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On the Fixed Point Theorem for Large Contraction Mappings with Applications to Delay Fractional Differential Equations

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Abstract: This paper explores a new class of mappings and presents several fixed-point results for these mappings. We define these mappings by combining well-known mappings in the literature, specifically the large contraction mapping and Chatterjea's mapping. This combination allows us to achieve significant fixed-point results in complete metric spaces, both in a continuous and a non-continuous sense. Additionally, we provide an explicit example to validate our findings. Furthermore, we discuss a general model for fractional differential equations using the Caputo derivative. Finally, we outline the benefits of our study and suggest potential areas for future research.

Keywords: Chatterjea's map; fixed point; complete metric space; large contraction; Caputo operator

MSC: 47H10

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1. Introduction

The Banach fixed-point theorem [1], commonly known as the contraction mapping theorem in mathematics, plays a crucial role in studying metric spaces. It guarantees the presence and distinctiveness of fixed points for certain self-maps of metric spaces, and additionally provides a practical method for determining these fixed points. In science and engineering, fractional differential equations have experienced significant growth in recent decades due to their extensive range of applications [2–4]. The basis of the theory of fractional differential equations lies in the existence of solutions, prompting numerous researchers to employ fixed point theory as a valuable approach for proving the existence and uniqueness of solutions [5–13].

In 1972, Chatterjea [14] obtained a fixed point result that is a generalization of the Khanan fixed point (See [15,16]). Several mathematicians have generalized and extended Chatterjea's Theorem and Banach's Theorem of fixed points; for example, the authors in [17] presented necessary and sufficient conditions to establish the existence and uniqueness of fixed points of Chatterjea's maps in b-metric space. The authors in [18] introduced the notion of cyclic weakly Chatterjea-type contraction. In [19], the authors formulated Chatterjea contractions using graphs in metric spaces endowed with a graph. The fixed point results for large-Kannan mappings which are a combination of Kannan and large contraction mappings, have been introduced in [20]. Some other papers in this field are presented in [21–25].

Burton [26,27] noted that Banach's theorem gains greater significance when subjected to specific alterations in practical contexts. Consequently, he rephrased it in the sense of large contraction mapping. For this purpose, we have written this paper to combine Chatterjea's mapping with large contraction mapping to extract a new fixed point theorem for a new mapping.

The rest of this paper is divided as follows: Section 2 presents some preliminary results to help the analysis. In Section 3, we give our main results with an illustrative example. An application of our study to fractional differential equations is given in Section 4. Finally, we end this paper with a conclusion.

2. Preliminaries

This section presents a set of previous results on which our subsequent work is based.

Theorem 1 ([1])**.** *Consider the complete metric space (Ω, ρ) and a contraction mapping $S : \Omega \to \Omega$, i.e.,*

$$\rho(Su, Sv) \leq \lambda \rho(u, v),$$

for all $u, v \in \Omega$, where $\lambda \in (0, 1)$. Then, there is a unique fixed point v_0 in Ω for the map S. In addition, for each $u_0 \in \Omega$, the sequence of iterates $\{S^n u_0\}_n$ converges to v_0.

Theorem 2 ([14])**.** *Consider the complete metric space (Ω, ρ) and an application $S : \Omega \to \Omega$. If we consider the existence of $\lambda \in \left[0, \frac{1}{2}\right)$ such that*

$$\rho(Su, Sv) \leq \lambda[\rho(u, Sv) + \rho(v, Su)], \tag{1}$$

for all $u, v \in \Omega$, then we assure the existence of a unique fixed point w_0 in Ω for the map S. In addition, for each $u_0 \in \Omega$, the sequence of iterates $\{S^n u_0\}_n$ converges to w_0.

Definition 1 ([26])**.** *Consider a metric space (Ω, ρ) and let $S : \Omega \to \Omega$ be an application on Ω. We say that the application S is a large contraction, if for $u, v \in \Omega$, with $u \neq v$, we have $\rho(Su, Sv) < \rho(u, v)$, and if for all $\varepsilon > 0$, there exists $\varsigma < 1$ such that*

$$[u, v \in \Omega, \ \ \rho(u, v) \geq \varepsilon] \Longrightarrow \rho(Su, Sv) \leq \varsigma \rho(u, v).$$

Note that every contraction application is a large contraction. The following example in [26] shows that, in general, the converse is not true.

Example 1. *Let $(\Omega, \rho) = (\mathbb{R}, |.|)$ and let $S : \mathbb{R} \longrightarrow \mathbb{R}$, defined by $Su = u - u^3$. Then for $u_1, u_2 \in \mathbb{R}$, by using the Mean Value Theorem, we obtain*

$$|Su_1 - Su_2| = \left|u_1 - u_1^3 - u_2 + u_2^3\right| \leq \left|1 - 3c^2\right| |u_1 - u_2|,$$

where $c \in (\min\{u_1, u_2\}, \max\{u_1, u_2\})$.

Subsequently, it becomes apparent from the inequality mentioned above that there is a ς which is small enough, such that for any $u_1, u_2 \in [-\varsigma, \varsigma]$ ($u_1 \neq u_2$), we have $|Su_1 - Su_2| < |u_1 - u_2|$. Additionally, it was proved in [26] that for a given $\varepsilon > 0$, if $|u_1 - u_2| \geq \varepsilon$, then

$$|Su_1 - Su_2| \leq \left|1 - \frac{\varepsilon^2}{4}\right| |u_1 - u_2|.$$

Moreover, since $S0 = 0$ and $\lim_{u \to 0}\left|\frac{u - u^3}{u}\right| = 1$, we deduce that S is not a contraction application on $[-\varsigma, \varsigma]$.

Theorem 3 ([26]). *Consider a complete metric space (Ω, ρ) and let the application $S : \Omega \to \Omega$, which is a large contraction. If there exists $u_0 \in \Omega$ and a constant $L > 0$, such that*

$$\rho(u_0, S^n u_0) \leq L \text{ for all } n \geq 1, \tag{2}$$

then the application S has a unique fixed point in Ω.

In [20], the authors noted the following:

(i) If (Ω, ρ) is a compact metric space, then condition (2) can be neglected. Indeed, Edelstein's theorem guarantees the existence and uniqueness of the fixed point in this particular case.
(ii) If (Ω, ρ) is a complete metric space that is bounded, then condition (2) is satisfied, since $\rho(u_0, S^n u_0) \leq \varsigma(\Omega)$, where $\varsigma(\Omega)$ is the diameter of Ω.

3. Main Results

Motivated by [15,20,26], we introduce large-Chatterjea mappings in metric spaces in two senses.

Definition 2. *Consider the metric space (Ω, ρ) and let the application $S : \Omega \to \Omega$. We say that S is a large-Chatterjea contraction application (with continuous manner), if for $u, v \in \Omega$, with $u \neq v$, we have $\rho(Su, Sv) < \rho(u, v)$, and if for all $\varepsilon > 0$, there exists $\varsigma < \frac{1}{2}$ such that*

$$[u, v \in \Omega, \ \ \rho(u, v) \geq \varepsilon] \Longrightarrow \rho(Su, Sv) \leq \varsigma[\rho(u, Sv) + \rho(v, Su)].$$

It is worth noting that the condition $\rho(Su, Sv) < \rho(u, v)$, $(u \neq v)$ does not give the existence of the fixed point. To perceive this, it is sufficient to acquire $(\Omega, \rho) = (\mathbb{R}, |\cdot|)$ and $Su = \sqrt{u^2 + 1}$.

The following lemma reveals that the set of contraction mappings encompasses an infinite subset of Chatterjea mappings.

Lemma 1. *Consider the metric space (Ω, ρ). If we assume $S : \Omega \to \Omega$ satisfies*

$$\rho(Su, Sv) \leq \alpha \rho(u, v), \quad \alpha \in \left[0, \frac{1}{3}\right),$$

then the application S is a Chatterjea mapping with a constant of contraction equal to $\frac{\alpha}{1-\alpha}$.

Proof. Let $u, v \in \Omega$. Then, by assumption, we have

$$\rho(Su, Sv) \leq \alpha \rho(u, v),$$

where the constant α is within the interval $\left[0, \frac{1}{3}\right)$. However, when the triangle inequality is applied, we have

$$\rho(u, v) \leq \rho(u, Sv) + \rho(Sv, Su) + \rho(Su, v).$$

After multiplying the inequality by α as mentioned earlier, we can conclude the following

$$\rho(Su, Sv) \leq \alpha \rho(u, v) \leq \alpha(\rho(u, Sv) + \rho(Sv, Su) + \rho(Su, v)),$$

which implies that

$$\rho(Su, Sv) \leq \frac{\alpha}{1 - \alpha}(\rho(u, Sv) + \rho(v, Su)).$$

Since $\alpha \in \left[0, \frac{1}{3}\right)$, then $\frac{\alpha}{1-\alpha} \in \left[0, \frac{1}{2}\right)$. Consequently, S is a Chatterjea mapping. □

By the above lemma, we also conclude that if $S : \Omega \to \Omega$ is a large contraction application on Ω with $\varsigma \in \left[0, \frac{1}{3}\right)$, then S is a large-Chatterjea contraction mapping on Ω.

In the following, we will present the first result of fixed point theorem concerning large-Chatterjea contraction applications in the continuous sense.

Theorem 4. *Consider the complete metric space (Ω, ρ) and let the application $S : \Omega \to \Omega$ be a large-Chatterjea contraction mapping (with continuous manner). Then, the application S possesses a unique fixed point in Ω.*

Proof. Given $u_0 \in \Omega$, if there exists $m \in \mathbb{N}$, such that $S^m(u_0) = S^{m+1}(u_0)$, then $S(S^m u_0) = S^m u_0$, and $S^m u_0$ is a fixed point of S.

Now, assume that $S^n u_0 \neq S^{n+1} u_0$ for every $n \in \mathbb{N}$. Since the application S is large-Chatterjea contraction (with continuous manner), then

$$\rho\left(S^{n+1}u_0, S^n u_0\right) < \rho\left(S^n u_0, S^{n-1}u_0\right) < \dots < \rho(Su_0, u_0).$$

This shows that $\zeta_n = \rho\left(S^{n+1}u_0, S^n u_0\right)$ is a strictly decreasing sequence; therefore, $\lim\limits_{n \longrightarrow +\infty} \zeta_n = \gamma \geq 0$. If $\gamma > 0$, then for every $n \in \mathbb{N}$, we have

$$\rho\left(S^{n+1}u_0, S^n u_0\right) \geq \gamma.$$

Consequently, there exists $\varsigma < \frac{1}{2}$ such that

$$\begin{aligned}
\rho\left(S^{n+1}u_0, S^{n+2}u_0\right) &= \rho\left(S(S^n u_0), S\left(S^{n+1}u_0\right)\right) \\
&\leq \varsigma\left[\rho\left(S^n u_0, S^{n+2}u_0\right) + \rho\left(S^{n+1}u_0, S^{n+1}u_0\right)\right] \\
&= \varsigma\rho\left(S^n u_0, S^{n+2}u_0\right) \\
&\leq \varsigma\left[\rho\left(S^n u_0, S^{n+1}u_0\right) + \rho\left(S^{n+1}u_0, S^{n+2}u_0\right)\right].
\end{aligned}$$

This implies that

$$(1-\varsigma)\rho\left(S^{n+1}u_0, S^{n+2}u_0\right) \leq \varsigma\rho\left(S^n u_0, S^{n+1}u_0\right).$$

Thus, we conclude that

$$\begin{aligned}
\rho\left(S^{n+1}u_0, S^{n+2}u_0\right) &\leq \frac{\varsigma}{1-\varsigma}\rho\left(S^n u_0, S^{n+1}u_0\right) \\
&\leq \left(\frac{\varsigma}{1-\varsigma}\right)^2 \rho\left(S^{n-1}u_0, S^n u_0\right) \\
&\dots \\
&\leq \left(\frac{\varsigma}{1-\varsigma}\right)^n \rho\left(Su_0, S^2 u_0\right) \\
&\leq \left(\frac{\varsigma}{1-\varsigma}\right)^{n+1} \rho(u_0, Su_0).
\end{aligned} \tag{3}$$

Since $\varsigma < \frac{1}{2}$, then $k = \frac{\varsigma}{1-\varsigma} < 1$. So, by using (3), it follows that

$$\lim_{n \to \infty} \rho\left(S^n u_0, S^{n+1}u_0\right) = 0, \tag{4}$$

and this gives us a contradiction; so, $\gamma = 0$.

Next, we will show that $\{u_n\}_n$ defined by $u_n = S^n u_0$ is a Cauchy sequence in Ω. For this purpose, we assume the opposite, meaning we assume that $\{u_n\}_n$ is not a Cauchy sequence. Then, there exist a real number $\varepsilon > 0$ and subsequences (N_k) , (n_k), and (m_k) of integers such that

$$m_k > n_k > N_k, \ \ N_k \to \infty,$$

and

$$\varepsilon \le \rho\big(u_{m_k}, u_{n_k}\big). \tag{5}$$

Since S is large-Chatterjea mapping, there exists $\varsigma < \frac{1}{2}$ such that

$$\begin{aligned}
\rho\big(u_{m_k}, u_{n_k}\big) &= \rho\big(Su_{m_k-1}, Su_{n_k-1}\big) \\
&\le \varsigma\big[\rho\big(u_{m_k-1}, Su_{n_k-1}\big) + \rho\big(u_{n_k-1}, Su_{m_k-1}\big)\big] \\
&= \varsigma\big[\rho\big(u_{m_k-1}, u_{n_k}\big) + \rho\big(u_{n_k-1}, u_{m_k}\big)\big] \\
&\le \varsigma\big[\rho\big(u_{m_k-1}, u_{m_k}\big) + \rho\big(u_{m_k}, u_{n_k}\big) + \rho\big(u_{n_k-1}, u_{n_k}\big) + \rho\big(u_{n_k}, u_{m_k}\big)\big];
\end{aligned}$$

then, by using (5), we have

$$\begin{aligned}
\varepsilon &\le \rho\big(u_{m_k}, u_{n_k}\big) \\
&\le \frac{\varsigma}{1-2\varsigma}\big[\rho\big(u_{m_k-1}, u_{m_k}\big) + \rho\big(u_{n_k-1}, u_{n_k}\big)\big].
\end{aligned}$$

Letting $k \to \infty$, from (4), it follows that

$$\lim_{k\to\infty} \rho\big(u_{m_k-1}, u_{m_k}\big) = \lim_{k\to\infty} \rho\big(u_{n_k-1}, u_{n_k}\big) = 0.$$

Hence, $\lim\limits_{k\to\infty} \rho\big(u_{m_k}, u_{n_k}\big) = 0$, which is a contradiction. Thus, $\{u_n\}_n$ is a Cauchy sequence in the complete metric space Ω; then, there exists $w \in \Omega$ such that $\lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty} S^n u_0 = w$. By the continuity of the application S, we deduce that $S(w) = w$, which shows that w is a fixed point of the application S.

Now, to prove the uniqueness, we suppose that there is another fixed point, denoted by w', for the application S such that $w \neq w'$. Thus, $\rho(w, w') \ge \varepsilon_0$ for some $\varepsilon_0 > 0$. By the definition that the application S is a large-Chatterjea, there exists $\varsigma_0 < \frac{1}{2}$ such that

$$\begin{aligned}
\rho\big(w, w'\big) &= \rho\big(S(w), S(w')\big) \\
&\le \varsigma_0\big[\rho\big(w, S(w')\big) + \rho\big(w', S(w)\big)\big] \\
&= 2\varsigma_0 \rho\big(w, w'\big).
\end{aligned}$$

Since $1 - 2\varsigma_0 > 0$, we deduce that $\rho(w, w') = 0$, and this is a contradiction. Therefore, we must have $w = w'$. $\square$

Corollary 1. *Consider the complete metric space (Ω, ρ) and the application $S : \Omega \to \Omega$ such that for some integer $m_0 \ge 1$, S^{m_0} is a large-Chatterjea mapping (with continuous manner). Then, the application S possesses a unique fixed point in Ω.*

Proof. By using Theorem 4, we assure the existence of $w_0 \in \Omega$ such that $S^{m_0} w_0 = w_0$, then

$$S(S^{m_0} w_0) = S^{m_0+1} w_0 = S w_0.$$

This gives $S^{m_0}(S w_0) = S w_0$, which proves that $S w_0$ is a fixed point for the application S^{m_0}.

Second, by Theorem 4, the application S possesses a unique fixed point w_0 which satisfies $S w_0 = w_0$; so, if w_1 is another fixed point of the application S, then w_1 is a fixed point for the application S^{m_0}. Consequently, $w_0 = w_1$, which finishes the proof. $\square$

Example 2. *Consider the application* $S : [0,1] \to \mathbb{R}$*, given by* $S(u) = -u^4$ *and set*

$$\Omega = \left\{ (u_1, u_2) \in [0,1]^2 : |u_1 + u_2|^2 + \frac{1}{2}|u_1 - u_2|^2 \le \frac{1}{2} \right\}.$$

If $u_1, u_2 \in \Omega$*, then*

$$|u_1| = u_1 \le u_1 - \left(-u_2^4\right) = |u_1 - S(u_2)| \text{ and } u_1^2 + u_2^2 \le u_1 + u_2.$$

Hence,

$$\begin{aligned} |S(u_1) - S(u_2)| &= \left|-u_1^4 + u_2^4\right| \\ &\le |u_1 - u_2|\left|(u_1 + u_2)\left(u_1^2 + u_2^2\right)\right| \\ &\le (|u_1| + |u_2|)\left|(u_1 + u_2)\left(u_1^2 + u_2^2\right)\right| \\ &\le (|u_1 - S(u_2)| + |u_2 - S(u_1)|)|u_1 + u_2|^2 \\ &\le \left(\frac{1 - |u_1 - u_2|^2}{2}\right)[|u_1 - S(u_2)| + |u_2 - S(u_1)|]. \end{aligned}$$

Therefore, for a given (sufficiently small) $\varepsilon > 0$*, if* $u_1, u_2 \in \Omega$ *satisfy that* $|u_1 - u_2| \ge \varepsilon$*, we obtain*

$$|S(u_1) - S(u_2)| \le [|u_1 - S(u_2)| + |u_2 - S(u_1)|]\left(\frac{1 - \varepsilon^2}{2}\right).$$

To conclude that the application S is a large-Chatterjea, it suffices to take $\varsigma(\varepsilon) = \frac{1-\varepsilon^2}{2}$*, which finishes the proof.*

Now, we turn our attention to study the uniqueness fixed point for the large-Chatterjea applications in which the continuity is not necessary.

Definition 3. *Consider the metric space* (Ω, ρ) *and let the application* $S : \Omega \to \Omega$*. We say that S is a large-Chatterjea contraction application (with noncontinuous manner), if for* $u, v \in \Omega$*, such that* $u \neq v$*, we have*

$$\rho(Su, Sv) < \frac{1}{2}[\rho(u, Sv) + \rho(v, Su)], \tag{6}$$

and if for all $\varepsilon > 0$*, there exists* $\varsigma < \frac{1}{2}$ *such that we have*

$$[u, v \in \Omega,\ \ \rho(u, v) \ge \varepsilon] \Longrightarrow \rho(Su, Sv) \le \varsigma[\rho(u, Sv) + \rho(v, Su)]. \tag{7}$$

The following application given in [21] proves that an application satisfying that $\rho(Su, Sv) < \frac{1}{2}(\rho(u, Su) + \rho(v, Sv))$ may fail to have fixed points. By the same example, we can see that the mappings satisfying the inequality $\rho(Su, Sv) < \frac{1}{2}(\rho(u, Sv) + \rho(v, Su))$ may have no fixed points.

Example 3. *Consider the set* $\Omega = \{1 + \frac{1}{n}, n = 1, 2, ...\}$ *and* $\rho_0 : \Omega \times \Omega \longrightarrow [0, +\infty)$*, a metric given by*

$$\rho_0(u, v) = \begin{cases} 0, & \text{if } u = v, \\ u + v, & \text{if } u \neq v. \end{cases}$$

Thus, (Ω, ρ_0) *is a complete metric space. Moreover, let the application* $S : (\Omega, \rho_0) \longrightarrow (\Omega, \rho_0)$ *be defined by* $S(1 + \frac{1}{n}) = 1 + \frac{1}{n+1}$*. Then, S satisfies the inequality* $\rho_0(Su, Sv) < \frac{1}{2}(\rho_0(u, Sv) + \rho_0(v, Su))$ *for* $u \neq v$*, but S has no fixed points.*

Theorem 5. *Consider the complete metric space (Ω, ρ) and let the application $S : (\Omega, \rho) \longrightarrow (\Omega, \rho)$ be a large-Chatterjea (with noncontinuous manner). Then, the application S possesses a unique fixed point.*

Proof. First, we begin by proving the uniqueness. If the application S possesses two fixed points $u_1, u_2 \in \Omega, u_1 \neq u_2$, then

$$\begin{aligned} \rho(u_1, u_2) &= \rho(Su_1, Su_2) \\ &< \frac{1}{2}[\rho(u_1, Su_2) + \rho(u_2, Su_1)] \\ &= \frac{1}{2}[\rho(u_1, u_2) + \rho(u_2, u_1)] \\ &= \rho(u_1, u_2), \end{aligned}$$

which is a contradiction.

Second, we will prove the existence in the following steps:
Step 1:Given $u_0 \in \Omega$ and consider the sequence $\{u_n\}_n$ by $u_n = S^n u_0$ for all integers $n \in \mathbb{N}$. If there exists $m_0 \in \mathbb{N}$ that satisfies $S^{m_0}u_0 = S^{m_0+1}u_0$, then $S^{m_0}u_0$ is a fixed point for the application S.

Next, suppose that $u_n = S^n u_0 \neq S^{n+1}u_0 = u_{n+1}$ for $n \in \mathbb{N}$. We will show that the sequence $\varepsilon_n = \rho(u_n, u_{n+1})$ is strictly decreasing.

$$\begin{aligned} \rho(u_n, u_{n+1}) &= \rho\Big(S^n u_0, S^{n+1}u_0\Big) = \rho\Big(S\Big(S^{n-1}u_0\Big), S(S^n u_0)\Big) \\ &< \frac{1}{2}\Big[\rho\Big(S^{n-1}u_0, S^{n+1}u_0\Big) + \rho(S^n u_0, S^n u_0)\Big] \\ &= \frac{1}{2}\Big[\rho\Big(S^{n-1}u_0, S^n u_0\Big) + \rho\Big(S^n u_0, S^{n+1}u_0\Big)\Big]. \end{aligned}$$

So, we conclude that $\rho(u_n, u_{n+1}) < \rho(u_{n-1}, u_n)$, which proves that $\varepsilon_n = \rho(u_{n-1}, u_n)$ is a strictly decreasing sequence. Therefore, there exists $\varepsilon_0 \geq 0$ such that $\lim\limits_{n \longrightarrow +\infty} \rho(u_n, u_{n+1}) = \varepsilon_0$.
Step 2: Now, assume that $\varepsilon_0 > 0$; because the sequence $\varepsilon_n = \rho(u_n, u_{n+1})$ is decreasing, we obtain $\varepsilon_0 < \rho(u_n, u_{n+1})$ for $n \in \mathbb{N}$. Then, by assumption there exists $0 < \varsigma_0 < \frac{1}{2}$ such that

$$\begin{aligned} \rho(u_n, u_{n+1}) &= \rho\Big(S^n u_0, S^{n+1}u_0\Big) = \rho\Big(S\Big(S^{n-1}u_0\Big), S(S^n u_0)\Big) \\ &\leq \varsigma_0\Big[\rho\Big(S^{n-1}u_0, S^{n+1}u_0\Big) + \rho(S^n u_0, S^n u_0)\Big] \\ &= \varsigma_0\Big[\rho\Big(S^{n-1}u_0, S^n u_0\Big) + \rho\Big(S^n u_0, S^{n+1}u_0\Big)\Big], \end{aligned}$$

which gives

$$\rho(u_n, u_{n+1}) \leq \frac{\varsigma_0}{1 - \varsigma_0}\rho(u_{n-1}, u_n).$$

By induction, it follows that

$$\rho(u_n, u_{n+1}) \leq \left(\frac{\varsigma_0}{1 - \varsigma_0}\right)^n \rho(u_0, u_1).$$

Afterwards, since $0 < \varsigma_0 < \frac{1}{2}$, then $\frac{\varsigma_0}{1-\varsigma_0} < 1$. This proves that $\lim\limits_{n \longrightarrow +\infty} (\frac{\varsigma_0}{1 - \varsigma_0})^n = 0$ and implies $\lim\limits_{n \longrightarrow +\infty} \rho(u_n, u_{n+1}) = 0$, which is the opposite information to $\varepsilon_0 > 0$. Consequently, $\varepsilon_0 = 0$.

Step 3: Showing that $\{u_n\}_n$ is a Cauchy sequence in Ω:

If we have the contrary case, then there exists a real number α_0 and subsequences $\{N_k\}, \{n_k\}$, and $\{m_k\}$ such that

$$m_k > n_k > N_k \text{ and } \alpha_0 \leq \rho(u_{n_k}, u_{m_k}) = \rho(Su_{n_k-1}, Su_{m_k-1}),$$

which leads to the deduction that $u_{m_k-1} \neq u_{n_k-1}$.

Thus, by the same method used in the proof of Theorem 4, we obtain

$$\alpha_0 \leq \rho\big(u_{m_k}, u_{n_k}\big) \leq \frac{\varsigma}{1-2\varsigma}\big[\rho\big(u_{m_k-1}, u_{m_k}\big) + \rho\big(u_{n_k-1}, u_{n_k}\big)\big],$$

and by letting $k \to \infty$, it follows that

$$\lim_{k\to\infty} \rho\big(u_{m_k-1}, u_{m_k}\big) = \lim_{k\to\infty} \rho\big(u_{n_k-1}, u_{n_k}\big) = 0.$$

Then, $\lim_{k\to\infty} \rho\big(u_{m_k}, u_{n_k}\big) = 0$, which informs us that this is a contradiction. Therefore, $\{u_n\}_n$ is a Cauchy sequence, and because Ω is a complete metric space, there exists $w_0 \in \Omega$ such that $\lim\limits_{n\longrightarrow+\infty} u_n = w_0$.

Step 4: w_0 is a fixed point for S:

For this step, we suppose that $u_n \neq w_0$. Thus,

$$\begin{aligned}
\rho(w_0, Sw_0) &\leq \rho(w_0, Su_n) + \rho(Su_n, Sw_0) \\
&< \rho(w_0, Su_n) + \frac{1}{2}[\rho(u_n, Sw_0) + \rho(Su_n, w_0)] \\
&\leq \rho(w_0, Su_n) + \frac{1}{2}[\rho(u_n, w_0) + \rho(w_0, Sw_0) + \rho(Su_n, w_0)],
\end{aligned}$$

then,

$$0 \leq \rho(w_0, Sw_0) \leq 3\rho(u_{n+1}, w_0) + \rho(u_n, w_0).$$

Letting $n \longrightarrow +\infty$, we deduce

$$0 \leq \rho(w_0, Sw_0) \leq \lim_{n\longrightarrow+\infty} 3\rho(u_{n+1}, w_0) + \lim_{n\longrightarrow+\infty} \rho(u_n, w_0) = 0.$$

Hence $\rho(w_0, Sw_0) = 0$. As a consequence, w_0 is a fixed point of the application S, which ends the proof. □

Corollary 2. *Consider the complete metric space (Ω, ρ) and an application $S : \Omega \to \Omega$ such that S^{m_0} is a large-Chatterjea (with noncontinuous manner) for $m_0 \in \mathbb{N}$. Then, the application S possesses a unique fixed point in Ω.*

4. Application

The utilization of our previously established results in the preceding section empowers us to effectively address a range of fractional differential problems, as demonstrated in this concluding section. So, we consider a Caputo derivative operator in the following fractional differential equation

$$\begin{cases} {}^{c}D^{\alpha}(w(t) - \mu w(t-\theta)) = \phi(t, w(t)), t \in (0,1] \\ w(t) = \psi(t), \quad t \in [-\theta, 0] \end{cases} \tag{8}$$

where $\alpha \in (0,1)$, $\mu \in \left(0, \frac{1}{3}\right)$, $\theta > 0$, $w : [0,1] \to \mathbb{R}$, and $\phi(t, w(t)) : [0,1] \times \mathbb{R} \to (0, \infty)$ is continuous. Then, Equation (8) is immediately inverted as the very familiar integral equation

$$w(t) = \psi(0) - \mu\psi(-\theta) + \mu w(t-\theta) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-r)^{\alpha-1} \phi(r, w(r)) dr,$$

where Γ is the Gamma function.

Consider now the operator S defined on the Banach space $E = C([0,1], \mathbb{R})$ as

$$(Sw)(t) = \psi(0) - \mu\psi(-\theta) + \mu w(t-\theta) + (Fw)(t),$$

such that $SW \subset W$, where

$$W = \{w \in E : 0 \leq w \leq 1\},$$

and

$$(Fw)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-r)^{\alpha-1} \phi(r, w(r)) dr.$$

Under these assumptions, the following apply:

- The mapping F satisfies

$$\|Fw_1 - Fw_2\| < (1-\mu)\|w_1 - w_2\|, \forall w_1, w_2 \in W, \quad (w_1 \neq w_2);$$

- $\forall \varepsilon > 0, \exists \varsigma < \frac{1-3\mu}{2}$ such that if $w_1, w_2 \in W$ and $\|w_1 - w_2\| \geq \varepsilon$, we have for all $t \in [0,1]$

$$|(Fw_1)(t) - (Fw_2)(t)| \leq \varsigma(|w_1(t) - (Sw_2)(t)| + |w_2(t) - (Sw_1)(t)|),$$

and the operator S has a unique fixed point in W. Indeed, let $w_1, w_2 \in W$ such that $\|w_1 - w_2\| \geq \varepsilon$. Following this, with the help of our hypotheses, we acquire

$$\begin{aligned}
&|(Sw_1)(t) - (Sw_2)(t)| \\
&\leq \mu|w_1(t-\theta) - w_2(t-\theta)| + |(Fw_1)(t) - (Fw_2)(t)| \\
&\leq \mu|w_1(t-\theta) - w_2(t-\theta) + (Sw_1)(t-\theta) - (Sw_2)(t-\theta) + (Sw_2)(t-\theta) - (Sw_1)(t-\theta)| \\
&+ \varsigma(|w_1(t) - (Sw_2)(t)| + |w_2(t) - (Sw_1)(t)|) \\
&\leq \mu(\|w_1 - Sw_2\| + \|w_2 - Sw_1\| + \|Sw_1 - Sw_2\|) \\
&+ \varsigma(\|w_1 - Sw_2\| + \|w_2 - Sw_1\|),
\end{aligned}$$

which gives that

$$\|Sw_1 - Sw_2\| \leq \mu\|Sw_1 - Sw_2\| + (\mu + \varsigma)(\|w_1 - Sw_2\| + \|w_2 - Sw_1\|).$$

Hence, we have

$$\|Sw_1 - Sw_2\| \leq \frac{\mu + \varsigma}{1-\mu}(\|w_1 - Sw_2\| + \|w_2 - Sw_1\|).$$

Since $0 < \varsigma < \frac{1-3\mu}{2}$, then $\frac{\mu+\varsigma}{1-\mu} < \frac{1}{2}$ and the end result stems directly from Theorem 4.

5. Conclusions

This research issues new fixed point theorems based on the Chatterjea-type large contractions applications. Our results have been divided into two cases, the first is continuous and the second is noncontinuous. The benefit of this research is finding a particular application type that enables us to study some complex equations, as each Chatterjea mapping is a large-Chatterjea mapping, and each large contraction mapping is a large-Chatterjea mapping. However, the opposite is not valid in general.

The results are applied to delay equations with fractional derivatives to prove the existence of a unique solution. We look forward to larger applications of this work, especially for other spaces such as the b-metric space, ordered metric space, and so on.

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