1^{p-1}(t,s)}{p^p} \frac{2^{p-1}\varkappa(s) m(s)}{(\varrho s^2)^{p-1}}, \\ &= \frac{s}{4} \frac{2s}{\varrho s^2} = 1/2\varrho, \end{aligned}$$

where $\varrho \in (0,1)$. *Also, we see that*

$$\begin{aligned} \chi(s) &= \int_t^{\infty} \left(\frac{\varsigma}{m(\zeta)} \int_{\zeta}^{\infty} b(s) \mathrm{d}s \right)^{1/p-1} \mathrm{d}\zeta \\ &= b_0 \int_t^{\infty} \left(\frac{1}{\zeta} \int_{\zeta}^{\infty} 1/s \mathrm{d}s \right) \mathrm{d}\zeta. \end{aligned}$$

From Theorem 3, we see that (26) is oscillatory.

4. Conclusions

In this work, we aimed to present new theorems for (1). This investigation was conducted through the application of a comparison method and integral averaging and Riccati techniques, ultimately leading to the derivation of oscillation criteria. The study culminates in the establishment of a central theorem pertaining to the oscillation behavior of equations. Our results extend recent criteria for the same equations established previously by several authors. Additionally, some examples of these criteria were discussed.

In future work, we will study third-order differential equations in their noncanonical form to find their oscillatory properties, which will contribute to enriching oscillation theory.

Author Contributions: Methodology, N.A., O.B. and K.S.A.-G.; Investigation, O.B. and L.F.I.; Writing—original draft, N.A., O.B. and L.F.I.; Writing—review & editing, O.B. and N.A. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by the University of Oradea.

Data Availability Statement: No new data were created or analyzed in this study.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Baculikova, B.; Dzurina, J.; Graef, J.R. On the oscillation of higher-order delay differential equations. *Math. Slovaca* **2012**, *187*, 387–400. [CrossRef]
2. Bazighifan, O. On the Oscillation of Certain Fourth-Order Differential Equations with p-Laplacian Like Operator. *Appl. Math. Comput.* **2020**, *386*, 125475. [CrossRef]
3. Grace, S.; Dzurina, J.; Jadlovska, I.; Li, T. On the oscillation of fourth order delay differential equations. *Adv. Differ. Equ.* **2019**, *118*, 118. [CrossRef]
4. Gyori, I.; Ladas, G. *Oscillation Theory of Delay Differential Equations with Applications*; Clarendon Press: Oxford, UK, 1991.
5. Masood, F.; Cesarano, C.; Moaaz, O.; Askar, S.S.; Alshamrani, A.M.; El-Metwally, H. Kneser-Type Oscillation Criteria for Half-Linear Delay Differential Equations of Third Order. *Symmetry* **2023**, *15*, 1994. [CrossRef]
6. Hale, J.K. *Theory of Functional Differential Equations*; Springer: New York, NY, USA, 1977.
7. Bartusek, M.; Cecchi, M.; Dosla, Z.; Marini, M. Fourth-order differential equation with deviating argument. *Abstr. Appl. Anal.* **2012**, *2012*, 185242. [CrossRef]

8. Xu, Z.; Xia, Y. Integral averaging technique and oscillation of certain even order delay differential equations. *J. Math. Appl. Anal.* **2004**, *292*, 238–246. [CrossRef]
9. Agarwal, R.; Grace, S.; O'Regan, D. *Oscillation Theory for Difference and Functional Differential Equations*; Kluwer Acad. Publ.: Dordrecht, The Netherlands, 2000.
10. Bazighifan, O.; Alotaibi, H.; Mousa, A.A.A. Neutral Delay Differential Equations: Oscillation Conditions for the Solutions. *Symmetry* **2021**, *13*, 101. [CrossRef]
11. Tarasov, V.E. *Applications in Physics and Engineering of Fractional Calculus*; Springer: Berlin/Heidelberg, Germany, 2019.
12. Alatwi, M.; Moaaz, O.; Albalawi, W.; Masood, F.; El-Metwally, H. Asymptotic and Oscillatory Analysis of Fourth-Order NonlinearDifferential Equations withp-Laplacian-like Operators and Neutral Delay Arguments. *Mathematics* **2024**, *12*, 470. [CrossRef]
13. Alsharidi, A.K.; Muhib, A.; Elagan, S.K. Neutral Differential Equations of Higher-Order in Canonical Form: Oscillation Criteria. *Mathematics* **2023**, *11*, 3300. [CrossRef]
14. Nehari, Z. Oscillation criteria for second order linear differential equations. *Trans. Amer. Math. Soc.* **1957**, *85*,428–445. [CrossRef]
15. Philos, C. On the existence of nonoscillatory solutions tending to zero at ∞ for differential equations with positive delay. *Arch. Math.* **1981**, *36*, 168–178. [CrossRef]
16. Zhang, C.; Agarwal, R.P.; Bohner, M.; Li, T. New results for oscillatory behavior of even-order half-linear delay differential equations. *Appl. Math. Lett.* **2013**, *26*, 179–183. [CrossRef]
17. Zhang, C.; Li, T.; Saker, S. Oscillation of fourth-order delay differential equations. *J. Math. Sci.* **2014**, *201*, 296–308. [CrossRef]
18. Sugie, J.; Onitsuka, M. A non-oscillation theorem for nonlinear differential equations with p-Laplacian. *Proc. R. Soc. Edinb. A* **2006**, *136*, 633–647. [CrossRef]
19. Liu, S.; Zhang, Q.; Yu, Y. Oscillation of even-order half-linear functional differential equations with damping, Comput. *Math. Appl.* **2011**, *61*, 2191–2196.
20. Bazighifan, O.; Abdeljawad, T. Improved Approach for Studying Oscillatory Properties of Fourth-Order Advanced Differential Equations with p-Laplacian Like Operator. *Mathematics* **2020**, *8*, 656. [CrossRef]
21. Li, T.; Baculikova, B.; Dzurina, J.; Zhang, C. Oscillation of fourth order neutral differential equations with p-Laplacian like operators. *Bound. Value Probl.* **2014**, *56*, 41–58. [CrossRef]
22. Chatzarakis, G.E.; Dzurina, J.; Jadlovska, I. New oscillation criteria for second-order half-linear advanced differential equations. *Appl. Math. Comput.* **2019**, *347*, 404–416. [CrossRef]
23. Hassan, T.S. Kamenev-type oscillation criteria for second order nonlinear dynamic equations on time scales. *Appl. Math. Comput.* **2011**, *217*, 5285–5297. [CrossRef]
24. Agarwal, R.P.; Zhang, C.; Li, T. New Kamenev-type oscillation criteria for second-order nonlinear advanced dynamic equations. *Appl. Math. Comput.* **2013**, *225*, 822–828. [CrossRef]
25. Agarwal, R.P.; Bohner, M.; Li, T.; Zhang, C. Even-order half-linear advanced differential equations: Improved criteria in oscillatory and asymptotic properties. *Appl. Math. Comput.* **2015**, *266*, 481–490. [CrossRef]
26. Chatzarakis, G.E.; Elabbasy, E.M.; Bazighifan, O. An oscillation criterion in 4th-order neutral differential equations with a continuously distributed delay. *Adv. Differ. Equ.* **2019**, *336*, 1–9.
27. Cesarano, C.; Bazighifan, O. Oscillation of fourth-order functional differential equations with distributed delay. *Axioms* **2019**, *7*, 61. [CrossRef]
28. Agarwal, R.P.; Grace, S.R.; O'Regan, D. The oscillation of certain higher-order functional differential equations. *Math. Comput. Model.* **2003**, *37*, 705–728. [CrossRef]