Numerical Advancements: A Duel between Euler-Maclaurin and Runge-Kutta for Initial Value Problem

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Abstract

This work is dedicated to advancing the approximation of initial value problems through the introduction of an innovative and superior method inspired by the Euler-Maclaurin formula. This results in a higher-order implicit corrected method that outperforms the Runge-Kutta method in terms of accuracy. We derive an error bound for the Euler-Maclaurin higher-order method, showcasing its stability, convergence, and greater efficiency compared to the conventional Runge-Kutta method. To substantiate our claims, numerical experiments are provided, highlighting the exceptional efficiency of our proposed method over the traditional well-known methods. In conclusion, the proposed method consistently outperforms the Runge-Kutta method experimentally in all practical problems.

Keywords: Euler-Maclaurin formula; Runge-Kutta method; ODE; Darboux's formula; Approximations

1 Introduction

In our present era, marked by unprecedented progress in both experimental and applied sciences, the landscape of scientific exploration is continually expanding. A noteworthy facet of this evolution is the rapid strides in artificial intelligence, a transformative force that holds promise for addressing intricate mathematical challenges. In the dynamic realm of differential equations, researchers are dedicatedly engaged in the enhancement and modernization of classical methods for approximating both initial and boundary value problems.[1–7]

While the Runge-Kutta method maintains its supremacy as the go-to technique for solving differential equations, researchers find themselves at the intersection of tradition and innovation. The method, revered by many, serves as a robust benchmark against which emerging approaches are scrutinized, particularly in the intricate domain of chaotic systems. Yet, as we traverse this era of accelerating development, the exigencies of the moment compel us to not only acknowledge historical methodologies but also to push beyond established boundaries.

This contemporary epoch demands that we proactively propose and cultivate novel avenues for approximating Ordinary Differential Equations (O.D.E.) with heightened precision and efficiency. The quest for advancements in computational techniques becomes more pronounced as we endeavor to unlock deeper insights into

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complex mathematical models and systems, see[8–14] to get a future directions in relation to such techniques. In this quest for progress, we are challenged to explore uncharted territories, seeking methodologies that not only surpass the reliability of the Runge-Kutta method but also resonate with the evolving demands of modern scientific inquiry. As we stand at the connection of tradition and innovation, our pursuit is not merely about comparison but about carving new pathways that redefine the very structure of mathematical approximation in the era of artificial intelligence.

After navigating through the terrain of established methodologies, our exploration is poised to reach its zenith with the unveiling of a ground breaking approach for approximating solutions to Initial Value Problems (I.V.P.). This pioneering method endeavors to strike a nuanced equilibrium between precision and computational efficiency, offering a compelling alternative to the conventional techniques deliberated earlier. As we set forth on this transformative odyssey, we extend a warm invitation to readers, urging them to accompany us in unraveling the complexities of numerical methods. Together, let us pave the way for a new epoch in the realm of approximating solutions for I.V.P. In this direction, we recommend the reader refer to.[15–19]

The Euler-Maclaurin formula, a mathematical gem, stands as a testament to the intellectual prowess of Euler[20] and Maclaurin[21] during the 18th century. Euler and Maclaurin independently contributed to the development of the formula. Euler's motivation stemmed from the need to bridge the gap between discrete sums and continuous integrals, while Maclaurin's work built upon Euler's foundations. The collaborative efforts of these mathematicians gave rise to a formula that has since become a cornerstone in mathematical analysis. In fact, if the function $f(x)$ is analytic in the integration region, then the famous Euler-Maclaurin formula reads:

$$\sum_{k=1}^{n-1} f(k) = \int_0^n f(x)\,dx - \frac{f(0)+f(n)}{2} + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!}\left[f^{(2k-1)}(n) - f^{(2k-1)}(0)\right].$$

An elementary view of this formula was discussed extensively in.[22] The elegance of the Euler-Maclaurin formula lies in its derivation, grounded in the fundamental technique of integration by parts. By cleverly applying this method, Euler and Maclaurin created a formula that connects discrete sums to continuous integrals. The derivation involves manipulating the discrete sums, introducing integral terms, and carefully handling the boundary terms to obtain a remarkably expressive formula. This process showcases the ingenuity of these mathematicians in formulating a bridge between discrete and continuous mathematical concepts. The Euler-Maclaurin formula has garnered considerable attention among researchers, prompting a diverse exploration of various alternative formulations of the aforementioned theorem.

Darboux offered an alternative derivation, employing the mean value theorem to the integrals within the formula. This approach provides a fresh perspective, revealing the connection between discrete and continuous processes through the lens of the mean value theorem. Darboux's insight enhances our understanding of the formula, showcasing the various mathematical pathways leading to its elegant expression. Throughout this work, I is a real interval, $a, b \in I^\circ$ (the interior of I) with $a < b$. Let $\mathcal{P}_n(I)$ be the class of polynomials of degree n defined on an interval $I \subseteq \mathbb{R}$.

The origin of the Euler-Maclaurin formula could be noted in the celebrated Darboux formula: Let $f(x)$ be analytic at all points of the interval $[a, x]$, and let $\varphi(t) \in \mathcal{P}_n$. If $t \in [0,1]$ we have by differentiation:

$$\begin{aligned}\frac{d}{dt}\sum_{k=1}^{n} &(-1)^k (x-a)^k \varphi^{(n-k)}(t) f^{(k)}(a+t(x-a)) \\ &= -(x-a)\varphi^{(n)}(t) f'(a+t(x-a)) + (-1)^n (x-a)^{n+1}\varphi(t) f^{(n+1)}(a+t(x-a)) \quad (1)\end{aligned}$$

Since $\varphi^{(n)}(t) = \varphi^{(n)}(0) =$constant, we integrate from 0 to 1 with respect to t and obtain

$$\begin{aligned}\varphi^{(n)}(0)[f(x)-f(a)] &= \sum_{k=1}^{n} (-1)^{k-1}(x-a)^k \left\{\varphi^{(n-k)}(1) f^{(k)}(x) - \varphi^{(n-k)}(0) f^{(k)}(a)\right\} \\ &\quad + (-1)^n (x-a)^{n+1}\int_0^1 \varphi(t) f^{(n+1)}(a+t(x-a))\,dt \quad (2)\end{aligned}$$

which is known as Darboux's formula, see.[23] A clear discussion of this formula was also described significantly in.[24]

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The Euler-Maclaurin formula stands as a mathematical beacon, guiding researchers and practitioners through the intricacies of mathematical analysis. Its significance lies not only in its historical origins but also in its pervasive influence on contemporary mathematics and physics. Mathematicians, physicists, and engineers continue to rely on the formula for its ability to simplify intricate calculations and provide accurate approximations. As a testament to its enduring importance, the Euler-Maclaurin formula remains an indispensable tool in the mathematical toolkit, enriching our understanding of both discrete and continuous mathematical phenomena.

In his construction of the Darboux reached an interesting expansion that is not less important than the celebrated Euler-Maclaurin formula itself, indeed we have:[23]

$$\begin{aligned}(x-a)\,f'(a) &= f(x)-f(a)-\frac{x-a}{2}\left[f'(x)-f'(a)\right]\\ &+\sum_{m=1}^{n-1}\frac{(-1)^{m-1}B_m\,(x-a)^{2m}}{(2m)!}\left[f^{(2m)}(x)-f^{(2m)}(a)\right]-R_n\left(f,B_{2n}\right),\end{aligned}$$

such that

$$R_n\left(f,B_{2n}\right)=\frac{(x-a)^{2n+1}}{(2n)!}\int_0^1 B_{2n}(t)\,f^{(2n+1)}\left(a+t\,(x-a)\right)dt, \tag{3}$$

where $B_k(t)$ $(k=1,2,3,\cdots)$ are the Bernoulli polynomials, and B_k are the Bernoulli numbers. Since the odd Bernoulli numbers B_{2k-1} $(k=1,2,\cdots)$ are all zeros the then above expansion could be rewritten as:

$$\begin{aligned}f(x) &= f(a)+(x-a)\,f'(a)+\frac{(x-a)}{2}\left[f'(x)-f'(a)\right] \\ &-\sum_{m=1}^{n-1}(-1)^{m-1}\frac{B_{2m}\,(x-a)^{2m}}{(2m)!}\left[f^{(2m)}(x)-f^{(2m)}(a)\right]+R_n\left(f,B_{2n}\right).\end{aligned} \tag{4}$$

Accordingly; in this work, a general higher-order implicit method that outperforms the Runge–Kutta methods in terms of accuracy is derived. An error bound for the Euler-Maclaurin higher-order method, showcasing its stability, convergence, and greater efficiency compared to the conventional Runge-Kutta method is presented. To substantiate our claims, numerical experiments are provided, highlighting the exceptional efficiency of our proposed method over the traditional well-known methods.

2 The Euler-Maclaurin Method for Approximating Solutions of I.V.P.

This method aims to obtain a new approximation for the well-posed initial-value problem

$$\frac{dy}{dt}=f(t,y), \qquad a\le t\le b, \qquad y(a)=\alpha. \tag{5}$$

Suppose the solution $y(t)$ to the initial-value problem has $(2n+1)$-continuous derivatives. Expanding $y(t)$ in terms of its $(2n)$-th Euler-Maclaurin expansion about t_i and evaluate at t_{i+1}, we obtain

$$\begin{aligned}y(t_{i+1}) &= y(t_i)+(t_{i+1}-t_i)\,y'(t_i)+\frac{(t_{i+1}-t_i)}{2}\left[y'(t_{i+1})-y'(t_i)\right] \\ &-\sum_{m=1}^{n-1}(-1)^{m-1}\frac{B_{2m}\,(t_{i+1}-t_i)^{2m}}{(2m)!}\left[y^{(2m)}(t_{i+1})-y^{(2m)}(t_i)\right] \\ &+\frac{(t_{i+1}-t_i)^{2n+1}}{(2n)!}\int_0^1 B_{2n}(s)\,y^{(2n+1)}\left(t_i+s\,(t_{i+1}-t_i)\right)ds\end{aligned} \tag{6}$$

We commence by establishing the stipulation that the distribution of mesh points is uniform across the interval $[a,b]$. This requisite is guaranteed through the selection of a positive integer N, from which the mesh points are subsequently chosen.

$$t_i=a+ih, \qquad \text{for each} \quad i=0,1,2,\cdots,N.$$

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The step size or the uniform spacing between the points $h = \frac{b-a}{N} = t_{i+1} - t_i$. Suppose that the unique solution to (5), has $(2n+1)$ continuous derivatives on $[a, b]$, so that for each $i = 0, 1, 2, \ldots, N-1$. Also, since $y(t)$ satisfies the differential equation (6), Successive differentiation of the solution, $y(t)$, gives

$$y'(t) = f(t, y(t)), y''(t) = f'(t, y(t)), \ldots, y^{(k)}(t) = f^{(k-1)}(t, y(t)).$$

Substituting these results into (6) gives

$$\begin{aligned} y(t_{i+1}) = y(t_i) + hf(t_i, y(t_i)) + \frac{h}{2}\left[f(t_{i+1}, y(t_{i+1})) - f(t_i, y(t_i))\right] \\ - \sum_{m=1}^{n-1} (-1)^{m-1} \frac{B_{2m}h^{2m}}{(2m)!} \left[f^{(2m-1)}(t_{i+1}, y(t_{i+1})) - f^{(2m-1)}(t_i, y(t_i))\right] \end{aligned} \tag{7}$$

The difference-equation method corresponding to (7) is obtained by deleting the remainder term involving ξ_i.

$$\begin{aligned} w_0 &= \alpha \\ w_{i+1} &= w_i + hf(t_i, y(t_i)) + \frac{h}{2}\left[f(t_{i+1}, y(t_{i+1})) - f(t_i, y(t_i))\right] - h\mathcal{M}^{(n-1)}(w_i, w_{i+1}), \end{aligned} \tag{8}$$

for each $i = 0, 1, 2, \cdots N-1$, where

$$\mathcal{M}^{(n-1)}(w_i, w_{i+1}) := \sum_{m=1}^{n-1} (-1)^{m-1} \frac{B_{2m}h^{2m-1}}{(2m)!} \left[f^{(2m-1)}(t_{i+1}, y(t_{i+1})) - f^{(2m-1)}(t_i, y(t_i))\right],$$

In particular, we are interested in the following case of (8).

2.1 The Euler-Maclaurin Method of Order 7

Setting $n = 3$ in (8), we get

$$\begin{aligned} w_0 &= \alpha \\ w_{i+1} &= w_i + hf(t_i, w_i) + \frac{h}{2}\left[f(t_{i+1}, w_{i+1}) - f(t_i, w_i)\right] - \frac{h^2}{12}\left[f'(t_{i+1}, w_{i+1}) - f'(t_i, w_i)\right] \\ &\quad + \frac{h^4}{720}\left[f'''(t_{i+1}, w_{i+1}) - f'''(t_i, w_i)\right] - \frac{h^6}{30240}\left[f^{(5)}(t_{i+1}, w_{i+1}) - f^{(5)}(t_i, w_i)\right], \end{aligned} \tag{9}$$

for each $i = 0, 1, 2, \cdots N-1$.

Proposition 2.1. *The Euler-Maclaurin Method Order* (9) *is of order* 7.

Proof. Substituting the exact solution in the Taylor expansion and simplifying, we get

$$\begin{aligned} & y(t_{i+1}) - y(t_i) - hf(t_i, y(t_i)) - \frac{h}{2}\left[f(t_{i+1}, y(t_{i+1})) - f(t_i, y(t_i))\right] \\ & \quad + \frac{h^2}{12}\left[f'(t_{i+1}, y(t_{i+1})) - f'(t_i, y(t_i))\right] - \frac{h^4}{720}\left[f'''(t_{i+1}, y(t_{i+1})) - f'''(t_i, y(t_i))\right] \\ & \quad + \frac{h^6}{30240}\left[f^{(5)}(t_{i+1}, w_{i+1}) - f^{(5)}(t_i, w_i)\right] \\ & = y(t_i) + hy'(t_i) + \frac{h^2}{2}y''(t_i) + \frac{h^3}{6}y'''(t_i) + \frac{h^4}{24}y^{(4)}(t_i) + \frac{h^5}{120}y^{(5)}(t_i) + \frac{h^6}{720}y^{(6)}(t_i) + O\left(h^7\right) \\ & \quad - y(t_i) + \frac{h}{2}y'(t_i) - \frac{h^2}{12}y''(t_i) + \frac{h^4}{720}y^{(4)}(t_i) - \frac{h^6}{30240}y^{(6)}(t_i) \\ & \quad - \frac{h}{2}\left[hy''(t_i) + \frac{h^2}{2!}y'''(t_i) + \frac{h^3}{3!}y^{(4)}(t_i) + \frac{h^4}{4!}y^{(5)}(t_i) + \frac{h^5}{5!}y^{(6)}(t_i) + O\left(h^6\right)\right] \\ & \quad + \frac{h^2}{12}\left[hy'''(t_i) + \frac{h^2}{2!}y^{(4)}(t_i) + \frac{h^3}{3!}y^{(5)}(t_i) + \frac{h^4}{4!}y^{(6)}(t_i) + O\left(h^5\right)\right] \\ & \quad - \frac{h^4}{720}\left[hy^{(5)}(t_i) + \frac{h^2}{2!}y^{(6)}(t_i) + O\left(h^3\right)\right] + \frac{h^6}{30240} \cdot O(h) \\ & = O\left(h^7\right), \end{aligned}$$

which means that (9) is of order 7. □

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Remark 2.2. In general, using induction one can observe that the general Euler-Maclaurin Method is of $O\left(h^{2n+1}\right)$.

3 Convergence and Stability of the general Euler-Maclaurin method

To prove the convergence and the error bound of the general Euler-Maclaurin Method (8), we need the following key lemma.[25]

Lemma 3.1. *If s and t are positive real numbers, $\{a_i\}_{i=1}^{k}$ is a sequence satisfying $a_0 \geq -t/s$ and*

$$a_{i+1} \leq \exp\left((1+i)\, s\right)\left(a_0 + \frac{t}{s}\right) - \frac{t}{s}.$$

In the next result, we prove that the Euler-Maclaurin method of order $2n$ is convergent and an error bound is derived.

Theorem 3.2. *Suppose $f^{(k)}$ $(0 \leq k \leq 2n-1)$ are continuous and satisfy Lipschitz condition with constant L_k on*

$$D := \left\{(t, y) : a \leq t \leq b, -\infty < y < \infty\right\},$$

and that a constant M exists with $\left|f^{(2n)}\left(t, y(t)\right)\right| \leq M$, for all $t \in [a, b]$, where $y(t)$ denotes the unique solution to the initial-value problem

$$y' = f\left(t, y\right), \qquad a \leq t \leq b, \qquad y\left(a\right) = \alpha.$$

Let $w_0, w_1, \cdots, w_N$ be the approximations generated by the Euler-Maclaurin method (8) *for some positive integer N. Then, the general Euler-Maclaurin method described in* (8) *is convergent.*

Proof. When $i = 0$, the assertion is correct, as it holds that $y(t_0) = w_0 = \alpha$. Otherwise, from (6), we have

$$\begin{aligned} y\left(t_{i+1}\right) = y\left(t_i\right) + hf\left(t_i, y\left(t_i\right)\right) + \frac{h}{2}\left[f\left(t_{i+1}, y\left(t_{i+1}\right)\right) - f\left(t_i, y\left(t_i\right)\right)\right] \\ - \sum_{m=1}^{n-1}(-1)^{m-1}\frac{B_{2m}h^{2m}}{(2m)!}\left[f^{(2m-1)}\left(t_{i+1}, y\left(t_{i+1}\right)\right) - f^{(2m-1)}\left(t_i, y\left(t_i\right)\right)\right] \\ + \frac{h^{2n+1}}{(2n)!}\int_0^1 B_{2n}\left(s\right) f^{(2n)}\left(t_i + s\left(t_{i+1} - t_i\right)\right) ds \end{aligned}$$

for $i = 0, 1, \cdots, N-1$, and from the equations in (8),

$$\begin{aligned} w_{i+1} = w_i + hf\left(t_i, w_i\right) + \frac{h}{2}\left[f\left(t_{i+1}, w_{i+1}\right) - f\left(t_i, w_i\right)\right] \\ - \sum_{m=1}^{n-1}(-1)^{m-1}\frac{B_{2m}h^{2m}}{(2m)!}\left[f^{(2m-1)}\left(t_{i+1}, w_{i+1}\right) - f^{(2m-1)}\left(t_i, w_i\right)\right] \end{aligned}$$

for each $i = 0, 1, 2, \cdots N-1$. Utilizing the notations $y_i = y(t_i)$ and $y_{i+1} = y(t_{i+1})$, we deduce the following by subtracting these two equations:

$$\begin{aligned} y_{i+1} - w_{i+1} = y_i - w_i + hf\left(t_i, y_i\right) - hf\left(t_i, w_i\right) \\ + \frac{h}{2}\left[f\left(t_{i+1}, y_{i+1}\right) - f\left(t_{i+1}, w_{i+1}\right)\right] - \frac{h}{2}\left[f\left(t_i, y_i\right) - f\left(t_i, w_i\right)\right] \\ - \sum_{m=1}^{n-1}(-1)^{m-1}\frac{B_{2m}h^{2m}}{(2m)!}\left[f^{(2m-1)}\left(t_{i+1}, y_{i+1}\right) - f^{(2m-1)}\left(t_{i+1}, w_{i+1}\right)\right] \\ - \sum_{m=1}^{n-1}(-1)^{m-1}\frac{B_{2m}h^{2m}}{(2m)!}\left[f^{(2m-1)}\left(t_i, y_i\right) - f^{(2m-1)}\left(t_i, w_i\right)\right] \\ + \frac{h^{2n+1}}{(2n)!}\int_0^1 B_{2n}\left(s\right) f^{(2n)}\left(t_i + s\left(t_{i+1} - t_i\right)\right) ds \end{aligned}$$

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Employing the triangle inequality, we have

$$\begin{aligned}|y_{i+1}-w_{i+1}| &= |y_i-w_i| + h\left|f(t_i,y_i)-f(t_i,w_i)\right| \\ &+ \frac{h}{2}\left|f(t_{i+1},y_{i+1})-f(t_{i+1},w_{i+1})\right| + \frac{h}{2}\left|f(t_i,y_i)-f(t_i,w_i)\right| \\ &+ \sum_{m=1}^{n-1}\frac{B_{2m}h^{2m}}{(2m)!}\left|f^{(2m-1)}(t_{i+1},y_{i+1})-f^{(2m-1)}(t_{i+1},w_{i+1})\right| \\ &+ \sum_{m=1}^{n-1}\frac{B_{2m}h^{2m}}{(2m)!}\left|f^{(2m-1)}(t_i,y_i)-f^{(2m-1)}(t_i,w_i)\right| \\ &+ \frac{h^{2n}}{(2n)!}\left|f^{(2n)}(\mu_i,y(\mu_i))\right|\int_0^1 |B_{2n}(s)|\,ds.\end{aligned}$$

Now, function $f^{(m-1)}$ $(m=1,2,\cdots,2n-1)$ fulfills the Lipschitz condition in the second variable with a constant denoted as $L := \max_{1\le m\le 2n-1}\{L_k\}$, and $\left|f^{(2n+1)}(t,y(t))\right| \le M$, so

$$\begin{aligned}|y_{i+1}-w_{i+1}| &\le |y_i-w_i| + hL|y_i-w_i| + \frac{h}{2}L|y_{i+1}-w_{i+1}| + \frac{h}{2}L|y_i-w_i| \\ &+ L\cdot\sum_{m=1}^{n-1}\frac{|B_{2m}|h^{2m}}{(2m)!}|y_{i+1}-w_{i+1}| + L\cdot\sum_{m=1}^{n-1}\frac{|B_{2m}|h^{2m}}{(2m)!}|y_i-w_i| \\ &+ \frac{h^{2n}}{(2n)!}M\int_0^1 |B_{2n}(s)|\,ds.\end{aligned}$$

Combining the terms we get

$$\begin{aligned}|y_{i+1}-w_{i+1}| &\le \left(\frac{1}{2}hL + L\cdot\sum_{m=1}^{n-1}\frac{|B_{2m}|h^{2m}}{(2m)!}\right)|y_{i+1}-w_{i+1}| \\ &+ \left(1+\frac{3}{2}hL + L\cdot\sum_{m=1}^{n-1}\frac{|B_{2m}|h^{2m}}{(2m)!}\right)(|y_i-w_i|) \\ &+ \frac{h^{2n}}{(2n)!}M|B_{2n}|.\end{aligned}$$

where we used the fact $|B_{2n}(s)| < |B_{2n}|$, see.[26] Now, to seek simplicity, let us define

$$S_n(L,h) := \left(1+\frac{3}{2}hL + L\cdot\sum_{m=1}^{n-1}\frac{|B_{2m}|h^{2m}}{(2m)!}\right), \qquad C_n(L,h) := \left(1-\frac{1}{2}hL - L\cdot\sum_{m=1}^{n-1}\frac{|B_{2m}|h^{2m}}{(2m)!}\right),$$

and

$$E_n(h) := 2\sum_{m=1}^{n-1}\frac{|B_{2m}|h^{2m-1}}{(2m)!}$$

Before we go further, we need to remark that

$$\begin{aligned}\frac{1}{2}LhE_n(h) = L\sum_{m=1}^{n-1}\frac{|B_{2m}|h^{2m}}{(2m)!} &\le L\cdot\max_{1\le m\le n-1}\left\{h^{2m}\right\}\cdot\sum_{m=1}^{n-1}\frac{|B_{2m}|}{(2m)!} \\ &\approx L\cdot\max_{1\le k\le n-1}\left\{h^{2m}\right\}\cdot\sum_{m=1}^{n-1}\frac{2(2m)!}{(2\pi)^{2m}}\cdot\frac{1}{(2m)!} \\ &= K\cdot\left[\frac{2}{4\pi^2-1}+\frac{8\pi^2}{1-4\pi^2}\cdot\left(\frac{1}{4\pi^2}\right)^n\right],\end{aligned}$$

where the last sum is evaluated using Maple Software; before that, we note that we have used the asymptotic approximation of even Bernoulli numbers,[26] $(-1)^{m+1}B_{2m} \approx \frac{2(2m)!}{(2\pi)^{2m}}$, for every positive integer m. Moreover, as

$$\frac{1}{2}LhE_n(h) \le K\cdot\frac{2}{4\pi^2-1}, \qquad \text{as } n\to\infty.$$

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Considering our ultimate interest in allowing $h \to 0^+$, it is acceptable to presume that

$$\frac{1}{2} LhE_n(h) < K \cdot \frac{2}{4\pi^2 - 1}$$

where K is some fixed nonzero positive real number, without any adverse consequences. Consequently, we can infer that

$$\begin{aligned}|y_{i+1} - w_{i+1}| &\le \frac{S_n(L,h)}{C_n(L,h)} \cdot |y_i - w_i| + \frac{h^{2n}}{(2n)! C_n(L,h)} M |B_{2n}| \\ &= \left(1 + \frac{S_n(L,h) - C_n(L,h)}{C_n(L,h)}\right) \cdot |y_i - w_i| + \frac{h^{2n}}{(2n)! C_n(L,h)} M |B_{2n}| \\ &= \left(1 + \frac{L \cdot h \cdot E_n(h)}{C_n(L,h)}\right) \cdot |y_i - w_i| + \frac{h^{2n}}{(2n)! C_n(L,h)} M |B_{2n}|\end{aligned}$$

Employing Lemma 3.1, with $s(h) = \frac{L \cdot h \cdot E_n(h)}{C_n(L,h)}$, $t(h) = \frac{h^{2n}}{(2n)! C_n(L,h)} M |B_{2n}|$, and $a_j = |y_j - w_j|$, for each $j = 0, 1, 2, \cdots, N$, we observe that

$$|y_{i+1} - w_{i+1}| \le \exp\left((i+1) \cdot \frac{L \cdot h \cdot E_n(h)}{C_n(L,h)}\right) \left(|y_0 - w_0| + \frac{t(h)}{s(h)}\right) - \frac{t(h)}{s(h)}.$$

Since $|y_0 - w_0| = 0$,

$$\lim_{h \to 0^+} \frac{L \cdot h \cdot E_n(h)}{C_n(L,h)} = 0, \qquad \text{and} \qquad \lim_{h \to 0^+} \frac{t(h)}{s(h)} = 0.$$

then $\lim\limits_{h \to 0^+} \max\limits_{1 \le i \le N} |y_{i+1} - w_{i+1}| = 0$, which means that that w_{i+1} converges to y_{i+1}, and thus the Euler-Maclaurin Method of Order $2n$ is converge as required. □

Theorem 3.3. *Under the assumption of Theorem 3.2. We have*

$$|y_{i+1} - w_{i+1}| \le \frac{t(h)}{s(h)} \cdot \left(\exp\left((t_{i+1} - a) \frac{L \cdot hE_n(h)}{C_n(L,h)}\right) - 1\right) \tag{10}$$

for each $i = 0, 1, 2, \cdots, N-1$.

Proof. The inequality follows from the last inequality in the proof of Theorem 3.2, and since $(i+1)h = t_{i+1} - t_0 = t_{i+1} - a$, the error bound of this method is deduced from the last inequality in the proof of Theorem 3.2 which reduces to (10). □

Remark 3.4. According to the general theorem of stability of well-posed I.V.P., Theorem (3.2) implies that the general Euler-Maclaurin method described in (9) is stable and consistent.

The primary significance of the error-bound formula presented in Theorem 3.2 lies in its direct proportionality to the step size, h. As a result, reducing the step size should yield proportionally enhanced accuracy in the approximations.

4 Perturbations of the general Euler-Maclaurin method

Omitted from the findings of Theorems 3.2 & 3.3 is the consideration of the impact of round-off errors when selecting the step size. With diminishing h, an increased number of calculations is required, leading to a higher expectation of round-off errors. In practice, the difference equation given in (8) is not employed for the computation of the approximation to the solution, denoted as y_i, at a mesh point t_i. Instead, we employ an equation of the following structure

$$\begin{aligned} v_0 &= \alpha + \delta_0 \\ v_{i+1} &= v_i + h\widetilde{B}^{(n)}(t_i, v_i) + \delta_{i+1}, \end{aligned} \tag{11}$$

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for each $i = 0, 1, 2, \cdots N-1$, where

$$\widetilde{B}^{(n)}(t_i, v_i) := f(t_i, v_i) + \frac{1}{2}\left[f(t_{i+1}, v_{i+1}) - f(t_i, v_i)\right] - \sum_{m=1}^{n-1} \frac{B_{2m} h^{2m-1}}{(2m)!}\left[f^{(2m-1)}(t_{i+1}, v_{i+1}) - f^{(2m-1)}(t_i, v_i)\right]$$

for each $i = 0, 1, 2, \cdots N-1$. Here, δ_i represents the round-off error linked to the value v_i. Employing techniques akin to those applied in the demonstration of Theorem 3.2, we can derive an error threshold for the finite-precision approximations of y_i, as determined by the Euler-Maclaurin method. Consequently, it is feasible to formulate an analogous outcome to the following result.

Theorem 4.1. *Let $y(t)$ denote the unique solution to the initial-value problem*

$$y' = f(t, y), \qquad a \le t \le b, \qquad y(a) = \alpha. \tag{12}$$

Let $v_0, v_1, \cdots, v_N$ be the approximations generated by the Euler-Maclaurin method (8) *for some positive integer N. If $|\delta_i| < \delta$ for each $i = 0, 1, \cdot, N$ and the hypotheses of Theorem 3.2 hold for* (12)*, then*

$$|y_1 - v_i| \le \left(\frac{t(h)}{s(h)} + \frac{\delta C(n, h)}{LhE_n(h)}\right) \cdot \left(\mathrm{e}^{\left((t_i - a)\frac{L \cdot hE_n(h)}{C_n(L,h)}\right)} - 1\right) + |\delta_0| \, \mathrm{e}^{\left((t_i - a)\frac{L \cdot hE_n(h)}{C_n(L,h)}\right)} \tag{13}$$

for each $i = 0, 1, 2, \cdots, N$.

Proof. The proof is similar to the proof of Theorem 3.2 applied for the difference equation (11). □

On the other hand, it is convenient to note that the error bound (13) is no longer linear in h. In fact, since

$$\lim_{h \to 0^+}\left(\frac{t(h)}{s(h)} + \frac{\delta C_n(L, h)}{LhE_n(h)}\right) \to \infty$$

As the step size h tends toward infinitesimally small values, it is anticipated that the error will escalate. Moreover, as the step size h is reduced beyond this critical value, there is a tendency for the total error in the approximation to increase. Nevertheless, it is worth noting that, under typical circumstances, the magnitude of the error, denoted by δ, remains sufficiently small. Consequently, this established lower bound for h does not significantly impact the efficacy or accuracy of the Euler-Maclaurin method in its computational operation. Despite the theoretical considerations regarding the escalation of error with decreasing h, the practical implementation of the Euler-Maclaurin method remains robust within the determined range of step sizes.

5 Numerical Experiments

In this section, we apply the Euler-Maclaurin method of order 7 with various step sizes. to several I.V.P.

Example 5.1. The Euler-Maclaurin method of order 7 (9) is employed to approximate the solution of the initial-value problem

$$y'(t) = y - t^2 + 1, \qquad 0 \le t \le 2, \qquad y(0) = 0.5, \tag{14}$$

with specific parameters set to $N = 10$, $h = 0.2$, $t_i = 0.2i$, and $w_0 = 0.5$. This approximation is then compared with the exact solution provided by $y(t) = (t+1)^2 - 0.5\mathrm{e}^t$.

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Table 1: The table shows the absolute error in the Runge-Kutta (RK) method of order 6 and Euler-Maclaurin Method (EM) of order 7 applied in Example 5.1 with step size $h = 0.2$.

t_i	RK Error$\times 10^{-6}$	EM Error$\times 10^{-10}$
0.0	0.00000000	0.00000000
0.2	0.06348401	0.00259126
0.4	0.13430130	0.00633049
0.6	0.21258714	0.01159072
0.8	0.29817725	0.01887823
1.0	0.39046847	0.02883027
1.2	0.48823256	0.04224176
1.4	0.58936887	0.06021405
1.6	0.69057835	0.08404832
1.8	0.78693577	0.11551648
2.0	0.87133141	0.15674572

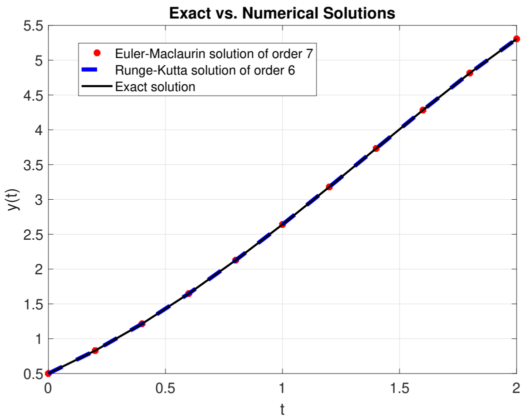

Figure 1: Example 5.1: The exact solution compared with the Runge-Kutta (RK) method of order 6 and the Euler-Maclaurin Method (EM) of order 7 with step size $h = 0.2$

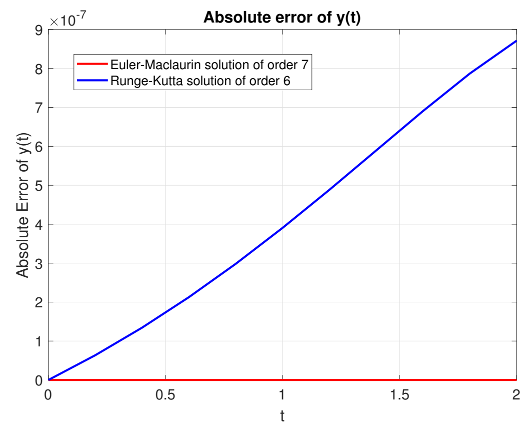

Figure 2: Example 5.1: Absolute errors of the RK method of order 6 and Euler-Maclaurin of order 7 with step size $h = 0.2$

As we can remark the Euler-Maclaurin Method (9) gives much better approximations compared with the celebrated Runge–Kutta method of order 6. Figures 1 and 2 show the comparison between the approximate solutions between the three methods and their corresponding absolute errors. To improve our results we consider two more examples.

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Example 5.2. The Euler-Maclaurin–Euler method (9) is employed to approximate the solution of the initial-value problem

$$y'(t) = \exp(t - y), \qquad 0 \le t \le 1, \qquad y(0) = 1, \tag{15}$$

with specific parameters set to $N = 10$, $h = 0.1$, $t_i = 0.1i$, and $w_0 = 1$. This approximation is then compared with the exact solution provided by $y(t) = \ln(\exp(t) + \exp(1) - 1)$.

Table 2: The table shows the absolute error in the Runge-Kutta (RK) method of order 6, and Euler-Maclaurin Method (EM) of order 7 applied in Example 5.2 with step size $h = 0.1$.

t_i	RK Error$\times 10^{-9}$	EM Error$\times 10^{-14}$
0.0	0.00000000	0.00000000
0.1	0.01797051	0.17763568
0.2	0.03922262	0.35527136
0.3	0.06406808	0.48849813
0.4	0.09274891	0.55511151
0.5	0.12541212	0.57731597
0.6	0.16208479	0.53290705
0.7	0.20265589	0.44408920
0.8	0.24686364	0.28865798
0.9	0.29429103	0.13322676
1.0	0.34437452	0.06661338

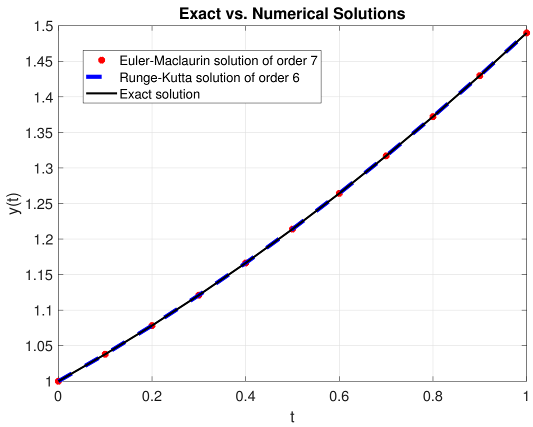

Figure 3: Example 5.2: The exact solution compared with the Euler-Maclaurin and Runge-Kutta methods of order 7 and 6; respectively, with stepsize $h = 0.1$

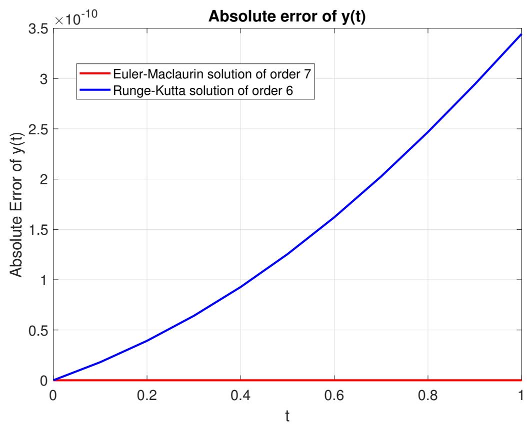

Figure 4: Example 5.2: Absolute errors of the Euler-Maclaurin's and Runge-Kutta methods of order 7 and 6; respectively, with step size $h = 0.1$.

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As we can remark the Euler-Maclaurin Method (9) gives much better approximations compared with the celebrated Runge–Kutta method of order 6. Figures 3 and 4 show the comparison between the approximate solutions between the three methods and their corresponding absolute errors. We consider the following example to enhance our outcomes and improve the Euler-Maclaurin method (9).

Example 5.3. The Euler-Maclaurin method (9) is employed to approximate the solution of the system of the linear initial-value problem

$$\begin{cases} z_1'(s) = z_2, & z_1(0) = 1 \\ z_2'(s) = -z_1 - 2\mathrm{e}^s + 1, & z_2(0) = 0 \\ z_3'(s) = -z_1 - \mathrm{e}^s + 1, & z_3(0) = 1 \end{cases}$$

for $0 \leq s \leq 2$, with specific parameters set to $N = 10$, $h = 0.2$ and $t_i = 0.2i$. This approximation is then compared with the exact solution provided by

$$\begin{cases} z_1(s) = \cos(s) + \sin(s) - \mathrm{e}^s + 1 \\ z_2(s) = -\sin(s) + \cos(s) - \mathrm{e}^s \\ z_3(s) = -\sin(s) + \cos(s) \end{cases}$$

Furthermore, a comparison is made between the classical RK's approach and our approximation. Specifically, Figures 5, 7, and 9 show the exact solution compared with the Euler-Maclaurin and RK methods with step size $h = 0.2$, whereas Figures 6, 8, 10 and Tables 3, 4, 5 show the absolute errors of the Euler-Maclaurin and RK methods of order 7 and 6, respectively, with the same step size. Roughly, the Euler-Maclaurin method gives outstanding approximations compared with the RK method.

Table 3: The table shows the comparison between the absolute error in both the Runge–Kutta (RK) of order 6 and the Euler-Maclaurin method (EM) of order 8 applied in Example 5.3 with step size $h = 0.2$ for $z_1(s)$.

t_i	RK Error	EM Error$\times 10^{-13}$
0.0	0.0000000	0.0000000
0.2	0.0426668	0.0022204
0.4	0.0927842	0.0077715
0.6	0.1468497	0.0166533
0.8	0.2008820	0.0277555
1.0	0.2506502	0.0466293
1.2	0.2919269	0.0677236
1.4	0.3207555	0.0954791
1.6	0.3337148	0.1298960
1.8	0.3281705	0.1663252
2.0	0.3024980	0.2065014

Table 4: The table shows the comparison between the absolute error in both the Runge–Kutta (RK) of order 6 and the Euler-Maclaurin method (EM) of order 7 applied in Example 5.3 with step size $h = 0.2$ for $z_2(s)$.

t_i	RK Error	EM Error$\times 10^{-13}$
0.0	0.0000000	0.0000000
0.2	0.0027999	0.0191513
0.4	0.0234673	0.0355271
0.6	0.0650996	0.0566213
0.8	0.1301210	0.0766053
1.0	0.2202233	0.0954791
1.2	0.3363741	0.1132427
1.4	0.4788972	0.1332267
1.6	0.6476335	0.1554312
1.8	0.8421832	0.1687538
2.0	1.0622309	0.1776356

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Table 5: The table shows the comparison between the absolute error in both the Runge–Kutta (RK) of order 6 and the Euler-Maclaurin method (EM) of order 7 applied in Example 5.3 with step size $h = 0.2$ for $z_3(s)$.

t_i	RK Error	OM Error$\times 10^{-14}$
0.0	0.0000000	0.0000000
0.2	0.0027999	0.0999200
0.4	0.0234673	0.1776356
0.6	0.0650996	0.2775557
0.8	0.1301210	0.3663735
1.0	0.2202234	0.4274358
1.2	0.3363742	0.4829470
1.4	0.4788973	0.5079270
1.6	0.6476337	0.5301314
1.8	0.8421834	0.5023759
2.0	1.0622312	0.4163336

As we can remark the Euler-Maclaurin method (9) gives much better approximations compared with both the celebrated Runge–Kutta method. Figures 5 and 6 show the comparison between the approximate solutions between the three methods and their corresponding absolute errors. Moreover, it is remarkable to note that the absolute error near discontinuity point $t = 1$ increases more rapidly in the Runge-Kutta method.

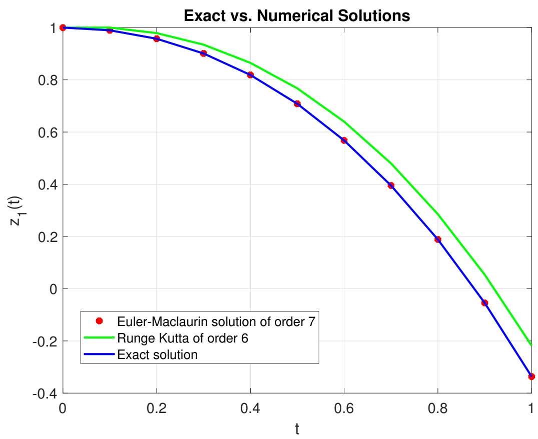

Figure 5: Example 5.3: The exact solution of $z_1(s)$ compared with the Euler-Maclaurin (EM) and Runge–Kutta (RK) methods of order 7 and 6; respectively, with step size $h = 0.2$.

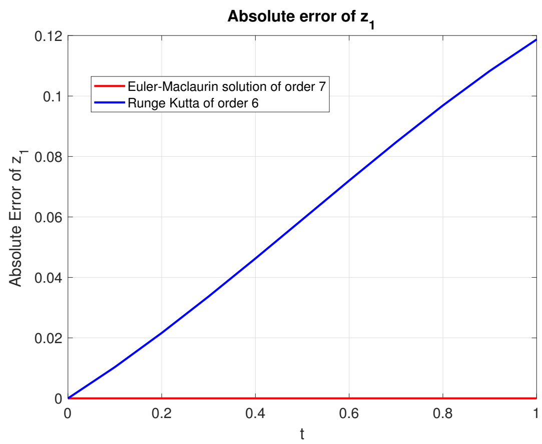

Figure 6: Example 5.3: Absolute errors of the Euler-Maclaurin's (EM) and Runge-Kutta (RK) methods of order 7 and 6; respectively, with step size $h = 0.2$ applied for $z_1(s)$.

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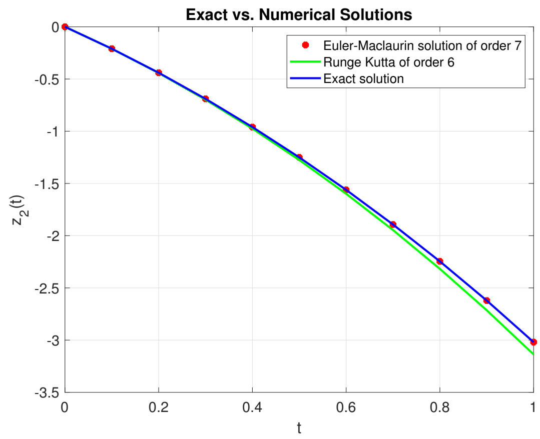

Figure 7: Example 5.3: The exact solution of $z_2(s)$ compared with the Euler-Maclaurin (EM) and Runge–Kutta (RK) methods of order 7 and 6; respectively, with step size $h = 0.2$.

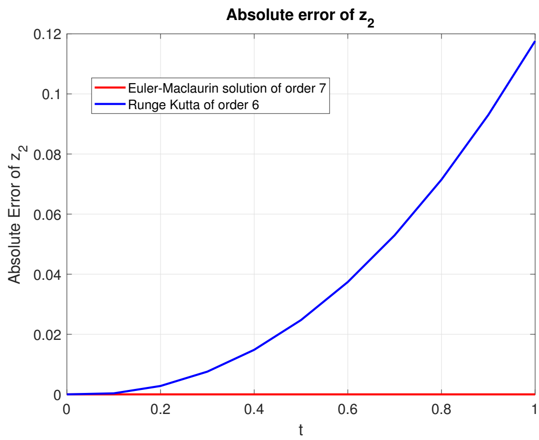

Figure 8: Example 5.3: Absolute errors of the Euler-Maclaurin's (EM) and Runge-Kutta (RK) methods of order 7 and 6; respectively, with step size $h = 0.2$ applied for $z_2(s)$

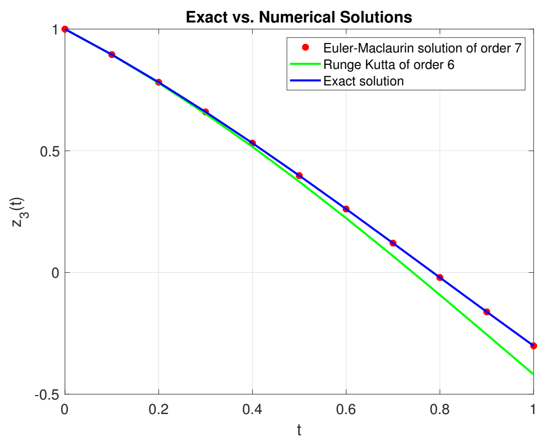

Figure 9: Example 5.3: The exact solution of $z_3(s)$ compared with the Euler-Maclaurin (EM) and Runge–Kutta (RK) methods of order 7 and 6; respectively, with step size $h = 0.2$.

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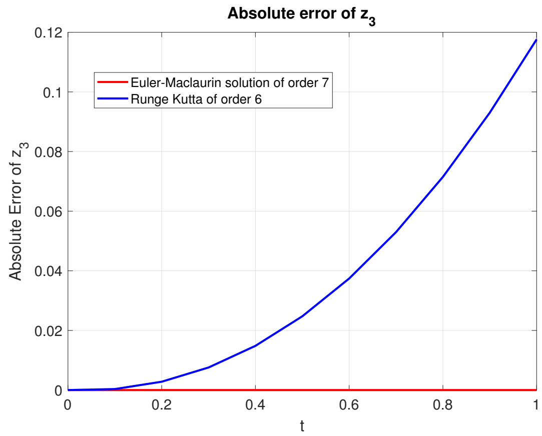

Figure 10: Example 5.3: Absolute errors of the Euler-Maclaurin's (EM) and Runge-Kutta (RK) methods of order 7 and 6; respectively, with step size $h = 0.2$ applied for $z_3(s)$.

6 Recommendation

In this study, we have introduced a novel approach for approximating I.V.P. Through the analysis of method (8) and the examination of relevant examples, it has been shown that the Euler-Maclaurin method surpasses previously acknowledged methods, notably the well-known Runge-Kutta method. Moreover, our extensive deliberations indicate that the Euler-Maclaurin method of order 7 outperforms the renowned Runge-Kutta methods of order 6 as long as the analytic solution is required. This is evidenced by the method's ability to yield superior outcomes with reduced absolute error.

The demonstrated superiority of the Euler-Maclaurin method extends beyond mere similarity, manifesting in heightened stability and accelerated convergence. The empirical evidence presented underscores the method's robustness and efficiency in addressing diverse contexts within mathematical modeling and analysis.

Over the long term, the Euler-Maclaurin method (8) of order $2n+1$ consistently outperforms the Runge-Kutta method, particularly when seeking analytic solutions. Additionally, the proposed method exhibits competitiveness in various scientific contexts, as exemplified by Example 5.3, providing clear evidence of its strong performance in approximating a system of linear I.V.P. compared to other known methods.

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