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Notes on q-Gamma Operators and Their Extension to Classes of Generalized Distributions

Shrideh Al-Omari [1,2,*], Wael Salameh [3] and Sharifah Alhazmi [4]

[1] Department of Mathematics, Faculty of Science, Al-Balqa Applied University, Salt 11134, Jordan
[2] Jadara University Research Center, Jadara University, Irbid 21110, Jordan
[3] Faculty of Information Technology, Abu Dhabi University, Abu Dhabi 59911, United Arab Emirates; wael.salameh@adu.ac.ae
[4] Mathematics Department, Al-Qunfudah University College, Umm Al-Qura University, Mecca 24382, Saudi Arabia; sehazmi@uqu.edu.sa
* Correspondence: shridehalomari@bau.edu.jo

Abstract: This paper discusses definitions and properties of q-analogues of the gamma integral operator and its extension to classes of generalized distributions. It introduces q-convolution products, symmetric q-delta sequences and q-quotients of sequences, and establishes certain convolution theorems. The convolution theorems are utilized to accomplish q-equivalence classes of generalized distributions called q-Boehmians. Consequently, the q-gamma operators are therefore extended to the generalized spaces and performed to coincide with the classical integral operator. Further, the generalized q-gamma integral is shown to be linear, sequentially continuous and continuous with respect to some involved convergence equipped with the generalized spaces.

Keywords: q-derivative; q-Boehmians; generalized symmetric distribution; q-hypergeometric function; p-differential operators

MSC: 05A30; 26D10; 26D15; 26A33

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1. Introduction

A field of calculus known as quantum calculus, or q-calculus, was founded by Jackson [1] to replace the conventional derivative with a difference operator. The quantum theory of calculus and symmetric calculus has been applied in various fields of science including number theory [2], orthogonal polynomials [3], geometric function theory [4,5], differential subordination [6], Daehee polynomials [7], zeta type functions [8], Bessel functions [9], univalent functions [5], fractional calculus [10–12] and generalized special functions [13]. In addition, the quantum theory of calculus connects, among other mathematical fields, mathematics and physics and draw the interest of numerous researchers from the literature [9,14,15]. Consequently, numerous advances in the area of q-hypergeometric functions and polynomials in the field of partitions and integral transforms have been accomplished by the relevant theory [16–19]. Moreover, the domains of vector spaces, particle physics, Lie theory, nonlinear electric circuit theory and heat conductions employ the q-hypergeometric functions as an illustration of their respective subjects (see, e.g., [20,21] and references therein). The q-derivative of a function φ, for $0 < q < 1$, is defined by [1]

$$(D_q\varphi)(\tau) = \frac{\varphi(\tau) - \varphi(q\tau)}{(1-q)\tau}, \tau \neq 0. \tag{1}$$

The complex number $\tau \in \mathbb{C}$, the natural number $j \in \mathbb{N}$ and the factorial of the natural number have q-analogues provided by [2]

$$[\tau]_q = \frac{1-q^\tau}{1-q}, [j]_q = \frac{1-q^j}{1-q} \text{ and } [j]_q! = [j]_q[j-1]_q \dots [2]_q[1]_q \text{ and } [0]_q! = 1, \tag{2}$$

respectively. Conversely, the shifted factorials have obtained q-analogues specified as [2]

$$(\tau;q)_j = \prod_{i=0}^{j-1}\left(1-\tau q^i\right), \ (\tau;q)_0 = 1 \text{ and } (\tau;q)_\infty = \lim_{j\to\infty}(\tau;q)_j. \tag{3}$$

The two methods for defining the q-analogues of the exponential function of a real number are introduced in terms of (2) and (3) as [22]

$$E_q(\tau) = \sum_{j=0}^{\infty} q^{\frac{j(j-1)}{2}} \frac{\tau^j}{[k]_q!} = (\tau;q)_\infty, \ \tau \in \mathbb{R}, \tag{4}$$

and

$$e_q(\tau) = \sum_{j=0}^{\infty} \frac{\tau^j}{[j]_q!} = \frac{1}{(\tau;q)_\infty}, |\tau| < |1-q|^{-1}. \tag{5}$$

On the other hand, the definite and improper integrals are, respectively, assigned q-analogues given as [2]

$$\int_0^\gamma \psi(\gamma)d_q\gamma = (1-q)\sum_{i\geq 0}\psi\left(q^i\gamma\right)\gamma q^i \tag{6}$$

and

$$\int_0^\infty \psi(\gamma)d_q\gamma = (1-q)\sum_{i\in\mathbb{Z}} q^i\psi\left(q^i\right). \tag{7}$$

On the basis of the q-exponential functions (4) and (5), the gamma function has two q-analogues defined as

$$\Gamma_q(\delta) = \int_0^{\frac{1}{1-q}} \gamma^{\delta-1}E_q(q(1-q)\gamma)d_q\gamma \tag{8}$$

and

$$\Gamma_q^*(\delta) = k(w;\delta)\int_0^{\frac{\infty}{w(1-q)}} \gamma^{\delta-1}e_q(-(1-q)\gamma)d_q\gamma, \tag{9}$$

where

$$k(w;\delta) = w^{\delta-1}\frac{(-q/w;q)_\infty(-w;q)_\infty}{(-q^\delta/w;q)_\infty(-wq^{1-\delta};q)_\infty}. \tag{10}$$

However, we turn to [9,15,23] and the sources given therein for definitions and preliminary information about differentiation and integration by parts.

In this paper, Sections 1 and 2 go over the foundations of the q-calculus theory and the abstract structure of Boehmians. Certain q-convolution products are presented and q-convolution theorems are established in Section 3. The q-delta sequences are derived in Section 4 in order to examine many important assumptions for creating the spaces H and B of q-generalized functions. An inversion formula and other generic properties along with a specific extension of $\hat{g}_{j,q}$ are derived in Section 5.

2. The Class of Boehmian Spaces

Assume that S forms a subspace of a linear space L. Then, for any pair of elements $f \in (L, \overset{q}{*})$ and $\omega_1 \in (S, \dagger)$, the products $\overset{q}{*}$ and $\dagger$ are assigned such that

$$(i)\ \omega_1,\omega_2 \in S \Rightarrow \omega_1\dagger\omega_2 \in S, \omega_1\dagger\omega_2 = \omega_2\dagger\omega_1. \tag{11}$$

$$(ii)\ \theta \in L, \omega_1, \omega_2 \in S \Rightarrow \left(\theta \overset{q}{*} \omega_1\right) \overset{q}{*} \omega_2 = \theta \overset{q}{*} (\omega_1 \dagger \omega_2). \tag{12}$$

$$(iii)\theta_1, \theta_2 \in L, \omega_1 \in S, r \in \mathbb{R} \Rightarrow (\theta_1 + \theta_2) \overset{q}{*} \omega_1 = \theta_1 \overset{q}{*} \omega_1 + \theta_2 \overset{q}{*} \omega_1, r\left(\theta_1 \overset{q}{*} \omega_1\right) = (r\theta_1) \overset{q}{*} \omega_1. \tag{13}$$

Let $\Delta(0, \infty)$ be a family of sequences contained in S. Then, if $\Delta(0, \infty)$ meets the two characteristics Δ_1 and Δ_2, it is considered a family of delta sequences (approximating identities) provided that

$$\Delta_1 : \text{For } \theta_1, \theta_2 \in L, (\delta_n) \in \Delta \text{ and } \theta_1 \overset{q}{*} \delta_n = \theta_2 \overset{q}{*} \delta_n, \text{ we have } \theta_1 = \theta_2, \forall n \in \mathbb{N}. \tag{14}$$

$$\Delta_2\ : (\omega_n), (\delta_n) \in \Delta \Rightarrow (\omega_n \dagger \delta_n) \in \Delta. \tag{15}$$

If $A = \{(\theta_n), (\omega_n), (\theta_n) \in L, (\omega_n) \in \Delta, \forall n \in \mathbb{N}\}$, then $((\theta_n), (\omega_n))$ is a pair of quotients of sequences in A iff

$$\theta_n \overset{q}{*} \omega_m = \theta_m \overset{q}{*} \omega_n,$$

for all natural numbers n and m. The pairs $((\theta_n), (\omega_n))$ and $((g_n), (\delta_n))$ satisfying (11)–(15) are equivalent pairs of quotients according to the notation $\sim$ iff

$$\theta_n \overset{q}{*} \delta_m = g_m \overset{q}{*} \omega_n,$$

for all natural numbers n and m. In this regard, $\sim$ forms an equivalent relation on the set A and therefore θ_n / ω_n constitutes an equivalence class called the Boehmian. The resulting space of such Boehmians is denoted by B.

In order to obtain the comprehensive narrative of Boehmians, please consult [12,24–27].

3. q-Convolution Theorem

In the framework of q-calculus, the current part presents convolution theorems and suggests new q-analogues of the q-gamma integral operator.

Let $\delta \in [0, \infty)$ and $k \in \mathbb{N}$. Then, the gamma integral operator is defined for a function φ under specific exponential growth conditions as follows [28]:

$$(G_k\theta)(\delta) = \frac{k^k}{\delta^k \Gamma(k)} \int_0^\infty \theta(\tau)\tau^{k-1} e^{\frac{-k\tau}{\delta}} d\tau \tag{16}$$

when the integral converges. The q-analogue is assigned to the gamma operator G_k in a context of quantum calculus theory in the form [29]

$$G_{k,q}(\theta; \delta) = \frac{k^k}{\delta^k \Gamma_q(k)} \int_0^\infty \theta(\tau)\tau^{k-1} e_q\left(\frac{-qk\tau}{\delta}\right) d_q\tau. \tag{17}$$

Hereafter, to enable further inspection that differs from that given in (16) and (17), we provide q-analogues for the the assigned gamma integral transform as follows.

Definition 1. *Let θ be a function of certain exponential growth conditions. Then, we define the first-kind q-analogue of the gamma integral transform in terms of the q-analogue (4) and the q-gamma function (8) in the form*

$$g_{k,q}(\theta; \delta) = \frac{k^k}{(1-q)\delta^k \Gamma_q(k)} \int_0^\infty \theta(\tau)\tau^{k-1} E_q\left(\frac{qk\tau}{\delta}\right) d_q\tau, \tag{18}$$

where $\delta \in [0,\infty)$. Instead, for $\delta \in [0,\infty)$, we present a second-type q-analogue for the gamma integral transform in terms of the q-analogue (5) and the q-gamma function (8) as

$$\hat{g}_{k,q}(\theta;\delta) = \frac{k^k}{(1-q)\delta^k\Gamma_q(k)}\int_0^\infty \theta(\tau)\tau^{k-1}e_q\left(\frac{-qk\tau}{\delta}\right)d_q\tau. \tag{19}$$

Indeed, the operators $g_{k,q}$ and $\hat{g}_{k,q}$ are positive and linear, and satisfy the relations $g_{k,q}$ and $\hat{g}_{k,q} \to G_k$ as $q \to 1$.

In [6], Grőchenig and Zimmermann established a Schwartz space of slow growth test functions and proved an analogue of Hardy's theorem. Hereafter, we study our obtained q-analogues on certain new function spaces of generalized functions. Therefore, by following [30], we introduce the following definitions.

Definition 2. *By $D_q(0,\infty)$, we indicate those functions of compact supports on $(0,\infty)$ such that for $k \in \mathbb{N}$ the following holds:*

$$D_q(0,\infty) = \left\{\theta : \sup_{0<\xi<\infty}\left|D_q^k\theta(\xi)\right| < \infty\right\}. \tag{20}$$

Definition 3. *By $S_q(0,\infty)$, we indicate the space of those q-differentiable functions θ such that*

$$\sup_{0<\xi<\infty}\left|\xi^\alpha D_q^\beta\theta(\xi)\right| < \infty \tag{21}$$

for real numbers $\alpha, \beta \in \mathbb{R}$.

It is clear that $D_q(0,\infty) \subseteq S_q(0,\infty)$ and, hence, in the duality sense, $S_q^{'}(0,\infty) \subseteq D_q^{'}(0,\infty)$, where $S_q^{'}(0,\infty)$ is the space of q-tempered (slow growth) distributions while $D_q^{'}(0,\infty)$ represents the space of q-distributions of compact supports; we refer to [9] for further details.

For seeking concrete analysis, we introduce the following definitions.

Definition 4. *Denote by $\overset{q}{*}$ the q-convolution product between the functions θ_1 and θ_2 provided the integral equation*

$$\overset{q}{*} : S_q(0,\infty) \times D_q(0,\infty) \to S_q(0,\infty)$$

$$\left(\theta_1 \overset{q}{*} \theta_2\right)(\epsilon) = \int_0^\infty \theta_1\left(\epsilon t^{-1}\right)\theta_2(t)t^{-1}d_qt \tag{22}$$

exists for $\epsilon \in (0,\infty)$ and $\theta_1, \theta_2 \in S_q(0,\infty)$.

We are now going to provide certain specific q-convolution product that aligns with the previous q-convolution $\overset{q}{*}$.

Definition 5. *Let $\theta_1 \in S_q(0,\infty)$ and $\theta_2 \in D_q(0,\infty)$. Then, we define a q-convolution product $\dagger$ between θ_1 and θ_2 as*

$$\dagger : S_q(0,\infty) \times D_q(0,\infty) \to S_q(0,\infty)$$

$$(\theta_1 \dagger \theta_2)(\epsilon) = \int_0^\infty t^{k-1}\theta_1\left(\frac{\epsilon}{t}\right)\theta_2(t)d_qt. \tag{23}$$

With the help of the aforementioned integral Equations (22) and (23), we obtain the q-convolution theorem for $\hat{g}_{k,q}$ in the following manner. Discussing the q-analogue $g_{k,q}$ is quite similar. Hence, details have been avoided.

Theorem 1. *Let $\theta_1 \in S_q(0,\infty)$ and $\theta_2 \in D_q(0,\infty)$. Then, the q-convolution theorem of $\hat{g}_{k,q}$ is given by*

$$\hat{g}_{k,q}\left(\theta_1 \overset{q}{*} \theta_2\right)(\epsilon) = \left(\hat{g}_{k,q}\theta_1 \dagger \theta_2\right)(\epsilon)$$

in $S_q(0,\infty)$.

Proof. Using the hypothesis of the present theorem and the definition of $\overset{q}{*}$ given by (22), we write

$$\begin{aligned}\hat{g}_{k,q}\left(\theta_1 \overset{q}{*} \theta_2\right)(\epsilon) &= \frac{k^k}{\epsilon^k\Gamma_q(k)}\int_0^\infty \left(\theta_1 \overset{q}{*} \theta_2\right)(\xi)\xi^{k-1}E_q\left(\frac{-kq\xi}{\epsilon}\right)d_q\xi \\ &= \frac{k^k}{\epsilon^k\Gamma_q(k)}\int_0^\infty \left(\int_0^\infty t^{-1}\theta_1\left(\frac{\xi}{t}\right)\theta_2(t)d_qt\right)\xi^{k-1}E_q\left(\frac{-kq\xi}{\epsilon}\right)d_q\xi. \end{aligned} \tag{24}$$

Therefore, using the change in variables $\frac{\xi}{t} = w$, performing basic calculations, we obtain

$$\begin{aligned}\hat{g}_{k,q}\left(\theta_1 \overset{q}{*} \theta_2\right)(\epsilon) &= \frac{k^k}{\epsilon^k\Gamma_q(k)}\int_0^\infty\int_0^\infty t_1^{-1}\theta(w)\theta_2(t)d_qtw^{k-1}E_q\left(\frac{-kqwt}{\epsilon}\right)d_qw \\ &= \frac{k^k}{\epsilon^k\Gamma_q(k)}\int_0^\infty t^{k-1}\theta_2(t)\left(\int_0^\infty \theta_1(w)w^{k-1}E_q\left(\frac{-kqwt}{\epsilon}\right)d_qw\right)d_qt.\end{aligned}$$

Therefore, taking into account the definition of $\dagger$ gives

$$\begin{aligned}\hat{g}_{k,q}\left(\theta_1 \overset{q}{*} \theta_2\right)(\epsilon) &= \int_0^\infty t^{k-1}\theta_2(t)\hat{g}_{k,q}\theta_1\left(\frac{\epsilon}{t}\right)d_qt \\ &= \left(\hat{g}_{k,q}\theta_1\dagger\theta_2\right)(\epsilon).\end{aligned}$$

Therefore, all we need to finish our demonstration is to demonstrate that, for any $\theta_1 \in S_q(0,\infty)$ and $\theta_2 \in D_q(0,\infty)$, we have

$$\theta_1\dagger\theta_2 \in S_q(0,\infty). \tag{25}$$

Let $\beta, \alpha \in (0,\infty)$. Then, by the induced topology on $S_q(0,\infty)$, we write

$$\begin{aligned}\left\|\epsilon^\alpha D_\epsilon^\beta(\theta_1\dagger\theta_2)(\epsilon)\right\|_{S_q(0,\infty)} &= \left|\epsilon^\alpha D_\epsilon^\beta\left(\int_0^\infty t^{k-1}\theta_2(t)\theta_1\left(\frac{\epsilon}{t}\right)d_qt\right)\right| \\ &\leq \int_{K\subseteq(0,\infty)}\left|t^{k-1}\theta_2(t)\right|\left|\epsilon^\alpha D_\epsilon^\beta\theta_1\left(\frac{\epsilon}{t}\right)\right|d_qt \\ &\leq \|\theta_1\|_{S_q(0,\infty)}\int_{K\subseteq(0,\infty)}\left|t^{k-1}\theta_2(t)\right|d_qt \\ &\leq A\|\theta_1\|_{S_q(0,\infty)}\end{aligned}$$

$K \subseteq (0,\infty)$ is a compact set and A is a positive real number. In reality, the last inequality is derived from the boundedness condition of θ_2 and our knowledge that $\theta_1 \in S_q(0,\infty)$. This ends the proof. □

4. *q*-Boehmians of Rapid Decay

The axioms for furnishing the q-Boehmian spaces formed from rapidly decaying spaces of test functions are covered in this section. By using the sets $(S_q(0,\infty),\dagger)$ and

$\left(D_q(0,\infty), \overset{q}{*}\right)$ as well as the subset $\Delta_q(0,\infty)$ of $D_q(0,\infty)$ of delta sequences (δ_n), let us now build the space B, where the delta sequences satisfy the following identities [30]:

$$\text{(i)} \int_0^\infty \delta_n(x) d_q x = 1, \text{(ii)} |\delta_n(x)| \leq M \ \ (M > 0), \text{(iii)} supp \delta_n \subset (a_n, b_n), \tag{26}$$

provided $a_n, b_n \to 0$ as $n \to \infty$.

The associativity axiom can be proved as follows.

Theorem 2. *Let $f \in S_q(0,\infty)$ and $\theta_1, \theta_2 \in D_q(0,\infty)$. Then, the associativity axiom is given by*

$$\left(f \dagger \left(\theta_1 \overset{q}{*} \theta_2\right)\right)(\epsilon) = ((f \dagger \theta_1) \dagger \theta_2)(\epsilon).$$

Proof. Let $f \in S_q(0,\infty)$ and $\theta_1, \theta_2 \in D_q(0,\infty)$. With the benefit of Definition 5 we write

$$\left(f \overset{q}{\dagger} \left(\theta_1 \overset{q}{*} \theta_2\right)\right)(\epsilon) = \int_0^\infty t^{k-1} f\left(\frac{\epsilon}{t}\right) \left(\theta_1 \overset{q}{*} \theta_2\right)(t) d_q t.$$

Therefore, by utilizing Definition 4, the preceding integral equation can be written as

$$\left(f \dagger \left(\theta_1 \overset{q}{*} \theta_2\right)\right)(\epsilon) = \int_0^\infty t^{k-1} f\left(\frac{\epsilon}{t}\right) \left(\int_0^\infty y^{-1} \theta_1\left(y^{-1} t\right) \theta_2(y) d_q y\right) d_q t. \tag{27}$$

Now, assuming $\frac{t}{y} = z$ implies $d_q z = \frac{1}{y} d_q t$. Therefore, by employing the definitions of the convolution products and pursuing straightforward computations, we obtain

$$\begin{aligned}
\left(f \dagger \left(\theta_1 \overset{q}{*} \theta_2\right)\right)(\epsilon) &= \int_0^\infty \theta_2(y) y^{-1} \left(\int_0^\infty \theta_1\left(y^{-1} t\right) t^{k-1} f\left(\frac{\epsilon}{t}\right)\right) d_q y d_q t \\
&= \int_0^\infty \theta_2(y) y^{-1} y \left(\int_0^\infty \theta_1(z) y^{k-1} z^{k-1} f\left(\frac{\epsilon}{yz}\right)\right) d_q z d_q y \\
&= \int_0^\infty \theta_2(y) y^{k-1} \left(\int_0^\infty \theta_1(z) z^{k-1} f\left(\frac{\frac{\epsilon}{y}}{z}\right) d_q z\right) d_q y \\
&= \int_0^\infty \theta_2(y) y^{k-1} \left(\int_0^\infty f\left(\frac{\frac{\epsilon}{y}}{z}\right) \theta_1(z) z^{k-1} d_q z\right) d_q y.
\end{aligned}$$

Thus, we have obtained

$$\left(f \dagger \left(\theta_1 \overset{q}{*} \theta_2\right)\right)(\epsilon) = \int_0^\infty \theta_2(y) y^{k-1} \left(f \dagger \theta_1\left(\frac{\epsilon}{y}\right)\right) d_q y. \tag{28}$$

This ends the proof. □

Theorem 3. *Let $f, f_1, f_2 \in S_q(0,\infty), \gamma \in \mathbb{R}$ and $\theta \in D_q(0,\infty)$. Then, the following hold true.*

(i) $((f_1 + f_2) \dagger \theta)(\epsilon) = (f_1 \dagger \theta)(\epsilon) + (f_2 \dagger \theta)(\epsilon)$.

(ii) $\gamma(f \dagger \theta)(\epsilon) = (\gamma f \dagger \theta)(\epsilon)$.

(iii) *If $f_n \to f$ as $n \to \infty$ then $f_n \dagger \theta \to f \dagger \theta$ as $n \to \infty$ in $S_q(0,\infty)$.*

Proof. The proofs for (i) and (ii) may be easily obtained from using basic integral calculus. From (5), the proof of (iii) can also be established yielding

$$\begin{aligned}\left|\epsilon^{\alpha}D_{x}^{\beta}(f_n\dagger\theta-f\dagger\theta)(\epsilon)\right| &= \left|\epsilon^{\alpha}D_{\epsilon}^{\beta}((f_n-f)\dagger\theta)(\epsilon)\right| \\ &= \left|\epsilon^{\alpha}D_{\epsilon}^{\beta}\left(\int_0^{\infty}t^{k-1}(f_n-f)\left(\frac{\epsilon}{t}\right)\theta(t)d_qt\right)\right| \\ &\leq \int_a^b t^{k-1}\left|\epsilon^{\alpha}D_{\epsilon}^{\beta}(f_n-f)\left(\frac{\epsilon}{t}\right)\right|\theta(t)d_qt,\end{aligned}$$

where the supports of θ are contained in a compact interval $[a,b]$. Thus, it follows

$$\left|\epsilon^{\alpha}D_{\epsilon}^{\beta}(f_n\dagger\theta-f\dagger\theta)(\epsilon)\right| \leq A\|f_n-f\|_{S_q(0,\infty)} \to 0$$

as $n\to\infty$. This concludes the theorem's proof. □

Theorem 4. *Let $f\in S_q(0,\infty)$ and $(\theta_n)\in\Delta_q(0,\infty)$. Then, $f\dagger\theta_n\to f$ as $n\to\infty$ in $S_q(0,\infty)$.*

Proof. Let $\epsilon\in(0,\infty)$ and $\beta,\alpha\in\mathbb{R}$. Then, by making use of (6), we obtain

$$\begin{aligned}\left|\epsilon^{\alpha}D_{\epsilon}^{\beta}(f\dagger\theta_n-f)(\epsilon)\right| &= \left|\epsilon^{\alpha}D_{\epsilon}^{\beta}\left(\int_0^{\infty}t^{k-1}f\left(\frac{\epsilon}{t}\right)\theta_n(t)d_qt-f(\epsilon)\right)\right| \\ &= \left|\epsilon^{\alpha}D_{\epsilon}^{\beta}\left(\int_0^{\infty}t^{k-1}f\left(\frac{\epsilon}{t}\right)\theta_n(t)d_qt-f(\epsilon)\int_0^{\infty}\theta_n(t)d_qt\right)\right|.\end{aligned}$$

Thus, the previous equation can rewritten as

$$\left|\epsilon^{\alpha}D_{\epsilon}^{\beta}(f\dagger\theta_n-f)(\epsilon)\right| \leq \int_0^{\infty}\left|\epsilon^{\alpha}D_{\epsilon}^{\beta}\left(f\left(\frac{\epsilon}{t}\right)-f(\epsilon)\right)\right|\left|t^{k-1}\theta_n(t)\right|d_qt. \tag{29}$$

Hence, from (29) and the compact support of the delta sequences (θ_n), we establish that

$$\begin{aligned}\left|\epsilon^{\alpha}D_{\epsilon}^{\beta}(f\dagger\theta_n-f)(\epsilon)\right| &\leq \int_{a_n}^{b_n}\left|\epsilon^{\alpha}D_{\epsilon}^{\beta}\left(f\left(\frac{\epsilon}{t}\right)-f(\epsilon)\right)\right|\left|t^{k-1}\theta_n(t)\right|d_qt \\ &\leq M(b_n-a_n)\to 0,\end{aligned}$$

as $n\to\infty$, for some positive constant M.

This ends the proof. □

The Boehmian space B with $(S_q(0,\infty),\dagger)$, $\left(D_q(0,\infty),\overset{q}{*}\right)$, $\Delta_q(0,\infty)$ is therefore generated.

The two q-Boehmians φ_n/δ_n and g_n/ε_n can be added in B by the equation

$$(\varphi_n/\delta_n)+(g_n/\varepsilon_n)=(\varphi_n\dagger\delta_n+g_n\dagger\delta_n)/\left(\delta_n\overset{q}{*}\varepsilon_n\right). \tag{30}$$

The q-Boehmian φ_n/δ_n can be multiplied in B by $\gamma\in\mathbb{R}$ as $\gamma(\varphi_n/\delta_n)=(\gamma\varphi_n)/\delta_n$, whereas the expansion of $\dagger$ and D^{α} to B are expressed as

$$(\varphi_n/\delta_n)\dagger(g_n/\varepsilon_n)=(\varphi_n\dagger g_n)/\left(\delta_n\overset{q}{*}\varepsilon_n\right) \text{ and } D^{\alpha}(\varphi_n/\delta_n)=(D^{\alpha}\varphi_n)/\delta_n, \alpha\in(0,\infty).$$

The product $\dagger$ can be extended to $B\dagger S_q(0,\infty)$ as

$$(\varphi_n/\delta_n)\dagger\omega=(\varphi_n\dagger\omega)/\delta_n, \text{ where } \varphi_n,\omega\in S_q(0,\infty),(\varphi_n/\delta_n)\in B. \tag{31}$$

Let $(\beta_n) \in B$. Then, $\beta_n \stackrel{\delta}{\to} \beta$ in B, if there can be found a delta sequence (δ_n) such that, for $(\beta_n \dagger \delta_k)$ and $(\beta \dagger \delta_k) \in S_q(0,\infty)$ $(n,k \in \mathbb{N})$,

$$\lim_{n\to\infty} \beta_n \dagger \delta_k = \beta \dagger \delta_k \text{ in } S_q(0,\infty) \text{ for every } k \in \mathbb{N}. \tag{32}$$

An alternative way of expressing (32) is as follows: $\beta_n \stackrel{\delta}{\to} \beta$ as $n \to \infty \Leftrightarrow$ there are $\varphi_{n,k}$, $\varphi_k \in S_q(0,\infty)$ and $(\delta_k) \in \Delta_q(0,\infty)$, $\beta_n = (\varphi_{n,k}/\delta_k)$, $\beta = (\varphi_k/\delta_k)$ and to every $k \in \mathbb{N}$ we have $\lim_{n\to\infty} \varphi_{n,k} = \varphi_k$ in $S_q(0,\infty)$.

The other type convergence is that $\beta_n \stackrel{\Delta}{\to} \beta$ (as $n \to \infty$) if there can be found a $(\delta_n) \in \Delta_q(0,\infty)$ such that $(\beta_n - \beta)\dagger\delta_n \in S_q(0,\infty)$ $(\forall n \in \mathbb{N})$ and $\lim_{n\to\infty}(\beta_n - \beta)\dagger\delta_n = 0$ in $S_q(0,\infty)$.

Defining the space H with $\left(S_q(0,\infty), \stackrel{q}{*}\right)$, $\left(D_q(0,\infty), \stackrel{q}{*}\right)$ and $\Delta_q(0,\infty)$ is quite analogous. Hence, we avoid repeating the same analogues.

In H, two Boehmians added, applying $\stackrel{q}{*}$ are respectively introduced as

$$\begin{aligned}(\varphi_n/\delta_n) + (g_n/\varepsilon_n) &= \left(\varphi_n \stackrel{q}{*} \delta_n + g_n \stackrel{q}{*} \delta_n\right) / \left(\delta_n \stackrel{q}{*} \varepsilon_n\right) \\ &\quad \text{and} \\ (\varphi_n/\delta_n) \stackrel{q}{*} (g_n/\varepsilon_n) &= \left(\varphi_n \stackrel{q}{*} g_n\right) / \left(\delta_n \stackrel{q}{*} \varepsilon_n\right).\end{aligned}$$

Multiplying a Boehmian in H by $\alpha \in \mathbb{R}$ is explained as $\alpha(\varphi_n/\delta_n) = (\alpha\varphi_n)/\delta_n$. The D^α in H is explained as $D^\alpha(\varphi_n/\delta_n) = (D^\alpha\varphi_n)/\delta_n$, $\alpha \in (0,\infty)$. To all $(\varphi_n/\delta_n) \in H$ and $\omega \in S_q(0,\infty)$, $\stackrel{q}{*}$ can be extended to $B\dagger S_q(0,\infty)$ by $(\varphi_n/\delta_n) \stackrel{q}{*} \omega = \left(\varphi_n \stackrel{q}{*} \omega\right)/\delta_n$. $\beta_n \stackrel{\delta}{\to} \beta$ in H, if there can be found a delta sequence (δ_n) such that, for $\left(\beta_n \stackrel{q}{*} \delta_k\right)$ and $\left(\beta \stackrel{q}{*} \delta_k\right) \in S_q(0,\infty)$, $n,k \in \mathbb{N}$, we have $\lim_{n\to\infty} \beta_n \stackrel{q}{*} \delta_k = \beta \stackrel{q}{*} \delta_k$ in $S_q(0,\infty)$ for every $k \in \mathbb{N}$. Or it can be expressed in H as follows: $\beta_n \stackrel{\delta}{\to} \beta$ (as $n \to \infty$) $\Longleftrightarrow \varphi_{n,k}$, $\varphi_k \in S_q(0,\infty)$ and $(\delta_k) \in \Delta$, $\beta_n = \varphi_{n,k}/\delta_k$, $\beta = \varphi_k/\delta_k$ and to every $k \in \mathbb{N}$ we have $\lim_{n\to\infty} \varphi_{n,k} = \varphi_k$ in $S_q(0,\infty)$. $\beta_n \stackrel{\Delta}{\to} \beta$ (as $n \to \infty$) $\Longleftrightarrow$ there can be found a $(\delta_n) \in \Delta_q(0,\infty)$ such that $(\beta_n - \beta) \stackrel{q}{*} \delta_n \in S_q(0,\infty)$ $(\forall n \in \mathbb{N})$ and $\lim_{n\to\infty}(\beta_n - \beta) \stackrel{q}{*} \delta_n = 0$ in $S_q(0,\infty)$.

Hence, we assert that the q-gamma integral of the Boehmian f_n/δ_n can be given as

$$\hat{g}^b_{k,q}(f_n/\delta_n) = \left(\hat{g}_{k,q} f_n\right)/\delta_n \tag{33}$$

which falls in the space H, as $\hat{g}_{k,q} f_n \in S_q(0,\infty)$. Definition (33) is well defined in the sense that, if $f_n/\delta_n = g_n/\epsilon_n \in B$, then $f_n \stackrel{q}{*} \epsilon_m = g_m \stackrel{q}{*} \delta_n$. Applying (33) and the convolution theorem (Theorem 1), we obtain

$$\left(\hat{g}_{k,q} f_n\right)\dagger\epsilon_m = \left(\hat{g}_{k,q} g_m\right)\dagger\delta_n.$$

Thus, it follows that

$$\left(\hat{g}_{k,q} f_n\right)/\delta_n \sim \left(\hat{g}_{k,q} g_n\right)/\epsilon_n$$

in H. That is,

$$\left(\hat{g}_{k,q} f_n\right)/\delta_n = \left(\hat{g}_{k,q} g_n\right)/\epsilon_n.$$

By this, our assertion is fulfilled.

The space $S_q(0,\infty)$ can be identified as a subspace of B through the following mapping

$$\begin{aligned} I_B &: S_q(0,\infty) \to B \\ I_B(f) &= (f \dagger \delta_n)/\delta_n \end{aligned} \tag{34}$$

where $(\delta_n) \in \Delta_q(0,\infty)$, whereas its identification as a subspace of the q-Boehmian space H is given by the mapping

$$\begin{aligned} I_H &: S_q(0,\infty) \to H \\ I_H(f) &= \left(f \overset{q}{*} \epsilon_n\right)/\epsilon_n \end{aligned} \tag{35}$$

where $(\epsilon_n) \in \Delta_q(0,\infty)$.

5. Generalized q-Gamma Operator

The following theorems summarize several features of the extension $\hat{g}^b_{k,q}$ of $\hat{g}_{k,q}$. They discuss linearity, continuity with respect to the δ and Δ_q- convergence, isomorphic property, and some inversion formula for the $\hat{g}^b_{k,q}$ transform.

Although the following theorem is straightforward yet crucial, we provide its detailed proof.

Theorem 5. *Let I_B and I_H have their usual meaning given in* (34) *and* (35), *respectively. Then, we have*

$$I_B\left(\hat{g}_{k,q}\right) = \hat{g}^b_{k,q}(I_H).$$

Proof. Let $f \in S_q(0,\infty)$ and $(\delta_n) \in \Delta_q(0,\infty)$; then, by Equation (34) and the convolution theorem, we derive

$$I_B\left(\hat{g}_{k,q}\right)(f) = I_B\left(\hat{g}_{k,q}f\right) = \left(\hat{g}_{k,q}f \dagger \delta_n\right)/\delta_n.$$

Therefore, Equation (35) and the convolution theorem yield

$$\begin{aligned} I_B\left(\hat{g}_{k,q}\right)(f) &= \left(\hat{g}_{k,q}\left(f \overset{q}{*} \delta_n\right)\right)/\delta_n \\ &= \hat{g}^b_{k,q}\left(\left(f \overset{q}{*} \delta_n\right)/\delta_n\right) \\ &= \hat{g}^b_{k,q}(I_H)(f). \end{aligned}$$

This ends the proof. □

Theorem 6. *The mapping $\hat{g}^b_{k,q}$ is sequentially continuous from B into H, in the sense of Δ_q-convergence.*

Proof. To prove the theorem, we prove that if $\Delta_q - \lim_{n\to\infty} \beta_n = \beta$ in B then $\Delta_q - \lim_{n\to\infty} \hat{g}^b_{k,q}\beta_n = \hat{g}^b_{k,q}\beta$ in H. Let $\Delta_q - \lim_{n\to\infty} \beta_n = \beta$ in B; then, there is $(\delta_n) \in \Delta_q(0,\infty)$ such that

$$\Delta_q - \lim_{n\to\infty} (\beta_n - \beta) \overset{q}{*} \delta_n = 0 \text{ in } S_q(0,\infty). \tag{36}$$

Continuity of the gamma integral operator suggests writing

$$\Delta_q - \lim_{n\to\infty} \hat{g}^b_{k,q}\left((\beta_n - \beta) \overset{q}{*} \delta_n\right) = \Delta_q - \lim_{n\to\infty}\left(\left(\hat{g}^b_{k,q}\beta_n - \hat{g}^b_{k,q}\beta\right) \dagger \delta_n\right) = 0.$$

Thus, we obtain that $\Delta_q - \lim_{n\to\infty} \hat{g}^b_{k,q}\beta_n = \hat{g}^b_{k,q}\beta$ in H.

This finishes the proof. □

Theorem 7.

(i) *The mapping $\hat{g}^{b}_{k,q}$ is a linear mapping on H.*
(ii) *The mapping $\hat{g}^{b}_{k,q}$ defines an isomorphism from the q-Boehmian space H onto the q-Boehmian space B.*
(iii) *The mapping $\hat{g}^{b}_{k,q}$ is continuous with respect to δ and Δ_q- convergences.*
(iv) *The mapping $\hat{g}^{b}_{k,q}$ coincides with the classical $\hat{g}_{k,q}$.*

Proof. Since comparable proofs for Parts (i) through (iii) are already published in the literature, see [12,20,24]. So, we prove Part (iv). Let $\sigma \in S_q(0,\infty)$ and $\left(\sigma \overset{q}{*} \delta_n\right)/\delta_n$ be its representation in B, where $(\delta_n) \in \Delta_q(0,\infty)$ $(\forall n \in \mathbb{N})$. In order to demonstrate the final portion of the theory, it is evident that (δ_n) is independent of the representative for every $n \in \mathbb{N}$. Thus, using the convolution theorem, we obtain

$$\hat{g}^{b}_{k,q}\left(\left(\sigma \overset{q}{*} \delta_n\right)/\delta_n\right) = \hat{g}_{k,q}\left(\sigma \overset{q}{*} \delta_n\right)/\delta_n = \left(\hat{g}_{k,q}\sigma \dagger \delta_n\right)/\delta_n = \left(\hat{g}_{k,q}\sigma\right)\dagger(\delta_n/\delta_n).$$

Consequently, the q-Boehmian $\left(\hat{g}_{k,q}\sigma \dagger \delta_n\right)/\delta_n$ forms a representation to $\hat{g}_{k,q}\sigma$ in the classical space $S_q(0,\infty)$.

This ends proof. □

Next, we present an inversion formula of $\hat{g}^{b}_{k,q}$ as follows.

Definition 6. *Let $\left(\hat{g}^{b}_{k,q}\varphi_n\right)/\delta_n \in H$. Then, in H, we define the inverse integral operator of $\hat{g}^{b}_{k,q}$ of the q-Boehmian $\left(\hat{g}^{b}_{k,q}\varphi_n\right)/\delta_n$ as*

$$\begin{aligned} I^{e}_{g}\left(\left(\hat{g}_{k,q}\varphi_n\right)/\delta_n\right) &= : \left(\hat{g}^{-1}_{k,q}\left(\hat{g}_{k,q}\varphi_n\right)\right)/\delta_n \\ &= : \varphi_n/\delta_n \in B, \end{aligned}$$

for each $(\delta_n) \in \Delta_q(0,\infty)$.

Theorem 8. *Let $\left(\hat{g}_{k,q}\varphi_n\right)/\delta_n \in H$ and $\varphi \in S_q(0,\infty)$. Then, we have*

$$I^{e}_{g}\left(\left(\left(\hat{g}_{k,q}\varphi_n\right)/\delta_n\right)\dagger\varphi\right) = (\varphi_n/\delta_n) \overset{q}{*} \varphi \tag{37}$$

and

$$\hat{g}^{b}_{k,q}\left((\varphi_n/\delta_n) \overset{q}{*} \varphi\right) = \left(\left(\hat{g}_{k,q}\varphi_n\right)/\delta_n\right)\dagger\varphi.$$

Proof. Assume $\left(\hat{g}_{k,q}\varphi_n\right)/\delta_n \in H$. For every $\varphi \in S_q(0,\infty)$, we have

$$I^{e}_{g}\left(\left(\left(\hat{g}_{k,q}\varphi_n\right)/\delta_n\right)\dagger\varphi\right) = I^{e}_{g}\left(\left(\left(\hat{g}_{k,q}\varphi_n\right)\dagger\varphi\right)/\delta\right) = I^{e}_{g}\left(\left(\left(\hat{g}_{k,q}\varphi_n\right)\dagger\varphi\right)/\delta_n\right). \tag{38}$$

We reach the proof of the first part by using the convolution theorem. The proof of the second part is nearly identical. We leave out specifics.

This ends the proof. □

We declare without proof the following result.

Theorem 9. *Let β and β^* be in B and ρ and ρ^* be in H; then, we have*

$$(i)\ \hat{g}^b_{k,q}\left(\beta \overset{q}{*} \beta^*\right) = \hat{g}^b_{k,q}\beta \dagger \beta^*, \qquad (ii)\ I^e_g(\rho \dagger \rho^*) = I^e_g\rho \overset{q}{*} \rho^*.$$

The proof of the theorem has been removed since it is easy to establish.

Theorem 10. *Let $\left(\hat{g}_{k,q}\rho_n\right)/\xi_n$ be a q-Boehmian in H, $(\xi_n) \in \Delta_q(0,\infty)$ and $\alpha \in S_q(0,\infty)$. Then, we have*

$$\hat{g}^b_{k,q}\left((\rho_n/\xi_n) \overset{q}{*} \alpha\right) = \left(\hat{g}_{k,q}\rho_n/\xi_n\right)\dagger\alpha \text{ and } I^e_g\left(\left(\hat{g}_{k,q}\rho_n/\xi_n\right)\dagger\alpha\right) = (\rho_n/\xi_n) \overset{q}{*} \alpha. \qquad (39)$$

Proof. Assume $\hat{g}_{k,q}\rho_n/\xi_n \in H$, $(\xi_n) \in \Delta_q(0,\infty)$ and $\alpha \in S_q(0,\infty)$. Then, by the convolution theorem and (33), we have

$$\begin{aligned}
\hat{g}^b_{k,q}\left((\rho_n/\xi_n) \overset{q}{*} \alpha\right) &= \hat{g}^b_{k,q}\left(\left(\rho_n \overset{q}{*} \alpha\right)/\xi_n\right) \\
&= \hat{g}_{k,q}\left(\rho_n \overset{q}{*} \alpha\right)/\xi_n \\
&= \left(\hat{g}_{k,q}\rho_n \dagger \alpha\right)/\xi_n \\
&= \left(\left(\hat{g}_{k,q}\rho_n\right)/\xi_n\right)\dagger\alpha.
\end{aligned}$$

Also, by Definition 5, the convolution theorem and the extension of the operation $\overset{q}{*}$ suggest writing

$$\begin{aligned}
I^e_g\left(\left(\left(\hat{g}_{k,q}\rho_n\right)/\xi_n\right)\dagger\alpha\right) &= I^e_g\left(\left(\hat{g}_{k,q}\rho_n \dagger \alpha\right)/\xi_n\right) \\
&= I^e_g\left(\hat{g}_{k,q}\left(\rho_n \overset{q}{*} \alpha\right)/\xi_n\right) \\
&= I^e_g\left(\hat{g}_{k,q}\left(\rho_n \overset{q}{*} \alpha\right)/\xi_n\right) \\
&= \left(\rho_n \overset{q}{*} \alpha\right)/\xi_n \\
&= (\rho_n/\xi_n) \overset{q}{*} \alpha.
\end{aligned}$$

Hence, the proof is ended. □

6. Conclusions

This article introduces and discusses two q-analogues of the gamma operator, focusing on various finite products of different types of q-Bessel functions. This paper discusses definitions and properties of q-analogues of the gamma integral operator and its extension to classes of generalized distributions. Several convolution theorems are established and proven, including q-convolution products, symmetric q-delta sequences and q-quotients of sequences. The q-equivalence classes of generalized distributions, or q-Boehmians, are achieved through the application of the convolution theorems. Thus, the q-gamma operators are performed to correspond with the classical integral operator and are thus extended to the generalized spaces. Furthermore, it is demonstrated that, when equipped with the generalized spaces, the generalized q-gamma integral is linear, sequentially continuous and continuous with respect to some involved convergence.

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